Chapter 1

Introduction and a Brief Review of Literature

1.1 Introduction

Reliability theory broadly talks about different characteristics of complex systems used in real-life. After the experience of numerous flaws in military systems during World War II, research contributions in reliability theory have substantially enhanced. With recent technological innovation, the importance of reliability theory is well accepted, across the disciplines, to produce/design high-quality products or high reliability systems. The reliability theory has been widely used in many engineering fields including industrial, financial, mechanical, electrical, computer, information and communication engineerings. This theory is useful to describe different aspects of a system, namely, whether a system is good or bad, how long a system may survive, what the failure rate of a system is, etc.

Most of the systems used in reality work in random environments. Consequently, a system continuously encounters shocks of different nature (i.e., harmful "instantaneous" events). These shocks can either cause the system's failure or decrease the system's lifetime. Thus, the study of systems' lifetimes subject to external shocks is one of the important problems in reliability theory. Shock models broadly describe the lifetime behavior of a system operating in a random environment. The term 'shock' is used in a generic sense and can be replaced by an appropriate terminology depending on the process under consideration. For example, a high dose of Ibuprofen drug can be considered as a shock for heart patients, and wind gusts can be viewed as shocks for wind turbines. Thus, the shock models have broad applications in modeling various real-life processes and systems. Some early applica-

tions of different shock models can be found in [Nakagawa, 2007; Cha and Finkelstein, 2018].

Nowadays, products/services, in any given sector, are expected to be more reliable with longer lifetimes and higher quality. Further, the systems used in industry are very costly and complicated in nature (e.g., aircraft, human space flight systems, submarines, wind turbines, bridges, etc.). The failure of such systems may create a substantial economic loss. Thus, the researchers have shown their keen interest to understand how shocks impact on the reliability or the average lifespan of a system. This study may be helpful to reduce errors in making critical decisions in a given scenario, for example, when and how one should go for replacement of a repairable system so that the total long-run average cost per unit time is minimum. Thus, the study of shock models is one of the frontier areas of research in reliability theory.

1.2 Review of Literature

A detailed literature survey concerned with the problems studied in this thesis is given in this section. We divide this section into five subsections. In Subsection 1.2.1, we discuss some important measures, namely, reliability function, failure rate function and mean residual life function. In Subsection 1.2.2, we give definitions, motivations and usefulness of different counting processes. In Subsection 1.2.3, we discuss different types of shock models with applications. In Subsection 1.2.4, we discuss the notions of stochastic orders and stochastic ageings. Lastly, in Subsection 1.2.5, we discuss some important distributions, namely, the phase-type distribution, the Mittag-Leffler distribution and the matrix Mittag-Leffler distribution.

1.2.1 Some important reliability measures

In this subsection, we discuss some important measures in reliability theory, namely, the reliability function, the failure rate function and the mean residual life function.

- **Reliability function**

For any given system, one of the natural questions that one may ask is how long the system can survive without failure, i.e., how reliable the system is. The *reliability* of a system can be defined as the probability of the system's survival under stated conditions for a stated time [see Nakagawa, 2006]. Below we give the basic definition of the reliability function.

Definition 1.2.1 *Let X be a nonnegative random variable. Then the function given by*

$$\bar{F}_X(t) = P(X > t),$$

is called the reliability function or the survival function of X.

Note that $\bar{F}_X(\cdot)$ is a non-increasing and bounded function. For a non-negative random variable X, its mean is given by

$$E(X) = \int_0^\infty t f_X(t)dt = \int_0^\infty \bar{F}_X(t)dt,$$

provided it exists. This is also known as the *mean time to failure* or the *mean lifetime* of X.

- **Failure rate function**

Reliability theory and other disciplines mainly deal with positive random variables which are often called lifetimes. A realization of a lifetime is viewed by failure, death or some other end-event. The failure rate function (also known as the hazard rate function) measures the probability of failure of an operating item in the next sufficiently small interval of time. In other words, this function mainly talks about the chance of instantaneous failure of a working system/component. Mathematically, the failure rate function of a random variable X is defined as

$$r_X(t) = \lim_{\Delta t \to 0+} \frac{P(t < X \leq t + \Delta t | X > t)}{\Delta t},$$

which can approximately be written as

$$r_X(t)\Delta t \approx P(t < X \leq t + \Delta t | X > t).$$

Thus, $r_X(t)\Delta t$ can be thought of as the conditional probability, given the survival up to time t, of failure of X in the interval $(t, t + \Delta t]$ when Δt (> 0) is very small. If X is an absolutely continuous random variable, then $r_X(t)$ can be represented as

$$r_X(t) = \begin{cases} \frac{f_X(t)}{\bar{F}_X(t)}, & \text{if } \bar{F}_X(t) > 0 \\ \infty, & \text{if } \bar{F}_X(t) = 0. \end{cases}$$

One important characteristic of the failure rate function is that it uniquely determines the distribution through the relation

$$\bar{F}_X(t) = \exp\left\{ -\int_0^t r_X(u)du \right\}.$$

The system reliability could be judged by the monotonic behavior of its failure rate function. For example, if the failure rate function is increasing, then the system is usually degrading in

some suitable probabilistic sense. Different monotonic behaviors of the failure rate function such as increasing, decreasing, bathtub-shaped, reversed bathtub-shaped, roller-coaster, etc. have been extensively studied in the literature [see Lai and Xie, 2006]. To know more on other properties of the failure rate function, we refer the reader to [Barlow and Proschan, 1965; Marshall *et al.*, 2011; Finkelstein, 2008].

- **Mean residual life function**

Mean residual life (also known as mean remaining life) function is another useful measure in reliability theory. It has vast applications in various areas of probability, statistics and related fields. In reliability theory, it is used to determine the optimal burn-in time. Although the shape of the failure rate function plays a vital role in replacement and repair strategies, the mean residual life function is found to be more relevant than the failure rate function. The failure rate represents the instantaneous chance of failure at any given point of time whereas the mean residual life summarizes the entire remaining life. Thus, the mean residual life has more intuitive appeal for modeling and analysis of failure data than the failure rate. The mean residual life function of a random variable X is defined as

$$m_X(t) = \begin{cases} E[X - t | X > t], & \text{for } t < t_0, \\ 0, & \text{otherwise,} \end{cases}$$

where $t_0 = \sup\{t : \bar{F}_X(t) > 0\}$. If X is an almost surely positive random variable, then $m_X(0) = E(X)$. By the finiteness of $E(X)$ we have that $m_X(t) < \infty$, for all $t < \infty$. However, it is possible that $\lim_{t\to\infty} m_X(t) = \infty$. It is worthwhile to mention here that

$$m_X(t) = \frac{1}{\bar{F}_X(t)} \int_t^\infty \bar{F}_X(u)du, \quad \text{when } t_0 = \infty.$$

Similar to the failure rate function, the mean residual life function also uniquely determines the distribution through the relation

$$\bar{F}_X(t) = \frac{E(X)}{m_X(t)} \exp\left\{ -\int_0^t \frac{du}{m_X(u)} \right\} \quad \text{over } \{t : P(X > t) > 0\}.$$

In literature, different properties of the mean residual life function have been studied by different researchers [see Shaked and Shanthikumar, 2007; Marshall *et al.*, 2011; Finkelstein, 2008; Barlow and Proschan, 1975; Lai and Xie, 2006, and the references therein].

1.2.2 Counting Processes

In this subsection, we discuss renewal processes, Poisson processes, and their various generalizations. We start this subsection with the following definition of a counting process [see Cha and Finkelstein, 2018].

Definition 1.2.2 *A stochastic process* $\{N(t) : t \geq 0\}$ *is said to be a counting process if* $N(t)$ *represents the total number of events in the interval* $(0, t]$ *with* $N(0) = 0$. □

Counting processes are generally used to model the occurrences of various types of events. We usually represent the occurrence of an event as an *arrival*. In practice, counting processes occur in a variety of situations, namely, the number of cars parked at a commercial centre in a time interval $[0, t]$, the number of incoming calls in a given time period, the number of natural disasters occured in a year, the number of customers arrived at a service station (e.g., workshops, filling stations, supermarkets, etc.) by a given time, etc. Clearly, the state space of the counting process $\{N(t) : t \geq 0\}$ is given by $\{0, 1, 2, \dots\}$. For any $t_1, t_2 \geq 0$ with $t_1 < t_2$, the increment

$$N(t_1, t_2) \equiv N(t_2) - N(t_1)$$

is equal to the number of events occurred in $(t_1, t_2]$, and $N(t_1) \leq N(t_2)$ for $t_1 < t_2$. Now, we discuss two important characteristics of a counting process $\{N(t) : t \geq 0\}$.

- *Independent increments property:* If the numbers of events in disjoint intervals are independent random variables, i.e., $N(t + \tau) - N(\tau)$ is independent of $N(s)$, for all $s < \tau$, $t > 0$, then the process is said to have the independent increments property.
- *Stationary increments property:* If, for $t_1 < t_2$, $N(t_2) - N(t_1)$ and $N(t_2 + \tau) - N(t_1 + \tau)$ have the same distribution, for all $\tau \geq 0$, then the process is said to have the stationary increments property.

Throughout the thesis, we only consider *orderly* counting processes as follows [see Cha and Finkelstein, 2018].

Definition 1.2.3 *A counting process* $\{N(t) : t \geq 0\}$ *is said to be an orderly counting process if* $P(N(t + \Delta t) - N(t) \geq 2) = o(\Delta t)$ *for small enough* Δt*, i.e., the probability of the occurrence of more than one event in a small interval of length* Δt *is* $o(\Delta t)$*; here* $o(\Delta t)$ *stands for any function* f *satisfying* $\lim_{x \to 0} f(x)/x = 0$. □

In what follows, we define the intensity function or the rate of an orderly counting process [see Cha and Finkelstein, 2018].

Definition 1.2.4 *The intensity function (or rate)* $\lambda(t)$ *of an orderly counting process* $\{N(t) : t \geq 0\}$ *is defined as*

$$\begin{aligned} \lambda(t) &= \lim_{\Delta t \to 0} \frac{P(N(t + \Delta t) - N(t) = 1)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{E(N(t + \Delta t)) - E(N(t))}{\Delta t} = \frac{dE(N(t))}{dt}. \end{aligned}$$

□

From the above definition, $\lambda(t)dt$ can be interpreted as an approximate probability of an event occurrence in the interval $(t, t+\Delta t]$ or the approximate expected number of events in this interval. Further, the mean number of events in $(0, t]$ is given by the cumulative rate

$$E(N(t)) \equiv \Lambda(t) = \int_0^t \lambda(x)dx.$$

Note that any orderly counting process can not entirely be determined by its intensity function $\lambda(\cdot)$. Thus, a more detailed description is used to characterize any counting process. There can be different characterizations of point processes. The most commonly used ones are as follows: (i) via the joint distributions of times between successive events (ie., the interval characterization), (ii) via the joint distributions of the numbers of events in all finite sets of disjoint intervals on the line (i.e., counting characterization), and (iii) via intensity process. Some of the counting processes considered in this thesis are defined via the intensity process. Thus, it is important to define the notion of the intensity process for a counting process. Below we give a formal definition of an intensity process [see Aven and Jensen, 1999; Anderson *et al.*, 1993; Cha and Finkelstein, 2018].

Definition 1.2.5 *An intensity process (stochastic intensity) λ_t, $t \geq 0$, of an orderly counting process $\{N(t) : t \geq 0\}$ is defined through the following limit:*

$$\lambda_t = \lim_{\Delta t \to 0} \frac{P(N(t, t+\Delta t) = 1 | H_{t^-})}{\Delta t} = \lim_{\Delta t \to 0} \frac{E(N(t, t+\Delta t) | H_{t^-})}{\Delta t},$$

where $H_{t^-} = \{N(s) : 0 \leq s \leq t\}$ is an history of the counting process in $[0, t)$, i.e., the set of all point events in $[0, t)$. □

Let the random variable T_i represent the arrival time of the i-th event, $i \in \mathbb{N}$. Then, the following order holds true:

$$0 \equiv T_0 < T_1 < T_2 < \cdots < T_i < T_{i+1} < \ldots.$$

Let the random variable X_i denote the time between $(i-1)$-th and i-th arrivals (also called the inter-arrival time), i.e., $X_i = T_i - T_{i-1}$, $i \in \mathbb{N}$. Then

$$T_0 \equiv 0, \quad T_n = \sum_{i=1}^{n} X_i, \quad n \in \mathbb{N},$$

and $N(t)$ can be represented as

$$N(t) = \sup\{n : T_n \leq t\} = \sum_{n=1}^{\infty} I(T_n \leq t),$$

where $I(T_n \leq t)$ is an indicator random variable, i.e., $I(T_n \leq t) = 1$, if $T_n \leq t$, and is equal to 0, otherwise. Note that there is a one-to-one relationship between $N(t)$ and T_n as

follows: $T_n \leq t$ is the same as $N(t) \geq n$. This relationship is called the *count-time duality* because it connects a discrete random variable $N(t)$ (*counts* the number of arrivals) with a continuous random variable T_n (*time* of the n-th arrival)) [see Blitzstein and Hwang, 2015].

Now we discuss some important counting processes, which are used in further study, namely, renewal processes, Poisson processes (homogeneous Poisson process and non-homogeneous Poisson process), and several mixed Poisson processes (Pólya process, generalized Pólya process and Poisson generalized gamma process).

• **Renewal process**

Renewal processes are commonly used in different maintenance modelings. For example, in a replacement policy, a system is replaced by a new one (statistically identical with the original system) as and when a failure is observed. In this context, the replacement of a failed system is often called the renewal. The sequence of the system's lifetimes after renewals generates a renewal process. A renewal may correspond to a job arrival in a server, a telephone call arrival to a trunk, etc. For more detailed mathematical information and applications of renewal processes, one may refer to [Cha and Finkelstein, 2018; Beichelt, 2006; Nakagawa, 2006]. Below we give the basic definition of a renewal process [see Ross, 2003].

Definition 1.2.6 *Let $\{N(t) : t \geq 0\}$ be a counting process, and let X_n denote the time between the $(n-1)$-th and the n-th arrival, $n \in \mathbb{N}$. If the sequence of nonnegative random variables $\{X_n : n \in \mathbb{N}\}$ is independent and identically distributed, then the counting process $\{N(t) : t \geq 0\}$ is said to be a renewal process.*

• **Poisson process**

Poisson processes have been widely used in various applications to model recurrent point events. These processes are mathematically tractable and hence, explicit results for these processes are readily available [see Ross, 1996; Grandell, 1997; Beichelt, 2006; Cha and Finkelstein, 2018, to name a few]. Different applications of Poisson processes could be found in reliability theory, risk theory, queueing theory, industrial engineering, finance, etc. We now quickly go through the homogeneous Poisson process (HPP) which is one of the popular counting processes used in stochastic modelings.

Definition 1.2.7 *A counting process $\{N(t) : t \geq 0\}$ is said to be the HPP with rate λ, $\lambda > 0$, if*

(*i*) $N(0) = 0$;

(*ii*) *The process* $\{N(t) : t \geq 0\}$ *has independent increments;*

(*iii*) *The number of events in any interval of length* t *is Poisson distributed with mean* λt*, i.e., for all* $s, t \geq 0$*,*

$$P(N(t+s)-N(s) = n) = \frac{(\lambda t)^n}{n!} \exp\{-\lambda t\}, \quad n = 0, 1, 2, \ldots \qquad \square$$

From the above definition, one can observe that the HPP possesses the stationary increments as well. Moreover,

$$E(N(t)) = Var(N(t)) = \lambda t.$$

Now, by using the above equality in Definition 1.2.4, the rate of the HPP is obtained as

$$\lambda(t) = \frac{dE(N(t))}{dt} = \lambda.$$

Further, by using Definition 1.2.7 (*ii*) and (*iii*) in Definition 1.2.5, the stochastic intensity of the HPP is obtained as $\lambda_t = \lambda$ (rate of the process), for all $t \geq 0$.

Remark 1.2.1 *From Definition 1.2.7, it can be noted that the inter-arrival times* X_i*,* $i = 1, 2, \ldots$*, of the HPP with rate* λ *are iid exponential random variables with mean* $1/\lambda$*. This property is often considered as the definition of the HPP. Further, this property shows that the HPP is indeed a renewal process (see Definition 1.2.6).* $\square$

In the HPP, the intensity λ is assumed to be a constant. In place of this, if we assume a time-dependent intensity, i.e., $\lambda = \lambda(t)$, we get the non-homogeneous Poisson process with rate $\lambda(t)$. One may note that the HPP is widely used in various applications primarily due to its simplicity and appealing properties. As discussed earlier, it has the stationary increments property with a constant rate. However, in reality, the occurrences of point events are often not stationary and hence, these cases can be described more appropriately by using the NHPP with a time-dependent rate. A formal definition of the NHPP is given below.

Definition 1.2.8 *A counting process* $\{N(t) : t \geq 0\}$ *is said to be the NHPP with rate* $\lambda(t)$*,* $t \geq 0$*, if*

(*i*) $N(0) = 0$*;*

(*ii*) *The process* $\{N(t) : t \geq 0\}$ *has independent increments;*

(*iii*) *The number of events in any interval of length* t *is Poisson distributed with mean* $\Lambda(t)$*, i.e., for all* $s, t \geq 0$*,*

$$P(N(t+s)-N(s) = n) = \frac{(\Lambda(t+s) - \Lambda(s))^n}{n!} \exp\{-(\Lambda(t+s)-\Lambda(s))\}, \quad n = 0, 1, 2, \ldots,$$

where $\Lambda(t) = \int_0^t \lambda(x)dx$*.* $\square$

Clearly, from part (iii) of the above definition, one can conclude that the NHPP does not possess the stationary increments property. Further, from Definition 1.2.5, the stochastic intensity of the NHPP with rate $\lambda(t)$ is given by $\lambda_t = \lambda(t)$, for all $t \geq 0$.

In the following proposition, we give the expression of the conditional joint distribution of $(T_1, \ldots, T_{N(t)}|N(t))$, where T_i's are the arrival times [see Cha and Finkelstein, 2018].

Proposition 1.2.1 *For the NHPP with rate $\lambda(t)$, the conditional joint probability density function of $(T_1, T_2, \ldots, T_{N(t)})$, given that $N(t) = n$, is given by*

$$f_{(T_1,T_2,\ldots,T_{N(t)}|N(t))}(t_1, t_2, \ldots, t_n|n) = n! \prod_{i=1}^{n} \left(\frac{\lambda(t_i)}{\Lambda(t)} \right), \quad 0 < t_1 \leq t_2 \leq \cdots \leq t_n \leq t,$$

where $\Lambda(t) = \int_0^t \lambda(x)dx$.

• Mixed Poisson process

In the literature, mixed Poisson processes were initially introduced for modeling claim number processes in accident and sickness insurance [see Dubourdieu, 1938]. In view of their flexibility, these processes are now used in many other applications. For example, in reliability theory, it is used to model the minimal repair process under heterogeneous population setting. In the literature, a mixed Poisson process is sometimes called the *weighted* Poisson process. For more information on this topic, one may refer to [Grandell, 1997]. Below we give the formal definition of a mixed Poisson process [see Cha and Finkelstein, 2018].

Definition 1.2.9 *Let Z be an absolutely continuous random variable with non-negative support. If*

$$(\{N(t) : t \geq 0\}|Z = z) \sim NHPP(\lambda(t, z)),$$

then the point process $\{N(t) : t \geq 0\}$ is called the mixed Poisson process or conditional Poisson process with mixing distribution Z.

Remark 1.2.2 *From the above definition, it can be easily shown that, if the mixing distribution Z is a non-degenerate random variable, then the increments of the mixed Poisson process $\{N(t) : t \geq 0\}$ are not independent.* □

In what follows, we discuss some important mixed Poisson processes, namely, Pólya, generalized Pólya, and Poisson generalized gamma processes. We start with the definition of the Pólya process, which is a mixed Poisson process with the gamma as a mixing distribution.

Definition 1.2.10 *A mixed Poisson process* $\{N(t), t \geq 0\}$ *is said to be the Pólya process with a set of parameters* $\{\beta, b\}$, $\beta > 0$, $b > 0$, *if the counting distribution is given by*

$$\begin{aligned} P(N(t) = n) &= \frac{\Gamma(\beta + n)}{\Gamma(\beta) n!} \left(\frac{t}{t+b}\right)^n \left(\frac{b}{t+b}\right)^\beta \\ &= \int_0^\infty \exp\{-\lambda t\} \frac{(\lambda t)^n}{n!} dH(\lambda), \quad n = 0, 1, 2, \ldots, \end{aligned}$$

where H, *to be called the structure distribution, is the gamma distribution with the density*

$$dH(\lambda) = \frac{b^\beta}{\Gamma(\beta)} \lambda^{\beta-1} \exp\{-b\lambda\} d\lambda. \qquad \square$$

Pólya process enjoys some interesting properties. The inter-arrival times, X_1, X_2, ..., of a Pólya process are exchangeable random variables. Further, this process has the dependent increments property, i.e., for $0 = t_0 < t_1 < \cdots < t_n$, the random variables $N(t_1) - N(t_0)$, $N(t_2) - N(t_1)$, ..., $N(t_n) - N(t_{n-1})$ are dependent. Note that, the marginal distribution of X_i is the Pareto distribution with the cdf given by

$$F_{X_i}(t) = 1 - \left(\frac{b}{b+t}\right)^\beta, \quad t \geq 0, \ i \in \mathbb{N}.$$

The Pareto distribution is well known in the literature due to its heavy-tailed behavior. Consequently, the Pólya process is also very useful in modeling extreme events in many real-life applications where the inter-arrival times are heavy-tailed. Further, a repairable system may have the Pareto inter-failure distribution [see Last and Sxekli, 1998]. Thus, the Pólya process is useful for studying the lifetime behavior of a repairable system. As stated by Beichelt [2006], the three reasons why the Pólya process is increasingly used for modeling real-life point processes are as follows: (*i*) the finite-dimensional distributions of this process are explicitly available, (*ii*) dependent increments occur more frequently than independent ones, and (*iii*) the two free parameters b and β of this process allow its adaptation to various data sets.

Next we discuss the generalized Pólya process (GPP) introduced by Konno [2010]. It was further studied, in depth, and characterized by Cha [2014]. This process is a generalization of the commonly used counting processes, namely, the HPP, the NHPP and the Pólya process. It is suitable in many applications (e.g., in maintenance modeling and other related areas) due to its flexibility and the property of dependent increments [see Cha and Finkelstein, 2018; Finkelstein and Cha, 2013, and the references therein]. In what follows, we give the formal definition of the GPP [see Cha, 2014].

Definition 1.2.11 *A counting process* $\{N(t) : t \geq 0\}$ *with stochastic intensity* λ_t *is said to be the GPP with a set of parameters* $\{\lambda(t), \alpha, \beta\}$, $\alpha \geq 0$, $\beta > 0$, *if*

(*i*) *N(0) = 0;*

(*ii*) $\lambda_t = (\alpha N(t^-) + \beta)\lambda(t)$,

where $N(t^-)$ *is defined as the total number of events in the interval* $(0, t)$. □

Below we state some important results of the GPP. In the following theorem, we give the expressions of the pmf of $N(t)$ and the conditional joint distribution of $(T_1, T_2, \cdots T_{N(t)} | N(t) = n)$ [see Cha, 2014].

Theorem 1.2.1 *For the GPP with the set of parameters* $\{\lambda(t), \alpha, \beta\}$, $\alpha > 0$, $\beta > 0$, *the following results hold:*

(*a*) *The distribution of* $N(t)$ *is given by*

$$P(N(t) = n) = \frac{\Gamma(\frac{\beta}{\alpha} + n)}{\Gamma(\frac{\beta}{\alpha})n!}(1 - exp\{-\alpha\Lambda(t)\})^n (exp\{-\alpha\Lambda(t)\})^{\frac{\beta}{\alpha}}, \quad n = 0, 1, 2, \ldots;$$

(*b*) *The conditional joint distribution of* $(T_1, T_2, \ldots, T_{N(t)})$, *given that* $N(t) = n$, *is given by*

$$f_{(T_1, T_2, \cdots, T_{N(t)} | N(t))}(t_1, t_2, \cdots, t_n | n) = n! \prod_{i=1}^{n} \frac{\alpha\lambda(t_i) exp\{\alpha\Lambda(t_i)\}}{exp\{\alpha\Lambda(t)\} - 1}, \ 0 \le t_1 \le t_2 \le \cdots \le t_n \le t,$$

where $\Lambda(t) = \int_0^t \lambda(u)du$. □

The following theorem shows that the GPP possesses the positive dependence property, i.e., the more significant number of events in the past often implies the larger number of events in the future [see Cha and Finkelstein, 2016].

Theorem 1.2.2 *The GPP* $\{N(t), t \ge 0\}$ *possesses the positive quadrant dependent increments property, i.e.,*

$$P(N(t_4) - N(t_3) > x, N(t_2) - N(t_1) > y) \ge P(N(t_4) - N(t_3) > x)P(N(t_2) - N(t_1) > y)$$

for all x, y, $t_1 < t_2 < t_3 < t_4$. □

The following theorem characterizes the GPP as a subclass of mixed Poisson processes [see Cha and Finkelstein, 2018].

Theorem 1.2.3 *Let*

$$(\{N^*(t) : t \ge 0\} | Z = z) \sim NHPP(z\nu(t)),$$

where

$$\nu(t) = \lambda(t) \exp\{\alpha\Lambda(t)\}$$

and the pdf of Z is given by

$$f_Z(z) = \frac{b^a z^{a-1}}{\Gamma(a)} \exp\{-bz\}, \quad z \geq 0, \text{ with } a = \beta/\alpha,\ b = 1/\alpha.$$

Further, let $\{N(t) : t \geq 0\}$ be the GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$. Then $\{N(t) : t \geq 0\}$ and $\{N^*(t) : t \geq 0\}$ share the same stochastic properties.

Remark 1.2.3 *The following statements are true.*

(*i*) *The GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$, where $\lambda(t) = \lambda$ (> 0), $\alpha = 0$ and $\beta = 1$, is the HPP with the intensity λ;*

(*ii*) *The GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$, where $\alpha = 0$ and $\beta = 1$, is a NHPP with the intensity function $\lambda(\cdot)$;*

(*iii*) *The GPP with the set of parameters $\{\lambda(t), \alpha, \beta\}$, where $\lambda(t) = 1/(b + t)$, $b > 0$, and $\alpha = 1$, is the Pólya process with the set of parameters $\{\beta, b\}$.* □

Now we discuss the PGGP introduced by Cha and Mercier [2021]. This process possesses the dependent increments property with a generalized dependence structure and hence, it may be applicable to a wider class of problems. Moreover, it contains many well-known processes (namely, the HPP, the NHPP, the Pólya process and the GPP) as the particular cases. In what follows, we give the formal definition of the PGGP. First, we recall the definition of the generalized gamma distribution [see Agarwal and Kalla, 1996].

Definition 1.2.12 *A random variable Q is said to have the generalized gamma distribution with the set of parameters $\{\nu, k, \alpha, l\}$, $\nu \geq 0$, $k, \alpha, l > 0$, denoted by $Q \sim GG(\nu, k, \alpha, l)$, if its pdf is given by*

$$f_Q(q) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \frac{q^{k-1} \exp\{-\alpha q\}}{(q + l)^\nu}, \quad q > 0, \tag{1.2.1}$$

where

$$\Gamma_\nu(k, \alpha l) = \int_0^\infty \frac{y^{k-1} \exp\{-y\}}{(y + \alpha l)^\nu} dy = \int_0^\infty \frac{\alpha^{k-\nu} y^{k-1} \exp\{-\alpha y\}}{(y + l)^\nu} dy. \tag{1.2.2}$$

Definition 1.2.13 *A counting process $\{N(t), t \geq 0\}$ is said to be the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ for all $t \geq 0$, $\nu \geq 0$, $k, \alpha, l > 0$, if*

(*i*) $\{N(t), t \geq 0 | Q = q\} \sim NHPP(q\lambda(t))$;

(*ii*) $Q \sim GG(\nu, k, \alpha, l)$.

Moreover, when $\lambda(t) = \lambda$ (> 0), the PGGP is called the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. □

The PGGP is a mixed Poisson process with the generalized gamma as a mixing distribution. Indeed, it is a Poisson process with a random intensity function defined as a product of a deterministic intensity function and a random variable.

In the following theorem, we give the expressions of the pmf of $N(t)$ and the conditional joint distribution of $(T_1, T_2, \cdots, T_{N(t)}|N(t) = n)$ [see Cha and Mercier, 2021].

Theorem 1.2.4 *For the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$, *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$, *the following results hold.*

(a) *The distribution of* $N(t)$ *is given by*

$$P(N(t) = n) = \frac{\alpha^{k-\nu}}{(\alpha + \Lambda(t))^{k+n-\nu}} \frac{\Gamma_\nu(k+n, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)} \frac{(\Lambda(t))^n}{n!}, \quad n = 0, 1, 2, \ldots;$$

(b) *The conditional joint pdf of* $(T_1, T_2, \ldots, T_{N(t)})$, *given that* $N(t) = n$, *is given by*

$$f_{(T_1,T_2,\cdots,T_{N(t)}|N(t))}(t_1, t_2, \cdots, t_n|n) = n! \prod_{i=1}^{n} \left(\frac{\lambda(t_i)}{\Lambda(t)} \right), \ 0 < t_1 \leq t_2 \leq \cdots \leq t_n \leq t,$$

where $\Lambda(t) = \int_0^t \lambda(u)du$. □

The following remark shows that the PGGP is a generalization of the HPP, the NHPP, the Pólya process and the GPP.

Remark 1.2.4 *The following statements are true.*

(a) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, *where* $\lambda(t) = \lambda$ (> 0), $\nu = 0, \alpha = k$ *and* $k \to \infty$, *is the HPP with the intensity* λ, *regardless of* l;

(b) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, *where* $\nu = 0, \alpha = k$ *and* $k \to \infty$, *is the NHPP with the intensity function* $\lambda(\cdot)$, *regardless of* l;

(c) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, *where* $\lambda(t) = 1/b$ (> 0), $\nu = 0, k = \beta, \alpha = 1$, *is the Pólya process with the set of parameters* $\{\beta, b\}$, *regardless of* l;

(d) *The PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, *where* $\nu = 0, k = \tau/\zeta, \alpha = 1/\zeta$ *and* $\lambda(t) = \eta(t) \exp\{\zeta \int_0^t \eta(x)dx\}$, *is the GPP with the set of parameters* $(\eta(t), \zeta, \tau)$, *regardless of* l.

1.2.3 Shock models

In this subsection, we discuss various shock models. The word "shock" is broadly used to describe an immediate and potentially harmful event. Some examples are as follows: electrical impulses of large magnitudes, demands for energy in biological objects, insurance claims in finance, cyber attacks on computer systems, random missile attacks in warfare, voltage surges in power generation systems, wind gusts for wind turbines, earthquakes for

various structures (for example, bridges), failure of a cooling system that results in a sharp rise of temperature of the main system, virus attacks on human body, high dose of Ibuprofen drug for heart patients, etc.

Most of the systems used in real-life work under random (stochastic) environments. One of effective ways to model the impact of this environment is via the external shocks encountered by the system under consideration. Shock models are widely used in reliability, finance, and related fields for stochastic descriptions of the lifetimes of systems operating in random environments. In shock models, arrivals of shocks are modelled through different point processes.

We often use the term 'system', but we don't explain what it is. In this thesis, we consider systems that are made up of several units and carry out specific tasks. A unit may have different meaning depending on the context, namely, item, component, device, subsystem, equipment, circuit, material, structure, machine, etc. A system could be a mechanical system, or a living organism, for example, airplane, bridge, wind turbine, human body, etc. When shocks occur on a system there are two fundamental questions that one has to keep in mind in any shock modeling:

(i) *What is the stochastic pattern of arrivals of shocks?*

(ii) *How do the occurrences of shocks affect or decrease the lifetime of a system?*

Stochastic patterns of arrivals of shocks are modeled by different counting processes, namely, HPP, NHPP, renewal processes, etc. The answer of the second question depends on which shock model is used in modeling the lifetime of the system. Based on the consequences of shocks on a system, shock models are usually classified into four broad classes: extreme shock models, cumulative shock models, run shock models, and δ-shock models. Now, we briefly discuss these shock models one by one.

- **Extreme shock model**

In the literature, one of the popular shock models is the extreme shock model. In the classical extreme shock model, a system fails if the magnitude of a single shock exceeds the pre-determined threshold value γ (say) [see Gut and Husler, 1999, 2005; Shanthikumar and Sumita, 1983, 1984; Cha and Finkelstein, 2011, and the references therein]. Let L be a random variable representing the lifetime of a system under the extreme shock model. Further, let Y_i be the random variable representing the magnitude of the i-th shock, $i \in \mathbb{N}$. Then, according to this model, the conditional probability of the event "the system survives till time t, given that n shocks occurred" is given by

$$P(L > t|N(t) = n) = P(Y_1 \leq \gamma, Y_2 \leq \gamma, \ldots, Y_{n-1} \leq \gamma, Y_n \leq \gamma).$$

Under the 'iid' assumption of Y_i's, we can write

$$P(L > t|N(t) = n) = q^n,$$

where $q = P(Y_1 \leq \gamma)$. Further, if we assume the arrivals of shocks according to the NHPP with rate $\lambda(t)$, then the survival function of the system is given by

$$\bar{F}_L(t) = \exp\{-p\Lambda(t)\},$$

where $\Lambda(t) = \int_0^t \lambda(x)dx$ and $p = 1 - q$. Although the classical extreme shock model was extensively studied in the literature, its applications are limited due to the restrictive assumptions that the system's survival probability at any time does not depend on arrival times of shocks and the history of a shock process. To overcome this limitation, a time and history dependent extreme shock model was proposed by Cha and Finkelstein [2011]. This model was further generalized by Cha and Finkelstein [2013] by considering that both the system's survival probability and the shock process depend on the history. Later, Cha and Finkelstein [2016] studied an extreme shock model based on the GPP. Recently, Eryilmaz and Kan [2021] defined and studied an extreme shock model under the assumption that there is a change in the distribution of the magnitudes of shocks. Two applications of the extreme shock model are given below.

- A vehicle axle fails when the depth of a crack exceeds a given threshold value. In the actual situation, a train axle is replaced at a distance traveled or the number of revolutions [see Nakagawa, 2007]. Here, the lifetime of this system can be modeled by the extreme shock model.

- A wind turbine is a system that converts the kinetic energy in the wind into electrical energy. The power generated by the wind turbine depends on the wind speed. If the wind speed exceeds the cut-out wind speed (or a fixed threshold), then the turbine is stopped. Consequently, it indicates a caveat of mechanical failure and hence, no electrical power is generated. In this setup, one important question is: How much time does the turbine actively work without any break? [see Eryilmaz and Kan, 2021]. This question can be answered by modeling the lifetime of the wind turbine using the extreme shock model.

- **Cumulative shock model**

In the cumulative shock model, a system fails if the additive damage due to shocks exceeds the predetermined threshold value γ (say) [see A-Hameed and Proschan, 1973; Esary *et al.*, 1973; Gut, 1990, to name a few]. Let L be a random variable representing the lifetime of a system under the cumulative shock model. Further, let Y_i be the random

variable representing the damage by the i-th shock, $i \in \mathbb{N}$. Let M be a random variable representing the fatal shock, i.e., the number of shocks arrived till the failure of the system. Then, according to this model, a system fails at i-th shock if the cumulative damage till the occurrence of i-th shock exceeds the predetermined threshold value γ and the cumulative damage till the occurrence of the $(i-1)$-th shock is less than γ, i.e., $\sum_{j=1}^{i} Y_j > \gamma$ and $\sum_{j=1}^{i-1} Y_j \leq \gamma$ hold. Therefore

$$\begin{aligned} P(M = m) &= P\left(\sum_{j=1}^{m-1} Y_j \leq \gamma, \sum_{j=1}^{m} Y_j > \gamma\right) \\ &= F^{(m-1)}(\gamma) - F^{(m)}(\gamma), \end{aligned}$$

where $F^{(m)}(\cdot)$ is the cdf of $\sum_{j=1}^{m} Y_j$. Then we can write

$$L = \sum_{i=1}^{M} X_i, \tag{1.2.3}$$

where X_i represents the inter-arrival time between $(i-1)$-th shock and i-th shock, $i \in \mathbb{N}$. Let shocks occur on the system according to the counting process $\{N(t) : t \geq 0\}$. Then, the cumulative damage of the system due to shocks at time t is given by $\sum_{i=1}^{N(t)} Y_i$. Consequently, it holds that

$$\{L \leq t\} \text{ if and only if } \left\{\sum_{i=1}^{N(t)} Y_i > \gamma\right\}. \tag{1.2.4}$$

Note that (1.2.3) and (1.2.4) give two equivalent representations of the lifetiem of a system under the cumulative shock model. For more detailed information on the cumulative shock models, one may refer to [Nakagawa, 2007; Cha and Finkelstein, 2018; Gong *et al.*, 2020; Ranjkesh *et al.*, 2019, and the references therein]. Below we discuss two applications of this model.

- Each electrical appliance connected with an electric system can be considered as a kind of shock; here the appliance's power is the magnitude of the shock. When the cumulative shock size exceeds a threshold, the system will be overloaded, leading to power failure [see Gong *et al.*, 2020].

- The strength of fibers determines the strength of a fibrous carbon composite. The fibers may break within the material when a composite specimen is placed under tensile stress. Such materials are broken based on cumulative damage [see Nakagawa, 2007].

- **Run-shock model**

In this model, a system fails when the magnitudes of r consecutive shocks exceed the predetermined threshold value γ (say) [see Mallor and Omey, 2001; Gong *et al.*, 2018]. Let L be a random variable representing the lifetime of a system under the run shock model. Further, let Y_i be the random variable representing the magnitude of the i-th shock, $i \in \mathbb{N}$. Then, the lifetime of the system can be written as

$$L = \sum_{i=1}^{M} X_i,$$

where X_i represents the inter-arrival time between $(i-1)$-th shock and i-th shock, $i \in \mathbb{N}$, and M is the waiting time for the first r consecutive shocks with magnitudes more than the threshold value γ, i.e.,

$$M = min\{n : Y_{n-r+1} > \gamma, \ldots, Y_n > \gamma\}.$$

Note that the random variable M follows the geometric distribution of order r with the success probability p, where p is the probability that the magnitude of a single shock exceeds the threshold value γ. Recently, Tuncel and Eryilmaz [2018] showed that the distribution of M can equivalently be represented as a discrete PH distribution [see Definition 1.2.19] i.e., $M \sim DPH(a, A)$, where $a = (1, 0, 0, \ldots, 0)_{1\times r}$ and

$$A = \begin{pmatrix} 1-p & p & 0 & \ldots & 0 \\ 1-p & 0 & p & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p & 0 & 0 & \ldots & p \\ 1-p & 0 & 0 & \ldots & 0 \end{pmatrix}_{r\times r}.$$

In the classical run-shock model, only one critical level γ was considered. As a generalization of this, Gong *et al.* [2018] have studied a generalized run shock model in which two critical thresholds γ_1 and γ_2 $(\gamma_1 < \gamma_2)$ were considered. According to this model, for given two positive integers r_1 and r_2, the system is assumed to fail if, at least r_1 consecutive shocks with magnitudes above γ_1 or r_2 consecutive shocks with magnitudes above γ_2 occur. Another generalization of the classical run shock model was studied by Yalcin *et al.* [2018]. In this model, a system fails due to the occurrences of n non-overlapping runs of r consecutive shocks. Below we give some applications of the run shock model.

- A shock can be viewed as a load that might correspond to mechanical stress, a voltage, or internally generated stress such as temperature. A single mechanical stress of small size may not cause the failure of a device, while a certain number of consecutive stresses of small sizes may cause the failure of a device.

- A micro-engine consists of several orthogonal linear comb drive actuators, which are mechanically joined to a rotating gear. In shock tests on micro-engines, the fracture of the springs is observed when k_1 consecutive external shocks with size of at least d_1 or k_2 successive external shocks with size of at least d_2 occur, where $d_1 < d_2$, $k_1 > k_2$ [see Wen *et al.*, 2022].

- **δ-shock model**

The δ-shock model was defined for the first time by Li *et al.* [1999a,b]. In this model, a system fails if the inter-arrival time between two successive shocks is less than the threshold value δ, i.e., the recovery time of a system from a shock is δ. It is worth mentioning that the δ-shock model is different, in nature, from other shock models because the δ-shock model deals mainly with the frequency of shocks, whereas magnitudes of shocks play a crucial role in other shock models. Let M be a random variable representing the fatal shock that causes the system failure. Further, let X_i denote the inter-arrival time between $(i-1)$-th shock and i-th shock, $i \in \mathbb{N}$. Then, according to this model,

$$P(M = m) = P(X_1 > \delta, X_2 > \delta, \ldots, X_{m-1} > \delta, X_m \leq \delta), \quad m \in \mathbb{N}.$$

By assuming X_i's as iid random variables, one can write

$$P(M = m) = (P(X_1 > \delta))^{m-1} P(X_1 \leq \delta) = (P(X_1 > \delta))^{m-1}(1 - P(X_1 > \delta)), \quad m \in \mathbb{N}.$$

This implies that M follows the geometric distribution with parameter q, where $q = P(X_1 > \delta)$. Let L be a random variable representing the lifetime of a system under the δ-shock model. Then, L can be written as

$$L = \sum_{i=1}^{M} X_i.$$

By assuming that the shock process is the HPP with parameter λ, Li *et al.* [1999a] derived the reliability function of L given by

$$\bar{F}_L(t) = \exp\{-\lambda t\} \sum_{n=0}^{\lfloor \frac{t}{\delta} \rfloor} \lambda^n \frac{(t - n\delta)^n}{n!},$$

where n is a non-negative integer. One may refer to Li and Kong [2007]; Wang and Zhang [2005]; Lam and Zhang [2004]; Eryilmaz [2012, 2017b]; Kus *et al.* [2021], and the references therein for an extensive information on the δ-shock model. Below we give some applications of this model.

- In electrical engineering, it is a well-known fact that overheating is one of the common reasons for a capacitor failure. In this case, the capacitor fails if a shock causes the

heat and the inter-arrival time between two successive shocks is too short so that the capacitor don't get sufficient time to cool down to the operating temperature. Thus, the δ-shock model can be used to model the lifetime of an electrical device wherein capacitors are used [see Bohlooli-Zefreh *et al.*, 2021].

- Consider a light bulb used to illuminate a house. If the time between turning on and off the bulb is too short, the bulb may be damaged. If each turning on is considered as a shock, one can find that the lifetime of the light bulb can be modeled using the δ-shock model [see Lorvand and Kelkinnama, 2022].

- Suppose that 20 minutes is good enough to do customer service at a bank counter. If the inter-arrival time between two successive customers is less than 15 minutes, there may be a problem in service. However, if this time is more than 15 minutes, there will be no problem. If a customer's arrival is considered as a shock, then the service disruption can be thought of as a failure of the system (i.e., the counter is unable to give the service). This scenario can be modeled by using the δ-shock model [see Lorvand and Nematollahi, 2022].

- Consider the used time of Ibuprofen drug which is very important for heart patients. If the time between two Ibuprofen drugs is less, it can cause severe problems for a patient. Usually, experts prescribe Aspirin daily for heart patients. If these patients use Ibuprofen to control their pain, they should pay attention to the time of taking the drug. Failure to pay attention to this fact can fix irreparable effects on the patient. If a patient takes Ibuprofen within a little time (less than 8 hours), it will reduce the effect of Aspirin in protecting the heart. Particularly, if this work is repeated for several times, it can cause more dangerous problems, e.g., death. We can model this problem using the δ-shock model, where a dose can be considered as a shock, and the time of taking the drug can be considered as the arrival time of a shock [see Lorvand *et al.*, 2020].

Apart from the aformentioned shock models, there are various mixed shock models, which are the combinations of two or more shock models, namely, the extreme shock model with the cumulative shock model [see Cha and Finkelstein, 2009], the extreme shock model with the run shock model [see Eryilmaz and Tekin, 2019], the extreme shock model with the δ-shock model [see Parvardeh and Balakrishnan, 2015; Wang and Zhang, 2005], the cumulative shock model with the run shock mode [see Mallor *et al.*, 2006], the cumulative shock model with the δ-shock model [see Parvardeh and Balakrishnan, 2015], the run shock model with the δ-shock model [see Eryilmaz, 2012], etc.

1.2.4 Stochastic orders and stochastic ageings

Suppose that we have two different systems and want to compare their reliabilities. Then the natural question arises how one can conclude that one system is more reliable than the other one. One of the the plausible solutions to this problem can be obtained by using the notion of stochastic orders. Stochastic orders are frequently used, as effective tools, to compare two or more random variables/vectors. In the literature, different types of stochastic orders (namely, usual stochastic order, hazard rate order, etc.) were introduced to study different problems. Each stochastic order has its own importance. For the comprehensive theory of stochastic orderings, we refer the reader to [Shaked and Shanthikumar, 2007]. Below we give the definition of the usual stochastic order introduced by Mann and Whitney [1947]. This is one of the simplest and most useful stochastic orders.

Definition 1.2.14 *Let Y_1 and Y_2 be two random variables. Then Y_1 is said to be greater than Y_2 in the usual stochastic (st) order, denoted by $Y_1 \geq_{st} Y_2$, if*

$$\bar{F}_{Y_1}(t) \geq \bar{F}_{Y_2}(t) \text{ for all } t \in (-\infty, \infty). \qquad \square$$

In what follows, we give the definitions of some other stochastic orders that are exclusively used in this thesis [see Shaked and Shanthikumar, 2007; Li *et al.*, 2009].

Definition 1.2.15 *Let Y_1 and Y_2 be two absolutely continuous non-negative random variables. Then Y_1 is said to be greater than Y_2 in the*

(*i*) *mean residual life (mrl) order, denoted by $Y_1 \geq_{mrl} Y_2$, if $m_{Y_1}(x) \geq m_{Y_2}(x)$ for all $x > 0$;*

(*ii*) *hazard rate (hr) order, denoted by $Y_1 \geq_{hr} Y_2$, if $\bar{F}_{Y_1}(x)/\bar{F}_{Y_2}(x)$ is increasing in $x > 0$;*

(*iii*) *likelihood ratio (lr) order, denoted by $Y_1 \geq_{lr} Y_2$, if $f_{Y_1}(x)/f_{Y_2}(x)$ is increasing in $x > 0$;*

(*iv*) *Laplace transform (lt) order, denoted by $Y_1 \geq_{lt} Y_2$, if $L_{Y_2}(x) \geq L_{Y_1}(x)$ for all $x > 0$;*

(*v*) *Laplace transform ratio (lt-r) order, denoted by $Y_1 \geq_{lt-r} Y_2$, if $L_{Y_1}(x)/L_{Y_2}(x)$ is decreasing in $x > 0$;*

(*vi*) *reversed Laplace transform ratio (r-lt-r) order, denoted by $Y_1 \geq_{r-lt-r} Y_2$, if $[1 - L_{Y_1}(x)]/[1 - L_{Y_2}(x)]$ is decreasing in $x > 0$;*

(*vii*) *differentiated Laplace transform ratio (d-lt-r) order, denoted by $Y_1 \geq_{d-lt-r} Y_2$, if $L'_{Y_1}(x)/L'_{Y_2}(x)$ is decreasing in $x > 0$.* $\square$

The hazard rate, likelihood ratio and mean residual life orders are equally important with the usual stochastic order. These orders are commonly used in system reliability to study the lifetime behavior of a complex system. Further, stochastic orders based on Laplace

transforms have also many applications in reliability, insurance, actuarial science and many other fields. For instance, suppose that an item, with the survival function $\bar{F}$, produces one unit of output per hour when it is operating. Then the *present value* of one unit produced at time t is given by $1.\exp\{-st\}$, where s is the discount rate. Further, the expected present value of the total output produced during the lifetime of the item is $\int_0^\infty \exp\{-st\}\bar{F}(t)dt$. Consequently, $Y_1 \geq_{lt} Y_2$ implies that an item with a lifetime described by the survival function $\bar{F}_{Y_2}(\cdot)$ produces the smaller expected total present value as compared with an item with the survival function $\bar{F}_{Y_1}(\cdot)$ [see Alzaid *et al.*, 1991]. Similarly, $Y_1 \geq_{r-lt-r} Y_2$ implies that the expected present value of an item with the survival function $\bar{F}_{Y_1}(\cdot)$, relative to the expected present value of an item with the survival function $\bar{F}_{Y_2}(\cdot)$, increases as s gets smaller [see Shaked and Wong, 1997]. Other applications of these orders may be found in [Alzaid *et al.*, 1991; Li *et al.*, 2009, and the references therein]. Below we give a proposition that talks about different relationships between Laplace transform orders.

Proposition 1.2.2 *The following results hold true.*

(*i*) *If* $Y_1 \geq_{st} Y_2$ *then* $Y_1 \geq_{lt} Y_2$ *(Theorem 5.A.16 in Shaked and Shanthikumar [2007]);*

(*ii*) *If* $Y_1 \geq_{lt-r} Y_2$ *or* $Y_1 \geq_{r-lt-r} Y_2$*, then* $Y_1 \geq_{lt} Y_2$ *(Theorem 5.B.10 in Shaked and Shanthikumar [2007]);*

(*iii*) *If* $Y_1 \geq_{d-lt-r} Y_2$*, then* $Y_1 \geq_{lt-r} Y_2$ *and* $Y_1 \geq_{r-lt-r} Y_2$ *(Theorem 3.2 of Li* et al. *[2009]).* □

Like stochastic orders, the orderings between real vectors are also very useful in establishing various inequalities. Below we give the definition of an ordering notion between real vectors. This is exclusively used in Chapter 5.

Definition 1.2.16 *Let* $\boldsymbol{u} = (u_i)$ *and* $\boldsymbol{v} = (v_i)$ *be two real vectors of the same dimension. Then,* $\boldsymbol{u}$ *is said to be greater (resp. less) than* $\boldsymbol{v}$ *in the component order, denoted by* $\boldsymbol{u} \geq_{comp}$ *(resp.* $\leq_{comp}$*)* $\boldsymbol{v}$*, if* $u_i \geq$ *(resp.* $\leq$*)* v_i *for all* i*. We say that* $\boldsymbol{u}$ *is equal to* $\boldsymbol{v}$ *in the component order, denoted by* $\boldsymbol{u} =_{comp} \boldsymbol{v}$*, if both* $\boldsymbol{u} \geq_{comp} \boldsymbol{v}$ *and* $\boldsymbol{u} \leq_{comp} \boldsymbol{v}$ *hold.*

An ageing concept largely describes - how a system ages with time. In the literature, many different kinds of ageing classes (namely, IFR, DFR, DMRL, ILR, etc.) were introduced and studied. An encyclopaedic information on different ageing classes could be found in [Lai and Xie, 2006; Barlow and Proschan, 1975]. Below we give the definitions of a list of ageing classes that are used in this thesis.

Definition 1.2.17 *A non-negative random variable* Y *is said to have the*

(*i*) *increasing (resp. decreasing) likelihood ratio (ILR) (resp. DLR) property if* $f_X(x)$ *is log-concave (resp. log-convex) in* $x \geq 0$*;*

(*ii*) *increasing (resp. decreasing) failure rate (IFR) (resp. DFR) property if* $r_Y(x)$ *is increasing (resp. decreasing) in* $x \geq 0$*;*

(*iii*) *decreasing (resp. increasing) mean residual life (DMRL) (resp. IMRL) property if* $m_Y(x)$ *is decreasing (resp. increasing) in* $x \geq 0$.

1.2.5 Some useful distributions

Probability distributions are widely used in different stochastic modelings. Some of the well-known probability distributions include uniform, exponential, Erlang, gamma, Weibull, Pareto, phase-type, normal, lognormal, Mittag-Leffler, matrix Mittag-Leffler, binomial, negative binomial, geometric, and Poisson distributions. In this subsection, we discuss some important distributions that are used in this thesis, namely, the phase-type (PH) distribution, the Mittag-Leffler distribution, and the matrix Mittag-Leffler distribution.

- **Phase-type (PH) distribution**

As a generalization of the exponential distribution, the set of PH distributions was introduced by Neuts [1975]. The PH distribution has many interesting properties. For instance, "*The set of PH distributions is dense in the set of probability distributions on the non-negative half-line*" [see He, 2014]. It means that every lifetime distribution can be approximated by a PH distribution. Further, the PH distribution functions and other statistical measures (e.g., mean, variance, etc.) can be written in matrix forms that are convenient in computations when using the relevant mathematical packages [see, e.g., Eryilmaz, 2017a]. Thus, the set of PH distributions is mathematically tractable. These distributions have a large number of applications in various stochastic modelings [see Neuts, 1981; Neuts *et al.*, 2000; Pérez-Ocón and Montoro-Cazorla, 2004, to name a few]. Below we provide the formal definition of the PH distribution [see He, 2014].

Definition 1.2.18 *A non-negative random variable* X *is said to have a PH distribution if*

$$F_X(x) = 1 - \boldsymbol{\pi} \exp\{Tx\}\boldsymbol{e} = 1 - \boldsymbol{\pi} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} T^n \right) \boldsymbol{e}, \quad x \geq 0,$$

where

(*i*) $\boldsymbol{e}$ *is the column vector with all elements being one;*

(*ii*) $\boldsymbol{\pi}$ *is a substochastic vector of order* m*, i.e.,* $\boldsymbol{\pi}$ *is a row vector, all elements of* $\boldsymbol{\pi}$ *are nonnegative, and* $\boldsymbol{\pi}\boldsymbol{e} \leq 1$*, where* m *is a positive integer; and*

(*iii*) T *is a subgenerator of order* m*, i.e.,* T *is an* $m \times m$ *matrix such that:* (*a*) *all diagonal elements are negative;* (*b*) *all off-diagonal elements are nonnegative;* (*c*) *all row sums are non-positive; and* (*d*) T *is invertible.* □

We call T and the pair $(\boldsymbol{\pi}, T)$ as the PH generator and the PH representation of order m, respectively. We write $X \sim PH(\boldsymbol{\pi}, T)$ to indicate that X follows the PH distribution with the PH representation $(\boldsymbol{\pi}, T)$. Further, the corresponding pdf is given by

$$f_X(x) = \boldsymbol{\pi} \exp\{Tx\}\boldsymbol{T^0}, \quad x \geq 0, \tag{1.2.5}$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$. Furthermore, throughout the thesis we assume that $\boldsymbol{\pi e} = 1$.

Remark 1.2.5 *The following observations can be made [see He, 2014].*

(*i*) *A PH distribution with the PH representation* $(\boldsymbol{\pi}, T)$*, where* $\boldsymbol{\pi} = 1$ *and* $T = -\theta$*, is the exponential distribution with parameter* θ*;*

(*ii*) *A PH distribution with the PH representation* $(\boldsymbol{\pi}, T)$*, where*

$$\boldsymbol{\pi} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{1\times m} \text{ and } T = \begin{pmatrix} -\theta & 0 & \cdots & 0 & 0 \\ \theta & -\theta & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \theta & -\theta \end{pmatrix}_{m\times m},$$

is the Erlang distribution with the set of parameters $\{m, \theta\}$*.* □

Now, we define the class of discrete PH distributions [see Neuts, 1981; Eryilmaz, 2017a; Bozbulut and Eryilmaz, 2020]. Note that, a discrete PH distribution can be seen as the distribution of the time to absorption in an absorbing Markov chain. Further, this distribution is commonly used as the distribution of a fatal shock (see in Chapters 6 and 7).

Definition 1.2.19 *A discrete random variable* N *is said to have the discrete PH (DPH) distribution if its pmf is given by*

$$P(N = n) = \boldsymbol{a}Q^{n-1}\boldsymbol{u}, \text{ for } n = 1, 2, 3, \ldots,$$

where Q *is a matrix of order* m *such that* $I - Q$ *is non-singular, and* $\boldsymbol{u} = (I - Q)\boldsymbol{e}$*,* $\boldsymbol{ae} = 1$*, and* I *is the identity matrix.* □

We write $N \sim DPH(\boldsymbol{a}, Q)$ to represent that N has a discrete PH distribution with parameter set $\{\boldsymbol{a}, Q\}$. The survival function of N is given by $\bar{F}_N(n) = \boldsymbol{a}Q^n\boldsymbol{e}$.

- **Mittag-Leffler (ML) distribution**

The ML distribution is a generalization of the well-known exponential distribution. This distribution has many applications, e.g., the inter-arrival times of the fractional counting process follow the ML distribution, etc. [see Laskin, 2003; Beghin and Orsingher, 2009]. Below we give the definition of the ML distribution [see Pillai, 1990; Kataria and Vellaisamy, 2019].

Definition 1.2.20 *A random variable X is said to have the ML distribution with parameters α and λ, if its pdf is given by*

$$f_X(x) = \lambda x^{\alpha-1} E_{\alpha,\alpha}(-\lambda x^{\alpha}), \quad x \geq 0,$$

where $E_{\alpha,\beta}(x)$, $\alpha > 0, \beta > 0$, is the two parameter ML function defined as

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\beta + k\alpha)}, \quad x \in \mathbb{R}. \qquad \square$$

We write $X \sim ML(\alpha, \lambda)$ to represent that X follows the ML distribution with parameters α and λ. Note that, when $\alpha = 1$, this distribution reduces to the well known exponential distribution with parameter $\lambda > 0$. This distribution has heavy tailed behavior for $0 < \alpha < 1$.

Some useful properties of the ML distribution are given in the following proposition [see Pillai, 1990; Kataria and Vellaisamy, 2019].

Proposition 1.2.3 *Let $X \sim ML(\alpha, \lambda), \alpha > 0, \lambda > 0$. Then,*

(i) the survival function of X is given by $\bar{F}_X(x) = E_{\alpha,1}(-\lambda x^{\alpha})$;

(ii) the Laplace transform of X is given by $L_X(s) = \frac{\lambda}{\lambda + s^{\alpha}}, s > 0$.

- **Matrix Mittag-Leffler (MML) distribution**

The class of MML distributions, defined by Albrecher *et al.* [2020a], is a wide class of *heavy-tailed* distributions with attractive mathematical properties. Like in case of the PH distributions, it is also dense in the class of all lifetime distributions. Consequently, this distribution may be useful in modeling various real life scenarios where the inter-arrival times of shocks have heavier tails. Furthermore, the class of MML distributions is a fractional generalization of the PH distribution (note that, Albrecher *et al.* [2020b] named it as fractional PH distribution). This distribution contains many popular distributions as special cases, namely, the exponential distribution, the Erlang distribution, the PH distribution, the ML distribution, the fractional Erlang distribution (a fractional generalization of the Erlang distribution). Below we give the definition of the MML distribution.

Definition 1.2.21 *Let $(\boldsymbol{\pi}, T)$ be a PH reperesentation and $0 < \alpha \leq 1$. A random variable X is said to have a MML distribution, denoted by $X \sim MML(\alpha, \boldsymbol{\pi}, T)$, if its Laplace transform is given by*

$$L_X(u) = \boldsymbol{\pi}(u^{\alpha} I - T)^{-1} \boldsymbol{T^0},$$

where $\boldsymbol{T^0} = -T\boldsymbol{e}$. $\square$

The following lemma is borrowed from Albrecher *et al.* [2020a].

Lemma 1.2.1 *Let $X \sim MML(\alpha, \boldsymbol{\pi}, T)$. Then,*

(*i*) *the pdf of* X *is given by*

$$f_X(x) = x^{\alpha-1}\boldsymbol{\pi}E_{\alpha,\alpha}(Tx^{\alpha})\boldsymbol{T^0}, \quad \boldsymbol{T^0} = -T\boldsymbol{e};$$

(*ii*) *the cumulative distribution function (cdf) of* X *is given by*

$$F_X(x) = 1 - \boldsymbol{\pi}E_{\alpha,1}(Tx^{\alpha})\boldsymbol{e}.$$

Remark 1.2.6 *Let* $X \sim MML(\alpha, \boldsymbol{\pi}, T)$. *Then the following observations can be made.*

(*i*) *If* $\alpha = 1$ *then* $X \sim PH(\boldsymbol{\pi}, T)$;

(*ii*) *If* $\boldsymbol{\pi} = 1$ *and* $T = -\lambda$ *then* $X \sim ML(\alpha, \lambda)$, $\lambda > 0$;

(*iii*) *If* $\alpha = 1$, $\boldsymbol{\pi} = 1$ *and* $T = -\lambda$ *then* X *follows the well known exponential distribution with parameter* $\lambda > 0$;

(*iv*) *If* $(\boldsymbol{\pi}, T)$ *is a PH representation of the Erlang distribution then* X *follows the fractional Erlang distribution.*

1.3 A Brief Discussion on the Main Results of the Thesis

As discussed in the previous section, shock models have a large number of applications in reliability theory, operations research, economics, actuarial science, biological science, queuing theory, inventory, and related fields. In reliability theory, shock models are used to model the lifetimes of systems operating under random environments. Keeping the importance of shock models in mind, the present thesis is devoted to study some general shock models, and their applications in optimal maintainance.

The thesis consists of nine chapters of which Chapter 1 is introductory. Here we discuss a detailed survey of the literature related to the problems that are studied in this thesis. A brief discussion on the main results considered in Chapters 2 - 8 are presented below.

One of the most popular ways to model an impact of a random environment on system's performance characteristics is via the external shocks that either cause certain damages or immediately result in failures. There is a vast literature on this topic with respect to the relevant theoretical properties of the corresponding lifetime models as well as to the various practical applications. The δ-shock model is one of the widely discussed models in the literature that is described by the constant time of a system's recovery after a shock. However, this assumption is too restrictive and unrealistic to describe many real-life scenarios. Indeed, δ can obviously depend on other parameters, namely, magnitudes of

shocks, arrival times of shocks, etc. To overcome this limitation, some generalized δ-shock models are discussed in Chapters 2 - 4.

In Chapter 2, we consider a time-dependent δ-shock model, i.e., the recovery time becomes an increasing function of time. Moreover, we assume that shocks occur according to the GPP that contains the HPP, the NHPP and the Pólya process as particular cases. Novelty of this work is mainly in considering the time-dependent recovery time and a general arrival process of shocks that contains all previously considered in the literature special cases. For the defined shock model, we derive the survival function and the mean lifetime of a system. We also discuss various particular cases of the proposed model. We consider a special case, namely, the linear time-dependent δ-shock model. Finally, we discuss an application of the proposed model to optimal maintenance.

Note that, different systems, depending on their degradation states, have different recovery times even if they experience the same magnitude of shocks. Moreover, a system needs more (less) recovery time if a shock with large (small) magnitude occurs. For example, the recovery time of a bridge from the damage of an earthquake depends on the magnitude of a shock along with the age of the bridge (apart from other factors). If the magnitude of an earthquake is large, then the damage of the bridge is also substantial and hence, it takes more time to recover. Further, due to deterioration, the recovery time of the bridge from the damage caused by an earthquake increases as the age of the bridge increases. Moreover, if the magnitude of an earthquake exceeds a fixed threshold level, then the bridge entirely collapses. In Chapter 3, we introduce a general δ-shock model in which the recovery time depends on both the arrival times and the magnitudes of shocks. Moreover, we also consider a more general and flexible shock process, namely, the PGGP. For the defined model, we derive the survival function and the mean lifetime of a system. Further, we study some stochastic comparisons results. We study a special case where the recovery time is assumed as a linear function of arrival times and magnitudes of shocks. For this special case, we derive the reliability function, the mean lifetime and the distribution for a fatal shock. Finally, as an example of a possible application, the optimal replacement policy for the proposed model is considered.

In Chapter 4, we consider a history-dependent mixed shock model which is a combination of the history-dependent extreme shock model and the history-dependent δ-shock model. We assume that shocks occur according to the GPP that contains the HPP, the NHPP and the Pólya process as the particular cases. For the defined survival model, we derive the corresponding survival function, the mean lifetime and the failure rate. Further,

we study the asymptotic and monotonicity properties of the failure rate. Finally, some applications of the proposed model have also been included with relevant numerical examples.

The NHPP and its specific version, the HPP, are widely used in various applications for modeling of recurrent point events. These processes are mathematically tractable. However, they have important limitations in modeling of the real world phenomena. For example, these processes are characterized by the independent increments, which is often not realistic in practice. Moreover, the mean and the variance for the number of events in $(0, t]$ are equal, which is also restrictive. Some generalizations for overcoming these limitations were suggested in the literature in the form of various point processes, for instance, mixed Poisson processes. The Pólya process, the GPP and the PGGP are mixed Poisson processes. But, these processes are based on the specific mixing distribution and, therefore, can be not suitable in some applications. This motivates us to consider a more general set up based on a mixing distribution that can approximate any continuous lifetime distribution. In Chapter 5, we introduce a new counting process to be called the Poisson phase-type process (PPHP), which is the mixed NHPP with the continuous PH mixing distribution. It is well known that "*The set of PH distributions is dense in the set of probability distributions on the non-negative half-line*". It means that every lifetime distribution can be approximated by the PH distribution and consequently, any mixed Poisson process can also be approximated by the PPHP. This unique property of the PPHP makes it special among other mixed Poisson processes considered in the literature. We study some basic stochastic properties of the new process and discuss relevant applications by considering the extreme shock model, the stochastic failure rate model and the δ-shock model.

Shocks arrivals are usually modeled by relevant point processes. Most of the studies devoted to shock modeling consider renewal processes or Poisson processes (HPP and NHPP) as the corresponding shock point processes affecting engineering systems. Note that the inter-arrival times in the renewal process are iid, which is very restrictive in many real life scenarios, whereas the Poisson process has independent increments, which is also a restrictive assumption in many applications. For example, if there are larger number of shocks in the past, we may expect the same in the future (positive dependence). Thus, in many real life scenarios, it is more appropriate to consider more general shock processes, namely, the PGGP, the HPGGP, etc. In Chapter 6, we propose the unified class of shock models with the specified dependence structure and a general discrete failure model. This class contains many popular shock models, namely, the run shock model, the generalized run shock model, the generalized extreme shock model, some special mixed shock models, etc. We assume that shocks occur in accordance with the HPGGP, which is a rather general

process with dependent increments and dependent inter-arrival times, whereas the failure model is defined by the DPH distribution. For the proposed class of shock models, we study some important reliability indices, namely, the survival function, the failure rate function, the mean residual life function and the mean lifetime. Moreover, the derived results are in matrix forms that can easily be calculated using the relevant mathematical packages. We give some numerical examples to illustrate the developed results. Further, we illustrate our findings by considering the relevant optimal mission duration policy.

Counting processes based on heavy-tailed distributions (namely, the FHPP and the RPMML) have not yet been considered in the literature for modeling the occurrences of shocks. Therefore, in Chapter 7, some general shock models are considered under the assumption that shocks occur according to a renewal process with the MML distributed inter-arrival times. As the class of MML distributions is wide and well-suited for modeling the heavy tail phenomena, these shock models can be very useful for analysis of lifetimes of systems subject to random shocks with inter-arrival times having heavier tails. Some relevant stochastic properties of the introduced models are described. Finally, two applications, namely, the optimal replacement policy and the optimal mission duration are discussed.

In Chapter 8, we consider coherent systems subject to random shocks that can damage a random number of components of a system. Based on the distribution of the number of failed components, we discuss three models, namely, (i) a shock can damage any number of components (including zero) with the same probability, (ii) each shock damages, at least, one component, and (iii) a shock can damage, at most, one component. For the defined models, we obtain survival probabilities for coherent systems. These results hold for any counting process. Furthermore, we discuss in detail some special cases by considering three important counting processes, namely, the PPHP, the PGGP and the RPMML. The specific case of the k-out-of-n system is also considered. Our results are presented in the closed form that allow for straightforward computations. An optimal replacement policy for repairable systems is considered as an application of the proposed modeling.

Finally, conclusion of the thesis and some possibilities of future research are given in Chapter 9.

In order to make each chapter as independent as possible, a few definitions, descriptions of some notations and concepts, and some important results may be found to be repeated in the subsequent chapters.