

Selection of Driver Nodes in Discrete-Time Complex Networks Based on Region of Attraction

6.1 INTRODUCTION

In the previous chapter, the continuous-time models of the complex network were discussed. For most of the systems, controllers are implemented in digital domain due to ease of implementation and flexibility of usage. Discrete-time dynamics on complex network and associated problems need a detailed investigation as discrete-time dynamics is significantly different than its continuous-time counterpart. In this chapter, selection of an optimal driver node based on discrete-time model of complex networks is discussed. While considering the issue of controlling a discrete-time network, it is important to consider the limitations of driver nodes. Driver nodes have limited actuating capacity. This limitation has to be considered to achieve the control objective. Unstable eigenvalues of Adjacency matrix of discrete-time complex network may further increase the complexity. In such cases, region in state space, where stability is guaranteed, is finite. This region denotes Region of Attraction (ROA). The ROA depends on choice of driver node and control law. This chapter characterizes driver nodes and proposes a theory and algorithms such that the right driver node, among many, can be found to maximize ROA. In n -dimensional state space, the region is described by an ellipsoid. In context of linear discrete-time system with actuator saturation, many excellent works exist see [Gutman and Hagander, 1985; Saberi et al., 1996; Lauvdal and Fossen, 1997; Hu and Lin, 2001, 2002; Zhou and Duan, 2008, 2009] and the references therein. To control the discrete-time networks, a fixed number of the driver nodes is needed, i.e., set of driver nodes has fixed number of nodes, however, driver nodes are not unique. This raises a very important question about the choice of driver nodes when there are many nodes qualify to be considered as driver nodes. This chapter addresses two important issues the first issue is regarding the selection of driver node and the second considers driver node limitation. A theory and algorithms to select driver node such that the ROA corresponding to this driver node maximizes are proposed. In this work, to obtain largest possible region, LQ control law is used. The emphasis of this work is on the selection of right driver nodes rather than on control law. However, the presented theory is valid for any control law of the form $u = -Kx$. The presented algorithms can be used to identify the driver node which maximizes ROA in the power networks, robotic networks, biological networks and social networks etc.

The remaining sections of the chapter are organized as follows. The Section 6.2 describes the modeling of the networks of discrete-time dynamic with saturated actuator. Section 6.3 presents the theoretical results to characterize the maximum volume of the invariant ellipsoids of discrete-time anti-stable system and transformed discrete-time anti-stable subsystem. The simulation results corresponding to three network examples are given in Section 6.4. Finally, the summary of the chapter is presented in the Section 6.5.

6.2 MODELING OF DISCRETE-TIME COMPLEX NETWORKS WITH SATURATED ACTUATORS

A discrete-time network with n nodes can be represented by a graph $G := (V, E)$, where $V := \{1, 2, \dots, n\}$ is the set of nodes and $E \subseteq V \times V$ is the set of edges of the network. The weighted

Adjacency matrix of the discrete-time network $\mathbf{G} := (\mathbf{V}, \mathbf{E})$ is described as $\mathbf{A}$, where the elements $a_{ij} \in \mathbb{R}_{>0} \cup \{0\}$ are non zero if node i and j are adjacent, or, are 0 otherwise. The discrete-time network $\mathbf{G}$ is independently controllable by q sets of driver nodes from the superset $N_D = \{N_1, N_2, \dots, N_q\}$, where each set i.e. N_i , $i = 1, 2, \dots, q$ contains $m \leq n$ nodes and it can control the whole network. The cardinality (number of elements of the set) of a set of driver nodes is fixed for the discrete-time network $\mathbf{G}$. The superset of the control inputs corresponding to superset of the driver nodes N_D is defined as $\bar{U} := \{U_1, U_2, \dots, U_q\}$, where $U_i := \{u_{i1}, u_{i2}, \dots, u_{im}\}$, $i = 1, 2, \dots, q$. The cardinality of U_i is m . The set of control inputs U_i injected into the network corresponding to particular set of driver nodes N_i . With slight abuse of notation, the U_i denotes an m -dimensional vector depending on context in subsequent presentation, i.e. $U_i := [u_{i1}, u_{i2}, \dots, u_{im}]^T$, where $i = 1, 2, \dots, q$.

Consider a discrete-time complex network with control input vector U_i , and the corresponding input matrix as B_i , with this, discrete-time dynamics of networked systems with saturated actuators can be written as

$$x(k+1) = Ax(k) + B_i \text{sat}_i(U_i(k)) \quad (6.1)$$

where $A \in \mathbb{R}^{n \times n}$ is the Adjacency matrix, $x \in \mathbb{R}^{n \times 1}$ state vector and $U_i \in \mathbb{R}^{m \times 1}$ control input vector of the network $\mathbf{G}$. The input matrix corresponding to control input U_i is defined as $B_i \in \mathbb{R}^{n \times m}$ and each column of B_i is i^{th} non-unity vector of n -dimension corresponding to the set of driver nodes N_i , where $i = 1, 2, \dots, q$. The sat_i is a vector valued standard saturation function of appropriate dimensions i.e. $\text{sat}_i: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as $\text{sat}_i(U_i) = [\text{sat}_{i1}(u_{i1}), \text{sat}_{i2}(u_{i2}), \dots, \text{sat}_{im}(u_{im})]$ i.e., $\text{sat}_{ij}(u_{ij}) = \text{sign}(u_{ij}) \min\{u_{ij, \max}, |u_{ij}|\}$, where $u_{ij, \max} > 0$, $\forall i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$ is the saturation limit of the j^{th} component of the control input $U_i \in \mathbb{R}^m$. Without loss of generality, assume that $u_{ij, \max} = 1$, $\forall i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

Remark 6.1. Cardinality of a set of driver nodes for a single input ($m = 1$) controllable discrete-time networked system, is equal to 1, however, driver node may not be unique i.e. multiple choices of driver node may exist.

The control law for the discrete-time system defined in Eq.(6.1) is as follows

$$U(k) := -K_i x(k) = -(R + B_i^T P_i B_i)^{-1} B_i^T P_i A x(k) \quad (6.2)$$

where $R > 0$ is chosen matrix and matrix $P_i > 0$ is obtained by solving the following Discrete-Time Algebraic Riccati Equation (DARE) for some $Q > 0$

$$A^T P_i A - P_i - A^T P_i B_i (R + B_i^T P_i B_i)^{-1} B_i^T P_i A + Q = 0 \quad (6.3)$$

The control law Eq.(6.2) is divided into two parts, low gain and high gain, i.e. $U := U_L + U_H := -\frac{1}{\Phi} (R + B_i^T P_i B_i)^{-1} B_i^T P_i A x(k) - \frac{1}{\Psi} (R + B_i^T P_i B_i)^{-1} B_i^T P_i A x(k)$, where, $\Phi^{-1} + \Psi^{-1} = 1$ and $\Phi, \Psi > 0$. For a special case, the value of $\Phi = 2$ and $\Psi = 2$ are considered in this chapter. Hence, for simplicity, it is being assumed that $U_L = U_H = -\frac{1}{2} (R + B_i^T P_i B_i)^{-1} B_i^T P_i A x(k)$. The low gain part is not allowed to saturate, however, the overall control is allowed [Turner and Postlethwaite, 2001].

Remark 6.2. The chapter discusses low and high gain approach to obtain the better estimate of the ROA and performance for the discrete-time complex networks under the LQ control law [Turner and Postlethwaite, 2001], however, the presented theory is valid for any control law of the form $U = -Kx(k)$.

Under the control law Eq.(6.2), the discrete-time system defined in Eq.(6.1) is asymptotically stable for all states such that these states lie in the invariant ellipsoid $\mathcal{E}(P_i, \delta_i)$ [Gutman and Hagander, 1985], and is given as

$$\mathcal{E}(P_i, \delta_i) = \{x(k) : x^T(k) P_i x(k) \leq \delta_i\} \quad (6.4)$$

where $\delta_i(>0)$ is the radius of the invariant ellipsoid $\mathcal{E}(P_i, \delta_i)$ for the discrete-time system defined in Eq.(6.1).

This radius is obtained by using the overall control law Eq.(6.2). For the discrete-time system Eq.(6.1), the radius of the invariant ellipsoid is given as

$$\delta_i := \min_j \left(\frac{(r_j + b_j^T P_i b_j)^2 u_{j,max}^2}{b_j^T P_i A P_i^{-1} A^T P_i b_j} \right) \quad (6.5)$$

where $i = 1, 2, \dots, q; j = 1, 2, \dots, m$, r_j is the j^{th} diagonal entry of chosen matrix $R > 0$, $u_{j,max}$ is the maximum limit of the j^{th} element of the control vector U_i and b_j is the j^{th} column vector of the input matrix $B_i \in \mathbb{R}^{n \times m}$.

Without loss of generality, assume $r_j = 1$ for $j = 1, 2, \dots, m$. The proof of Eq.(6.5) is given in [Gutman and Hagander, 1985; Henrion et al., 1999; Turner and Postlethwaite, 2001; Li and Lin, 2015]. It is obvious from Eqs.(6.3), (6.5), that with different sets of driver nodes, the corresponding input matrix B_i and matrix $P_i > 0$ will change and this eventually changes δ_i and the corresponding ROA.

Remark 6.3. Computational aspects for large discrete-time complex networks are similar to the computational aspects for continuous-time complex networks, as discussed in subsection 5.3.4 of Chapter 5.

6.3 SELECTION OF OPTIMAL DRIVER NODES

This section splits into three subsections. First, the problem statement is explained, second computes the volume of ROA of the discrete-time anti-stable networked system Eq.(6.1) (all eigenvalues of A are unstable) with saturated actuators and third computes the volume of ROA of unstable discrete-time networked subsystem of Eq.(6.1) with saturated actuators.

6.3.1 Problem Setup

A discrete-time complex network of homogeneous nodes with q sets of driver nodes from the superset $N_D = \{N_1, N_2, \dots, N_q\}$, where each set i.e. N_i , $i = 1, 2, \dots, q$ contains $m \leq n$ nodes, having capability to control the network acting alone, is considered. For a complex network with large number of nodes, all sets of driver nodes i.e. superset N_D can be obtained by applying the structural controllability theory [Lin, 1974], maximum matching algorithm [Liu et al., 2011], and exact controllability theory [Yuan et al., 2013]. However, all of above methods are computationally difficult to use [Jia and Barabási, 2013] and required a brute force search to obtain the driver nodes sets. Furthermore, consider different superset of the control inputs corresponding to sets of the driver nodes N_D as $\bar{U} := \{U_1, U_2, \dots, U_q\}$, where $U_i := \{u_{i1}, u_{i2}, \dots, u_{im}\}$, $i = 1, 2, \dots, q$ and cardinality of U_i is m , a region where stability, with different combinations of inputs, is guaranteed may expand or shrink. This chapter presents mathematical formulation and algorithms to find an optimal set of inputs such that the largest ROA for a given state-feedback control law of the form given in Eq.(6.2) can be achieved. The objective of this chapter is to characterize sets of driver nodes N_D such that ROA becomes largest in comparison to region obtained for all other sets of driver nodes from the superset of driver nodes N_D for given control law Eq.(6.2).

6.3.2 Region of Attraction and Selection of Optimal Set of Driver Nodes of Discrete-Time Anti-Stable Networked Systems With Saturated Actuators

This section computes a volume of ROA for the discrete-time networked system defined in Eq.(6.1) with all unstable eigenvalues of A (all eigenvalues $|\lambda(A)| \geq 1$). Consider the discrete-time

anti-stable networked system Eq.(6.1) (all eigenvalues of A are unstable) with saturated actuators after this a system with stable and unstable eigenvalues will be considered. The discrete-time networked system is controllable with q sets of driver nodes and cardinality of each set of driver nodes is m . The invariant ellipsoid (or ROA) corresponding to input U_i of the discrete-time networked system Eq.(6.1) depends on the positive definite matrix P_i which can be obtained by solving the DARE given in Eq.(6.3).

Theorem 6.1. For the discrete-time networked system defined in Eq.(6.1) with control input U_i , control law defined in Eq.(6.2) and radius of the invariant ellipsoid (or ROA) defined in Eq.(6.5):

(i) The volume of the ROA is given by

$$Vol_i = \frac{Vol(S(0, \sqrt{\delta_i}))}{\sqrt{\det(P_i)}} \quad (6.6)$$

where $Vol(S(0, \sqrt{\delta_i}))$ is the volume of n -dimensional Ball ($n \geq 3$) of radius $\sqrt{\delta_i}$ and center at origin, and $\det(P_i)$ is the determinant of the positive definite matrix P_i .

The value of $Vol(S(0, \sqrt{\delta_i}))$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009] is given as

$$Vol(S(0, \sqrt{\delta_i})) = \frac{(\pi \delta_i)^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \quad (6.7)$$

where n is the number of nodes in the discrete-time networked system Eq.(6.1) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\Re(\zeta) > 0$).

(ii) Largest ROA for the state-feedback control law of the form given in Eq.(6.2) corresponds to the i^{th} set of driver nodes for which the volume becomes the largest i.e.

Largest Volume = $\max_i \{Vol_i\}$, $i = 1, 2, \dots, q$.

Proof: To prove the first part, the outer boundaries of the contractively invariant ellipsoids $\mathcal{E}(P_i, \delta_i) = \{x(k) : x^T(k) P_i x(k) \leq \delta_i\}$ of the linear discrete-time anti-stable networked system Eq.(6.1) are obtained from the following equation

$$x^T(k) P_i x(k) = \delta_i \quad (6.8)$$

and the volume of the ROA is given by

$$Vol_i = \sum_{k=x_{jmin}}^{x_{jmax}} \underbrace{\dots \sum_{j=1, \dots, n} x_1(k) x_2(k) \dots x_n(k)} \quad (6.9)$$

where x_{jmin} and x_{jmax} are the minimum and maximum limits of state vector $x(k)$ in j^{th} direction.

Now assume that $x(k) = T_i y(k)$, where $T_i > 0$ is chosen a positive definite matrix and $y(k) \in \mathbb{R}^n$ state vector, then the Eq.(6.8) can be rewritten as

$$y^T(k) T_i^T P_i T_i y(k) = \delta_i \quad (6.10)$$

The positive definite matrix T_i is chosen in such that $T_i^T P_i T_i = I_{n \times n}$, where $I_{n \times n}$ is the identity matrix of dimension $n \times n$. This implies that $T_i = ((M_i \sqrt{D_i})^{-1})^T$, where M_i is the eigenvector matrix

of positive definite matrix P_i and matrix D_i is the diagonal matrix (elements are the eigenvalues of the matrix P_i).

Using the change of variables $x(k) = T_i y(k)$, it can be obtained $\underbrace{x_1(k) \ x_2(k) \ \dots \ x_n(k)}_{y_1(k) \ y_2(k) \ \dots \ y_n(k)} = (\frac{1}{\sqrt{\det(P_i)}})$. With this substitution, Eq.(6.9) becomes

$$Vol_i = (\frac{1}{\sqrt{\det(P_i)}}) \sum_{k=y_{jmin}, j=1, \dots, n}^{y_{jmax}} \underbrace{y_1(k) \ y_2(k) \ \dots \ y_n(k)}_{y_1(k) \ y_2(k) \ \dots \ y_n(k)} \quad (6.11)$$

The Eq.(6.9) becomes $y^T(k)y(k) = \delta_i$ i.e. $y_1^2(k) + y_2^2(k) + \dots + y_n^2(k) = \delta_i$ by using the change of variables.

Now, the volume of n -dimensional Ball ($n \geq 3$) is defined as

$$Vol(S(0, \sqrt{\delta_i})) := \sum_{k=y_{jmin}, j=1, \dots, n}^{y_{jmax}} \underbrace{y_1(k) \ y_2(k) \ \dots \ y_n(k)}_{y_1(k) \ y_2(k) \ \dots \ y_n(k)} \quad (6.12)$$

with center at origin and the radius $\sqrt{\delta_i}$.

Using Eq.(6.11) and Eq.(6.12), the Eq.(6.6) is obtained. Then using Eq.(6.6) and Eq.(6.7), the volume of ROA becomes

$$Vol_i = \frac{(\pi \delta_i)^{\frac{n}{2}}}{\sqrt{\det(P_i)} \ \Gamma(\frac{n}{2} + 1)} \quad (6.13)$$

After obtaining the volume of ROA corresponding to each set of driver nodes, the largest ROA for the state-feedback control law Eq.(6.2) is obtained corresponds to i^{th} set of driver nodes for which the volume of ROA becomes largest. This completes the proof of Theorem 6.1.

Corollary 6.1. For $n = 2$, the region is an area in state space. The area of the ellipse (region) is given by

$$Area_i = \frac{\pi \ \delta_i}{\sqrt{\det(P_i)}} \quad (6.14)$$

$Area_i > Area_j \iff \delta_j^2 \det(P_i) < \delta_i^2 \det(P_j)$, $i \neq j = 1, 2$. The i^{th} node is chosen as a driver node of the discrete-time networked system defined in Eq.(6.1), corresponding to the largest area.

The computation of $Area_{max} / Vol_{max}$ of the ROA for n nodes discrete-time anti-stable networked systems with q sets of driver nodes are summarized in the Algorithm 6.1.

Algorithm 6.1. Find maximum Area / Volume of the ROA and corresponding set of driver nodes of the discrete-time anti-stable networked system is defined in Eq.(6.1)

Input : Discrete-time anti-stable networked system $G := (V, E)$, number of nodes n , sets of driver nodes q , cardinality of driver nodes sets m , $Q > 0$ and $R = I_{m \times m}$

Output : Maximum Area / Volume of the ROA and its corresponding i^{th} set of driver nodes of the discrete-time networked system Eq.(6.1)

- 1: for $i = 1 : q$ do
- 2: Identify input matrix B_i corresponding to control input U_i
- 3: Obtain the positive definite matrix $P_i > 0$ by solving the DARE given in Eq.(6.3)

- 4: Calculate the radius δ_i of ROA using Eq.(6.5)
- 5: Calculate the $Area_i/Vol_i$ of ROA using Eq.(6.14) or Eq.(6.13)
- 6: end for
- 7: Calculate $Area_{max} / Vol_{max}$ as

$$Area_{max} / Vol_{max} = \max_i \{Area_i / Vol_i \mid i = 1, 2, \dots, q\}$$
- 8: index=1
- 9: for $i = 1 : q$ do
- 10: if $Area_{max} = Area_i$ or $Vol_{max} = Vol_i$ then
- 11: $max_{index}(index, 1) = i$
- 12: index=index+1
- 13: end if
- 14: end for
- 15: return $Area_{max} / Vol_{max}$ and the corresponding indices of sets of driver nodes

Remark 6.4. Whenever ROA of two or more sets of driver nodes equals to the largest ROA of the discrete-time networked system, then any one of them can be selected as the optimal set of driver nodes.

6.3.3 Region of Attraction and Selection of Optimal Set of Driver Nodes of Discrete-Time Unstable Networked Systems With Saturated Actuators

Now, this section considers discrete-time unstable networked systems which have stable eigenvalues (all eigenvalues $|\lambda(A)| < 1$, are inside the unit circle) and unstable eigenvalues (all $|\lambda(A)| \geq 1$) with saturated actuators. The discrete-time networked system is controllable with q sets of driver nodes and cardinality of each set of driver nodes is m . The original discrete-time networked system defined in Eq.(6.1) is transformed into two discrete-time subsystems. The first discrete-time subsystem has all unstable eigenvalues and the second has all stable eigenvalues. In what follows, the discrete-time subsystem with unstable eigenvalues is considered and the ROA is obtained. In this case, the original discrete-time networked system is transformed using a non singular transformation matrix V i.e $x(k+1) = Vz(k+1)$, where, the V can be obtained from eigenvectors of A .

The transformed discrete-time networked system is composed of an anti-stable subsystem and a stable subsystem of the original discrete-time networked system defined in Eq.(6.1) and it can be represented as

$$z(k+1) = \tilde{A}z(k) + \tilde{B}_i sat_i(U_i(k)) \quad (6.15)$$

where $U_i \in \mathbb{R}^{m \times 1}$; $|u_{ij,max}| = 1$; $i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

The Adjacency matrix $\tilde{A}$ can be partitioned as [Hu and Lin, 2001; Horn and Johnson, 2012]

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \quad (6.16)$$

where $\tilde{A}_1 \in \mathbb{R}^{l \times l}$ anti-stable and $\tilde{A}_2 \in \mathbb{R}^{(n-l) \times (n-l)}$ stable matrices of the discrete-time anti-stable subsystem and discrete-time stable subsystem respectively.

The control input matrix $\tilde{B}_i$ is partitioned as

$$\tilde{B}_i = V^{-1}B_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} \quad (6.17)$$

where $\tilde{B}_{1i} \in \mathbb{R}^{l \times m}$ and $\tilde{B}_{2i} \in \mathbb{R}^{(n-l) \times m}$, n is the number of nodes (order) in the discrete-time networked system Eq.(6.1) and l is the order of the discrete-time anti-stable subsystem respectively. Here, the

input matrix $\tilde{B}_{1i}$ is denoted as $\tilde{B}_{1i} = [\tilde{b}_{i1}^1, \tilde{b}_{i2}^1, \dots, \tilde{b}_{im}^1]$, where the column $\tilde{b}_{ij}^1$ represents the j^{th} column of the input matrix $\tilde{B}_{1i}$ of the transformed discrete-time anti-stable subsystem and similarly the input matrix $\tilde{B}_{2i}$ is denoted as $\tilde{B}_{2i} = [\tilde{b}_{i1}^2, \tilde{b}_{i2}^2, \dots, \tilde{b}_{im}^2]$, where the column $\tilde{b}_{ij}^2$ represents that the j^{th} column of the input matrix $\tilde{B}_{2i}$ of the transformed discrete-time stable subsystem.

The transformed discrete-time networked system defined in Eq.(6.15) can be rewritten as

$$\begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} sat_i(U_i(k)) \quad (6.18)$$

Now, the discrete-time anti-stable subsystem of Eq.(6.18) is described as follows

$$z_1(k+1) = \tilde{A}_1 z_1(k) + \tilde{B}_{1i} sat_i(U_i(k)) \quad (6.19)$$

Remark 6.5. To obtain the ROA of the discrete-time unstable networked systems, only discrete-time anti-stable subsystem Eq.(6.19) needs to be considered. For the discrete-time stable subsystem, global stabilization is possible with saturated actuators [Hu and Lin, 2001].

For discrete-time anti-stable subsystem Eq.(6.19), the control law is as follows

$$U_i(k) := -H z_1(k) \quad (6.20)$$

where $H = (R_1 + \tilde{B}_{1i}^T \tilde{P}_{1i} \tilde{B}_{1i})^{-1} \tilde{B}_{1i}^T \tilde{P}_{1i} \tilde{A}_1$ is a feedback gain matrix.

The positive definite matrix $\tilde{P}_{1i} > 0$ is the solution of the following DARE for the discrete-time anti-stable subsystem defined in Eq.(6.19)

$$\tilde{A}_1^T \tilde{P}_{1i} \tilde{A}_1 - \tilde{P}_{1i} - \tilde{A}_1 \tilde{P}_{1i} \tilde{B}_{1i} (R_1 + \tilde{B}_{1i}^T \tilde{P}_{1i} \tilde{B}_{1i})^{-1} \tilde{B}_{1i}^T \tilde{P}_{1i} \tilde{A}_1 + \tilde{Q}_1 = 0 \quad (6.21)$$

where $\tilde{Q}_1 > 0$ and $R_1 > 0$ are chosen positive definite and diagonal matrices respectively.

The radius ($\tilde{\delta}_{1i}$) of the invariant ellipsoid ($\mathcal{E}(\tilde{P}_{1i}, \tilde{\delta}_{1i}) = \{z_1 : z_1^T \tilde{P}_{1i} z_1 \leq \tilde{\delta}_{1i}\}$) of the discrete-time anti-stable subsystem defined in Eq.(6.19) is described as

$$\tilde{\delta}_{1i} = \min_j \left(\frac{(r_{1i} + \tilde{b}_{ij}^{1,T} \tilde{P}_{1i} \tilde{b}_{ij}^1)^2 u_{ij,max}^2}{\tilde{b}_{ij}^{1,T} \tilde{P}_{1i} \tilde{A}_1 \tilde{P}_{1i}^{-1} \tilde{A}_1^T \tilde{P}_{1i} \tilde{b}_{ij}^1} \right) \quad (6.22)$$

where $\tilde{b}_{ij}^1$ is the j^{th} column of the input matrix $\tilde{B}_{1i}$ of the transformed discrete-time anti-stable subsystems. Without loss of generality, assume $u_{ij,max} = 1$ and $r_{1i} = 1$ for $i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

Theorem 6.2. For the discrete-time anti-stable subsystem defined in Eq.(6.19) with control law Eq.(6.20) and the radius of invariant ellipsoid (or ROA) defined in Eq.(6.22):

(i) The volume of ROA of the discrete-time anti-stable subsystem Eq.(6.19) is given by

$$Vol_{1i} = \frac{(\pi \tilde{\delta}_{1i})^{\frac{l}{2}}}{\Gamma(\frac{l}{2} + 1) \sqrt{\det(\tilde{P}_{1i})}} \quad (6.23)$$

where $\det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$, l is the number of nodes in the discrete-time anti-stable subsystem Eq.(6.19) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\Re(\zeta) > 0$).

(ii) Largest ROA for the state-feedback control law of the form given in Eq.(6.20) corresponds to i^{th} set of driver nodes for which volume becomes largest i.e.

Largest Volume = $\max_i \{Vol_{1i}\}$, $i = 1, 2, \dots, q$.

Proof: To prove the first part, the outer boundaries of the contractively invariant ellipsoids $\varepsilon(\tilde{P}_{1i}, \tilde{\delta}_{1i}) = \{z_1(k) : z_1^T(k) \tilde{P}_{1i} z_1(k) \leq \tilde{\delta}_{1i}\}$ of the linear discrete-time anti-stable subsystem Eq.(6.19) are obtained from the following equation

$$z_1^T(k) \tilde{P}_{1i} z_1(k) = \tilde{\delta}_{1i} \quad (6.24)$$

and the volume of the ROA is given by

$$Vol_{1i} = \sum_{k=z_{1j_{min}}}^{z_{1j_{max}}} \dots \sum_{j=1, \dots, l} \underbrace{z_{1_1}(k) \ z_{1_2}(k) \ \dots \ z_{1_l}(k)} \quad (6.25)$$

where $z_{1j_{min}}$ and $z_{1j_{max}}$ are the minimum and maximum limits of state vector $z_1(k)$ in j^{th} direction.

Now assume that $z_1(k) = \tilde{T}_{1i} y_1(k)$, where $\tilde{T}_{1i} > 0$ is a chosen positive definite matrix and $y_1(k) \in \mathbb{R}^l$ state vector, then the Eq.(6.24) can be rewritten as

$$y_1^T(k) \tilde{T}_{1i}^T \tilde{P}_{1i} \tilde{T}_{1i} y_1(k) = \tilde{\delta}_{1i} \quad (6.26)$$

The positive definite matrix $\tilde{T}_{1i}$ is chosen in such a way that $\tilde{T}_{1i}^T \tilde{P}_{1i} \tilde{T}_{1i} = I_{l \times l}$, where $I_{l \times l}$ is the identity matrix of dimension $l \times l$. This implies that $\tilde{T}_{1i} = ((\tilde{M}_{1i} \sqrt{\tilde{D}_{1i}})^{-1})^T$, where $\tilde{M}_{1i}$ is the eigenvector matrix of positive definite matrix $\tilde{P}_{1i}$ and matrix $\tilde{D}_{1i}$ is the diagonal matrix (elements are the eigenvalues of the matrix $\tilde{P}_{1i}$).

Using the change of variables $z_1(k) = \tilde{T}_{1i} y_1(k)$, it can be obtained $\underbrace{z_{1_1}(k) \ z_{1_2}(k) \ \dots \ z_{1_l}(k)} = (\frac{1}{\sqrt{\det(\tilde{P}_{1i})}}) \underbrace{y_{1_1}(k) \ y_{1_2}(k) \ \dots \ y_{1_l}(k)}$. With this substitution, Eq.(6.25) becomes

$$Vol_{1i} = (\frac{1}{\sqrt{\det(\tilde{P}_{1i})}}) \sum_{k=y_{1j_{min}}}^{y_{1j_{max}}} \dots \sum_{j=1, \dots, l} \underbrace{y_{1_1}(k) \ y_{1_2}(k) \ \dots \ y_{1_l}(k)} \quad (6.27)$$

The Eq.(6.25) becomes $y_1^T(k) y_1(k) = \tilde{\delta}_{1i}$ i.e. $y_{1_1}^2(k) + y_{1_2}^2(k) + \dots + y_{1_l}^2(k) = \tilde{\delta}_{1i}$ by using the change of variables.

Now, the volume of l -dimensional Ball ($l \geq 3$) is defined as

$$Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}})) := \sum_{k=y_{1j_{min}}}^{y_{1j_{max}}} \dots \sum_{j=1, \dots, l} \underbrace{y_{1_1}(k) \ y_{1_2}(k) \ \dots \ y_{1_l}(k)} \quad (6.28)$$

with center at origin and the radius $\sqrt{\tilde{\delta}_{1i}}$.

Using Eq.(6.27) and Eq.(6.28), the volume of the ROA is obtained as

$$Vol_{1i} = \frac{Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}}))}{\sqrt{\det(\tilde{P}_{1i})}} \quad (6.29)$$

where $Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}}))$ is the volume of l -dimensional Ball ($l \geq 3$) of radius $\sqrt{\tilde{\delta}_{1i}}$ and center at origin, and $\det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$.

The value of $Vol_1(S(0, \sqrt{\tilde{\delta}_{li}}))$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009] is given as

$$Vol_1(S(0, \sqrt{\tilde{\delta}_{li}})) = \frac{(\pi \tilde{\delta}_{li})^{\frac{l}{2}}}{\Gamma(\frac{l}{2} + 1)} \quad (6.30)$$

where l is the number of nodes in the discrete-time subsystem Eq.(6.19) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\Re(\zeta) > 0$).

Using Eq.(6.29) and Eq.(6.30), the volume of ROA becomes

$$Vol_{li} = \frac{(\pi \tilde{\delta}_{li})^{\frac{l}{2}}}{\sqrt{\det(\tilde{P}_{li})} \Gamma(\frac{l}{2} + 1)} \quad (6.31)$$

After obtaining the volume of ROA corresponding to each set of driver nodes, the largest ROA for the state-feedback control law Eq.(6.20) is obtained corresponds to i^{th} set of driver nodes for which the volume of ROA becomes largest. This completes the proof of Theorem 6.2.

Corollary 6.2. For $l = 2$, it requires to compute area of the ROA of the discrete-time anti-stable subsystem defined in Eq.(6.19). The area of ROA is same as given in Eq.(6.14) with appropriate replacement of respective variables.

The computation of $Area_{max} / Vol_{max}$ of the ROA for the discrete-time unstable networked systems of order n with dimension of the discrete-time anti-stable subsystem l , $l < n$ are summarized in the Algorithm 6.2.

Algorithm 6.2. Find maximum Area/Volume of the ROA and the corresponding set of driver nodes of the discrete-time unstable networked system defined in Eq.(6.1)

Input : Discrete-time unstable networked systems $G := (V, E)$, number of nodes n , number of anti-stable nodes l , sets of driver nodes q , cardinality of driver nodes sets m , $\tilde{Q}_1 > 0$ and $R_1 = I_{m \times m}$, where $I_{m \times m}$ is the identity matrix of dimension $m \times m$.

Output : Maximum Area/Volume of the ROA and its corresponding i^{th} set of driver nodes of the discrete-time networked system defined in Eq.(6.1).

- 1: Partition matrix $\tilde{A}$ using Eq.(6.16)
- 2: for $i = 1 : q$ do
- 3: Identify $B_i \in \mathbb{R}^{n \times m}$ of the original discrete-time networked system defined in Eq.(6.1)
- 4: Partition $\tilde{B}_i$ using Eq.(6.17)
- 5: Calculate the positive definite matrix $\tilde{P}_{li}$ using Eq.(6.21)
- 6: Calculate the radius $\tilde{\delta}_{li}$ of the ROA using Eq.(6.22)
- 7: Calculate the $Area_i / Vol_i$ of the ROA using Eq.(6.14) or Eq.(6.23)
- 8: end for
- 9: Calculate $Area_{max} / Vol_{max}$ as

$$Area_{max} / Vol_{max} = \max_i \{Area_i / Vol_i \mid i = 1, 2, \dots, q\}$$
- 10: index=1
- 11: for $i = 1 : q$ do
- 12: if $Area_{max} = Area_i$ or $Vol_{max} = Vol_i$ then
- 13: $max_{index}(index, 1) = i$
- 14: index=index+1
- 15: end if
- 16: end for

17: return $Area_{max} / Vol_{max}$ and the corresponding indices of sets of driver nodes

Remark 6.6. Whenever largest ROA for two (or more) sets of driver nodes is equal, then any one set of them can be chosen as a set of driver nodes.

Remark 6.7. For some applications, the presented theories and algorithms can also be used to obtain a specific driver node (or a set of the specific driver nodes) based on the shape of the ROA.

6.4 SIMULATION RESULTS AND DISCUSSION

This section presents three numerical examples to verify the theoretical results. First example considers discrete-time anti-stable network which has unstable eigenvalues and other two examples consider discrete-time networked systems that have stable and unstable eigenvalues.

Example 6.1. Consider a weighted directed discrete-time network of two nodes as shown in Figure 6.1.

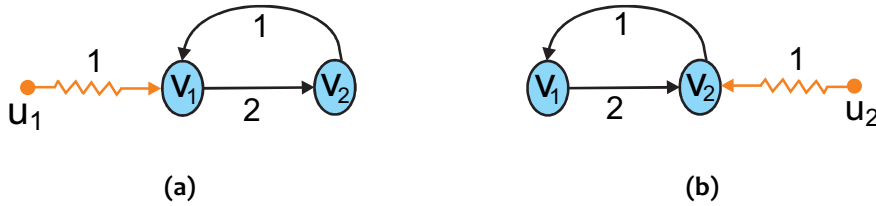


Figure 6.1: Two nodes of discrete-time network with input (a) u_1 , and (b) u_2

The Adjacency and control input matrices are given as:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Assume B_i is a variable input matrix which depends on i^{th} driver node in order to control the discrete-time network. Eigenvalues of the Adjacency matrix (A) of the discrete-time network are $\lambda(A) = \{1.4142, -1.4142\}$, which clearly indicates that the network is anti-stable (all eigenvalues (A) are outside the unit circle). Let $Q = I_2$, $R = 1$ and solve the DARE defined in Eq.(6.3) for different input matrices B_i . The radius of the invariant ellipsoids $\delta_1 = 1.5378$ and $\delta_2 = 2.3631$ are obtained using Eq.(6.5), for driver nodes 1 and 2 respectively. Compute the values of $\sqrt{\det(P_1)} = 4.8465$, $\sqrt{\det(P_2)} = 4.0335$, and also obtained the area of the two ellipsoids using Eq.(6.14), they are 0.9968 and 1.8405 respectively. Now, the maximum area of the invariant ellipsoid is obtained using Algorithm 6.1 is 1.8405 corresponding to node 2.

The ROA's of the discrete-time network considered in Example 6.1 for different driver nodes, are plotted in the Figure 6.2.

From the Figure 6.2, it is observed that the area of second ellipsoid (black color) is larger than first ellipsoid (blue color). In conclusion, the second node should be chosen as the driver node to control the entire network. Table 6.1, shows the effectiveness of the theoretical developments.

Example 6.2. Consider a weighted directed discrete-time network with four nodes as shown in Figure 6.3.

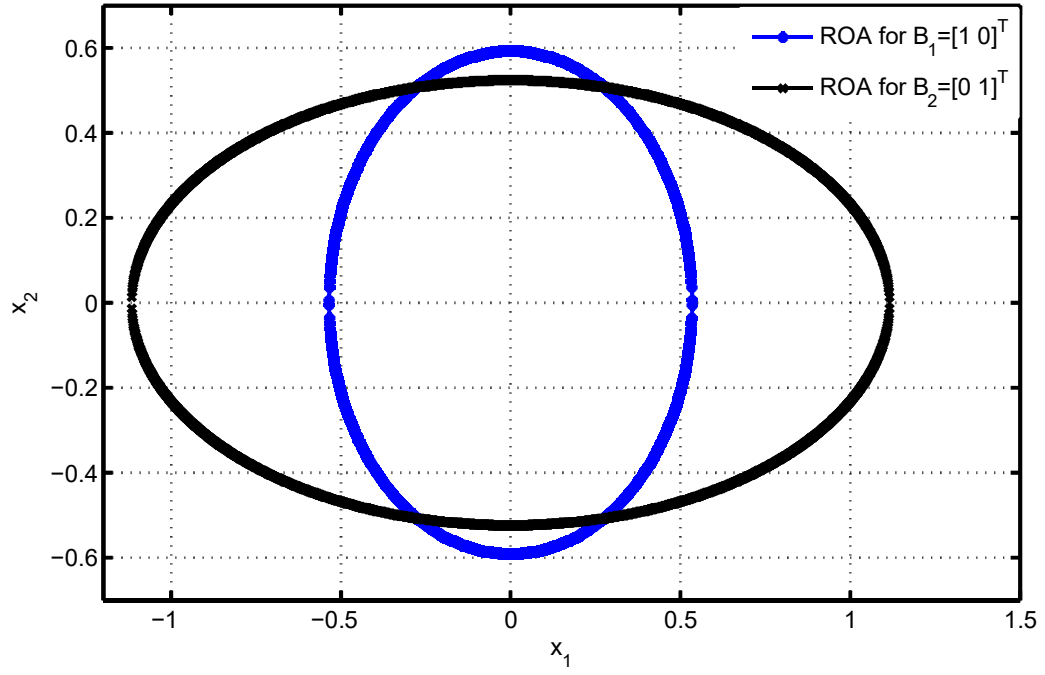


Figure 6.2 : Region of Attractions (ROA's) of the discrete-time network considered in Example 6.1 for different control input matrices B_i

Table 6.1 : Area of the ROA for different driver nodes of network considered in Example 6.1

Driver Node	Area of ROA	$\frac{Area_i}{Area_{max}}$	Remark
2	1.8405	1	Maximum Area
1	0.9968	0.5416	46% Reduction

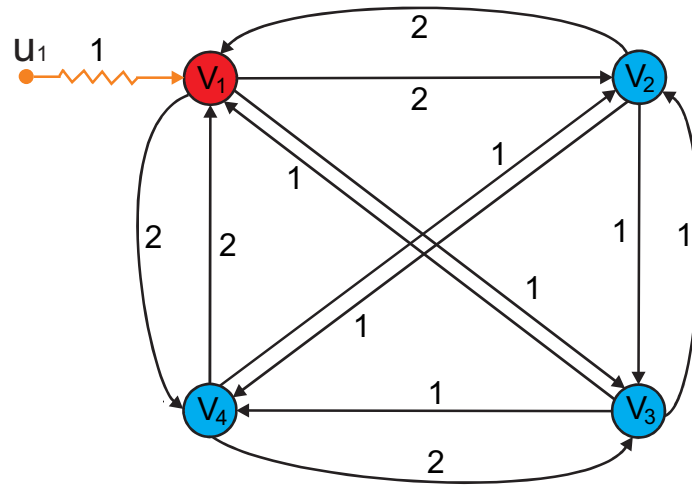


Figure 6.3 : Four nodes discrete-time network with control input u_1 and it can also be controlled by other individual driver node i.e. u_3 , & u_4

The Adjacency matrix and control input matrices are given as:

$$A = \begin{bmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 0 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Depending on choice of a particular node as a driver node, which makes the discrete-time network controllable, input matrix B_i has three possibilities. The eigenvalues of the discrete-time network shown in Figure 6.3, obtained as $\lambda(A) = \{4.2879, -2.4210, -0.8669, -0.9999\}$. Here two eigenvalues of A are unstable eigenvalues (outside the unit circle) so the network will be partitioned into two subsystem according to Eq.(6.18). Now, the discrete-time anti-stable subsystem Eq.(6.19) is analyzed. Let $\tilde{Q}_1 = I_2$, $R_1 = 1$ and solving the DARE Eq.(6.21) for different input matrices B_i corresponding to different driver nodes. Using the Algorithm 6.2, the radius of the invariant ellipsoids $\tilde{\delta}_{11} = 1.1059$, $\tilde{\delta}_{13} = 1.0406$ and $\tilde{\delta}_{14} = 1.0644$ are obtained. Also, the values of $\sqrt{\det(\tilde{P}_{11})} = 45.0967$, $\sqrt{\det(\tilde{P}_{13})} = 1.0266 \times 10^2$ and $\sqrt{\det(\tilde{P}_{14})} = 68.6931$ are obtained.

Now the ROA's of the discrete-time network considered in Example 6.2 for different driver nodes are plotted in the Figure 6.4. From the Figure 6.4, it has been obtained the largest area when first node is selected as driver node (blue color) vis-a-vis region corresponding to node 3 and 4 (red and magenta colors). In conclusion, the first node should be chosen as the driver node because it maximizes the ROA. From the Table 6.2, it is observed that the maximum area of the invariant ellipsoid 0.0770, corresponding to driver node 1. According to Table 6.2, results clearly validate the effectiveness of the theoretical developments.

Table 6.2 : Area of the ROA for different driver nodes of network considered in Example 6.2

Driver Node	Area of ROA	$\frac{Area_i}{Area_{max}}$	Remark
1	0.0770	1	Maximum Area
3	0.0318	0.4131	59% Reduction
4	0.0486	0.6319	37% Reduction

Example 6.3. Consider an undirected multiple-input discrete-time network of eight nodes with control constraint $u_{ij,max} = 1$ as shown in Figure 6.5.

The Adjacency matrix and input matrices are given as:

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_{1\{1,2\}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{2\{1,4\}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

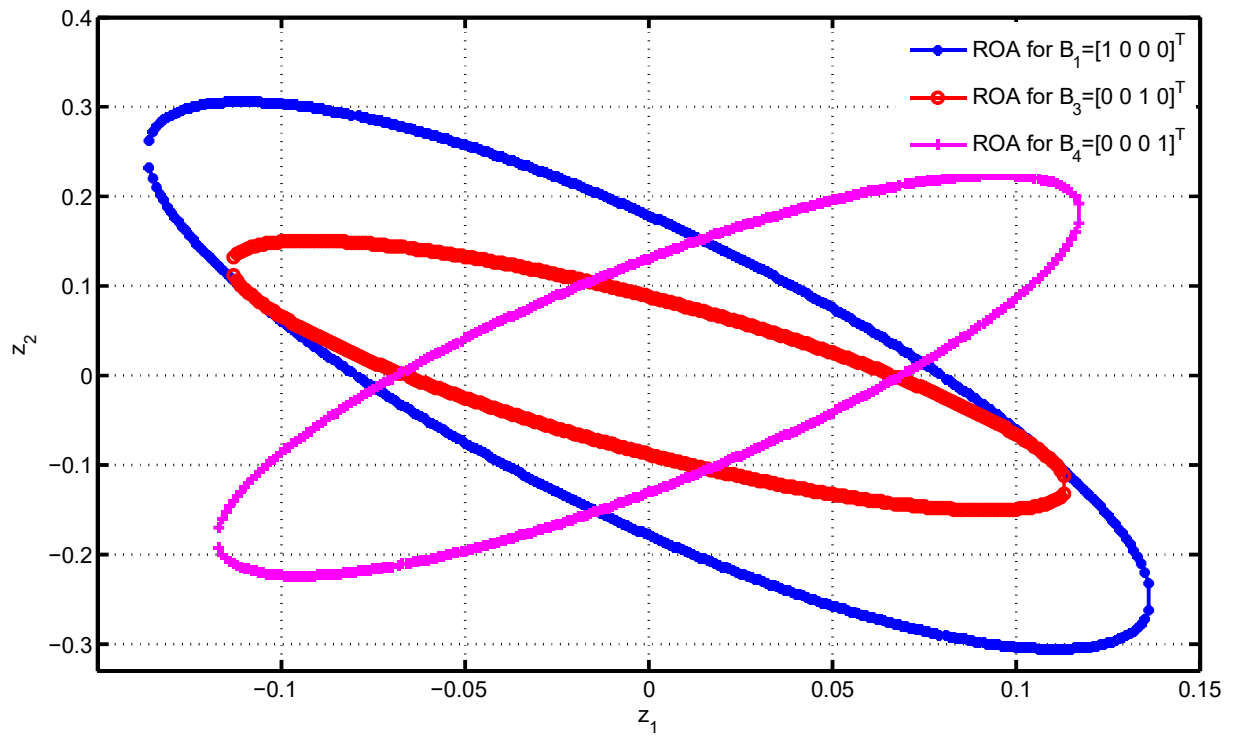


Figure 6.4 : Region of Attraction (ROA's) of the discrete-time network considered in Example 6.2 for different control input matrices B_i

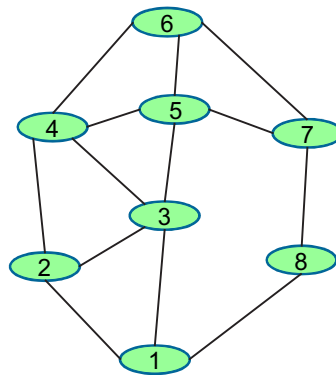


Figure 6.5 : Network of eight nodes independently controllable with different sets of two driver nodes

$$\begin{aligned}
B_{3\{1,7\}} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, B_{4\{1,8\}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_{5\{2,4\}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_{6\{2,6\}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_{7\{2,8\}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
B_{8\{4,6\}} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, B_{9\{4,7\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, B_{10\{6,7\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, B_{11\{6,8\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, B_{12\{7,8\}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

The exact controllability theory predicts that it needs minimum two driver nodes ($m = 2$) to control the entire discrete-time networked system shown in Figure 6.5. There are total twelve distinct choices of the set of driver nodes i.e. $N_1 = \{1, 2\}$, $N_2 = \{1, 4\}$, $N_3 = \{1, 7\}$, $N_4 = \{1, 8\}$, $N_5 = \{2, 4\}$, $N_6 = \{2, 6\}$, $N_7 = \{2, 8\}$, $N_8 = \{4, 6\}$, $N_9 = \{4, 7\}$, $N_{10} = \{6, 7\}$, $N_{11} = \{6, 8\}$, and $N_{12} = \{7, 8\}$. Depending on choice of a particular nodes set as a set of driver nodes, input matrix $B_i \in \mathbb{R}^{8 \times 2}$ has twelve possibilities B_1 to B_{12} . The eigenvalues of the multiple-input discrete-time networked system shown in Figure 6.5, obtained as $\lambda(A) = \{-2.1504, -1.7321, -1, -1, -0.4598, 1.1789, 1.7321, 3.4314\}$. Here seven eigenvalues are unstable (eigenvalues of (A) are outside or on the unit circle) and one stable (eigenvalues of (A) are inside the unit circle), so the system will be split into two subsystem according to Eq.(6.18). Now, the anti-stable subsystem Eq.(6.19) is analyzed. Using the LQ control design method with $\tilde{Q}_1 = I_7$, $R_1 = 1$ and solve the DARE Eq.(6.21) for different input matrices B_i corresponding to different set of driver nodes.

According to Table 6.3, the maximum volume of the ROA is obtained as 0.1114×10^{-3} corresponding to set of driver nodes $N_3 = \{1, 7\}$. In conclusion, choosing the third possibility of the set of driver nodes as the optimal driver nodes set is advantageous because the maximum volume of the ROA is obtained. Notice that the algorithms developed here ensures the identification of the optimal set of driver nodes among all possible set of driver nodes. According to Table 6.3, results clearly validate the effectiveness of the theoretical results.

6.5 SUMMARY

The chapter has presented a method to select an optimal set of driver nodes in the discrete-time networked system. A practical limitation on an actuator magnitude has been considered. The proposed theory and algorithms give most suitable driver nodes so that stability with saturated driver node can be ensured in the largest possible region. It has been shown, using numerical Example 6.3, 98% reduction in stability region occurs, in worst case, if driver node is not selected using the proposed algorithm. A close agreement between theoretical and simulation has been

Table 6.3 : Volume of the ROA for different set of driver nodes of network considered in Example 6.3

Set of Driver Nodes	Corresponding Input Matrix B_i	Volume of ROA	$\frac{Vol_i}{Vol_{max}}$	Remark
$N_3 = \{1, 7\}$	B_3	0.1114×10^{-3}	1	Maximum Volume
$N_1 = \{1, 2\}$	B_1	0.0126×10^{-3}	0.1135	89% Reduction
$N_2 = \{1, 4\}$	B_2	0.0031×10^{-3}	0.0275	98% Reduction
$N_4 = \{1, 8\}$	B_4	0.0291×10^{-3}	0.2617	74% Reduction
$N_5 = \{2, 4\}$	B_5	0.0185×10^{-3}	0.1665	84% Reduction
$N_6 = \{2, 6\}$	B_6	0.0323×10^{-3}	0.2900	71% Reduction
$N_7 = \{2, 8\}$	B_7	0.0057×10^{-3}	0.0514	95% Reduction
$N_8 = \{4, 6\}$	B_8	0.0185×10^{-3}	0.1665	74% Reduction
$N_9 = \{4, 7\}$	B_9	0.0031×10^{-3}	0.0275	98% Reduction
$N_{10} = \{6, 7\}$	B_{10}	0.0126×10^{-3}	0.1135	89% Reduction
$N_{11} = \{6, 8\}$	B_{11}	0.0057×10^{-3}	0.0514	95% Reduction
$N_{12} = \{7, 8\}$	B_{12}	0.0291×10^{-3}	0.2617	74% Reduction

obtained. The proposed work has the potential to be used for real world networks such as power networks, robotic networks, biological networks and social networks etc.

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