3

Structural balance and spectral properties of generalized corona product of signed graphs

In this chapter, I extend corona product of signed graphs [Adhikari *et al.*, 2023] into generalized corona product of signed graphs inspired by the generalized corona product of unsigned graphs. Then I study structural and spectral properties of these graphs. Utilizing the notion of coronal of a graph, I determine computable formulae of characteristic, Laplacian, and signless Laplacian polynomials of generalized corona product of signed graphs. Finally, I provide sufficient conditions for the generalized corona product of some distinct collections of signed graphs to be co-spectral.

3.1 GENERALIZED CORONA PRODUCT OF SIGNED GRAPHS AND STRUCTURAL BALANCE PROPERTIES

A signed graph is given by a triple $G^s = (V, E^s, \sigma)$, where V is a finite set of vertices, $E^s \subset V \times V$ is a set of edges, and $\sigma \colon E^s \longrightarrow \{+, -\}$ is a signature function. Then the canonical marking scheme $\mu : V \to \{1, -1\}$ on G^s is defined as

$$\mu(v) = \prod_{e \in E_v^s} \sigma(e) \tag{3.1}$$

where E_v^s is the set of signed edges adjacent at v [Beineke, 1978; LOWELL W. BEINEKE, 1978]. Throughout the Chapter, I consider a signed graph as a marked graph with the canonical marking scheme, and I denote such a graph as a quadruple $G_\mu^s = (V_\mu, E^s, \sigma, \mu)$. Now I recall the definition of corona product of two signed graphs as follows.

Definition 3.1.1. *[Adhikari* et al.*, 2023] Let $G_{\mu_1}^s = (V_{\mu_1}, E_1^s, \sigma_1, \mu_1)$ and $H_{\mu_2}^s = (V_{\mu_2}, E_2^s, \sigma_2, \mu_2)$ be signed graphs on n and k nodes respectively. Then corona product $G_{\mu_1}^s \circ H_{\mu_2}^s$ of $G_{\mu_1}^s, H_{\mu_2}^s$ is a signed graph by taking one copy of $G_{\mu_1}^s$ and n copies of $H_{\mu_2}^s$, and then forming a signed edge from i^{th} node of $G_{\mu_1}^s$ to every node of the ith copy of $H_{\mu_2}^s$ for all i. The sign of the new edge between i^{th} node of $G_{\mu_1}^s$, say u and j^{th} node in the ith copy of $H_{\mu_2}^s$, say v is given by $\mu_1(u)\mu_2(v)$ where μ_1 and μ_2 are a marking schemes defined by σ_1, σ_2 on $G_{\mu_1}^s$ and $H_{\mu_2}^s$ respectively.*

Based on the notion of corona of two unsigned graphs, the generalized corona product of two unsigned graphs is introduced in [Laali *et al.*, 2016]. Now I extend this definition of generalized corona product of unsigned graphs to signed graphs with the help of Definition 3.1.1 as follows.

Definition 3.1.2. *Let $G^s_\mu = (V_\mu, E^s, \sigma, \mu)$ be a signed graph on n vertices. Let $H^s_{\mu_l} = (V_{\mu_l}, E^s_l, \sigma_l, \mu_l)$ be signed graphs with $|V_{\mu_l}| = k_l$, $1 \le l \le n$. Then, the generalized corona product of G^s_μ and $H^s_{\mu_l}$, denoted by*

$$G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l},$$

is obtained when the l-th vertex of G^s_μ is linked with every node of $H^s_{\mu_l}$. If u is the l-th vertex of G^s_μ and v be any vertex of $H^s_{\mu_l}$ then the sign of the edge joining u and v is given by $\mu(u)\mu_l(v)$.

For example, see Figure 3.1. In what follows, I provide the statistics of signed edges and balanced/unbalanced triads in generalized signed corona graphs. I also determine necessary and sufficient conditions for a balanced generalized corona product of signed graphs in terms of patterns of signs of edges adjacent to certain types of marked vertices of the constituent signed graphs.

Keeping the notations in Definition 3.1.2, it is easy to verify that the number of nodes in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ is equal to $|V_\mu| + \sum_{l=1}^n |V_{\mu_l}|$ where $|V_G|$ and $|V_{H^s_{\mu_l}}|$ are the number of nodes in G^s_μ and $H^s_{\mu_l}$, respectively, $l = 1, 2, \ldots, n$. Let $M^+_{G^s_\mu}$ ($M^-_{G^s_\mu}$) and $M^+_{H^s_{\mu_l}}$ ($M^-_{H^s_{\mu_l}}$) denote the number of positively (negatively) marked nodes in G^s_μ and $H^s_{\mu_l}$, respectively. Then Table 3.1 lists the number of signed edges in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ that can be derived easily, where $|E^s_l|^+$ ($|E^s|^+$) is the number of edges with sign $s \in \{+,-\}$ that connect positively marked nodes, $|E^s_l|^\pm$ ($|E^s|^\pm$) is the number of edges with sign s that connect one positively marked node and one negatively marked node, and $|E^s_l|^-$ ($|E^s|^-$) is the number of edges with sign s that connect negatively marked nodes in $H^s_{\mu_l}$ (G^s_μ), where $s \in \{+,-\}$.

Edges	G^s_μ	$H^s_{\mu_l}$	$G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$
# of edges	$\lvert E^s\rvert$	$\lvert E^s_{\mu_l}\rvert$	$\lvert E^s\rvert + \sum_{l=1}^n \lvert E^s_l\rvert + \sum_{l=1}^n \lvert V_{\mu_l}\rvert$
# of+ edges	$\lvert E^+\rvert$	$\lvert E^+_{\mu_l}\rvert$	$\lvert E^+\rvert + \sum_{l=1}^n \lvert E^+_l\rvert + M^+_{G^s_\mu} \sum_{l=1}^n M^+_{H^s_{\mu_l}} + M^-_{G^s_\mu} \sum_{l=1}^n M^-_{H^s_{\mu_l}}$
# of− edges	$\lvert E^-\rvert$	$\lvert E^-_l\rvert$	$\lvert E^-\rvert + \sum_{l=1}^n \lvert E^-_l\rvert + M^+_{G^s_\mu} \sum_{l=1}^n M^-_{H^s_{\mu_l}} + M^-_{G^s_\mu} \sum_{l=1}^n M^+_{H^s_{\mu_l}}$

Table 3.1 : Statistics of number of signed edges in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$.

Now I consider counting triads of each type in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$. Let $|T_i(G^s_\mu)|$ and $|T_i(H^s_{\mu_l})|$ denote the number of triads, having i number of negative edges, in the graph G^s_μ and $H^s_{\mu_l}$, respectively, for $l = 1, 2, \ldots, n$. Then in table 3.2, I list the number of triads in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$, they can be easily deduced. Indeed, note that in addition to the triads present in G^s_μ and $H^s_{\mu_l}$, new triads are created due to the definition of corona product. A triad is formed when a node l in V_μ is linked to an edge (j,k) in $E^s_{\mu_l}$ and the type of this triad depends on the marking of nodes ($l, j,$ and k) and the sign of the edge (j,k). Thus, $|E_{H^s_{\mu_l}}|$ new triads are created by linking node $l \in V_\mu$ with all the nodes in V_{μ_l}, for $l = 1, 2, \ldots, n$. Hence, the total number of triads in $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$

triads	G^s_μ	$H^s_{\mu_l}$	$G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$
# of T_0	$\lvert T_0(G^s_\mu)\rvert$	$\lvert T_0(H^s_{\mu_l})\rvert$	$\lvert T_0(G^s_\mu)\rvert + \sum_{l=1}^{n}\lvert T_0(H^s_{\mu_l})\rvert + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^{s+}_l\rvert^+ + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^{s+}_l\rvert^-$
# of T_1	$\lvert T_1(G^s_\mu)\rvert$	$\lvert T_1(H^s_{\mu_l})\rvert$	$\lvert T_1(G^s_\mu)\rvert + \sum_{l=1}^{n}\lvert T_1(H^s_{\mu_l})\rvert + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^+_{H^s_{\mu_l}}\rvert^\pm + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^+ + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^+_{H^s_{\mu_l}}\rvert^\pm + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^-$
# of T_2	$\lvert T_2(G^s_\mu)\rvert$	$\lvert T_2(H^s_{\mu_l})\rvert$	$\lvert T_2(G^s_\mu)\rvert + \sum_{l=1}^{n}\lvert T_2(H^s_{\mu_l})\rvert + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^+_{H^s_{\mu_l}}\rvert^- + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^\pm + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^+_{H^s_{\mu_l}}\rvert^+ + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^\pm$
# of T_3	$\lvert T_3(G^s_\mu)\rvert$	$\lvert T_3(H^s_{\mu_l})\rvert$	$\lvert T_3(G^s_\mu)\rvert + \sum_{l=1}^{n}\lvert T_3(H^s_{\mu_l})\rvert + M^+_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^- + M^-_{G^s_\mu}\sum_{l=1}^{n}\lvert E^-_{H^s_{\mu_l}}\rvert^+$

Table 3.2 : Count of triads in $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$.

is

$$T(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}) = |T(G^s_\mu)| + \sum_{l=1}^{n}|T(H^s_{\mu_l})| + \sum_{l=1}^{n}|E^s_l|. \tag{3.2}$$

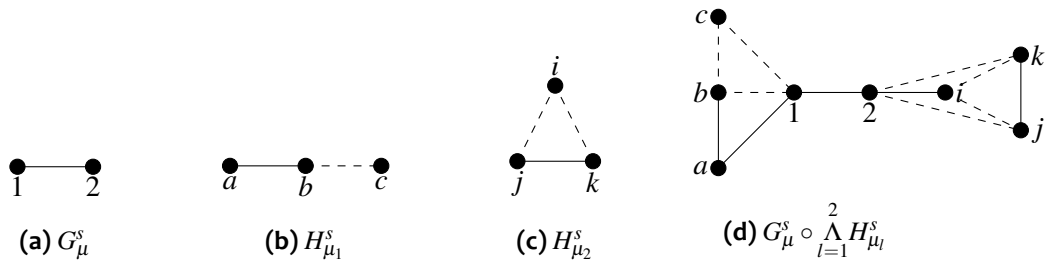
Towards investigating the structural balance properties of the generalized corona product of signed graphs, first, note that $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$, $l = 1,2,\ldots,n$ is unbalanced if any one graph among $G^s_\mu, H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_l}$ is unbalanced. Thus in [Adhikari *et al.*, 2023], conditions are derived under which corona product of balanced signed graphs becomes unbalanced. I extend this result for $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$, $l = 1,2,\ldots,n$ to be unbalanced when all the constituent signed graphs are balanced.

Theorem 3.1.1. *Let $G^s_\mu = (V_\mu, E^s, \sigma, \mu)$ and $H^s_{\mu_l} = (V_{\mu_l}, E^s_l, \sigma_l, \mu_l)$ be structurally balance signed graphs, $l = 1,2,\ldots,n$. Then $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ is unbalanced if there exists at least one $H^s_{\mu_j}$, $1 \le j \le n$, having one of the following types of edges*

1. *a positive edge that connects two opposite marked nodes,*
2. *a negative edge that connects two negatively marked nodes, and*
3. *a negative edge that connects two positively marked nodes.*

Proof. The proof is similar to Theorem 2.2 in [Adhikari *et al.*, 2023]. A positively marked node in G^s_μ will form a triad T_1 in $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ when an edge of type 1 and 3 is present in $H^s_{\mu_j}$, otherwise it forms a triad T_3 when an edge of type 2 is present in $H^s_{\mu_j}$. Similarly, a negatively marked node in G^s_μ will form a triad T_1 in $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ when an edge of type 1 and 2 is present in $H^s_{\mu_j}$, otherwise it forms a triad T_3 when an edge of type 3 is present in $H^s_{\mu_j}$. □

In Figure 3.1d, I give an example of an unbalanced generalized corona product graph of structurally balanced signed graphs $G^s_\mu, H^s_{\mu_1}$, and $H^s_{\mu_2}$.



(a) G^s_μ (b) $H^s_{\mu_1}$ (c) $H^s_{\mu_2}$ (d) $G^s_\mu \circ \overset{2}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$

Figure 3.1 : The generalized corona product $G^s_\mu \circ \overset{2}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ is shown in (d) using canonical marking scheme.

3.2 SPECTRAL PROPERTIES OF GENERALIZED CORONA PRODUCT OF GRAPHS

In this section, I define the coronal of a signed graph by generalizing the notion of coronal of an unsigned graph, which is a graph invariant introduced in [McLeman and McNicholas, 2011]. Further, I derive the characteristic polynomials for the adjacency matrix (characteristic polynomial), the Laplacian matrix (signed Laplacian polynomial), and the Signless Laplacian matrix (signed signless Laplacian polynomials). These polynomials correspond to the generalized corona product of signed graphs and are expressed in terms of specific polynomials associated with the constituent signed graphs that define the product. Additionally, I present criteria for generalized corona product of signed graphs to exhibit cospectral properties, including L-cospectrality and Q-cospectrality.

Definition 3.2.1. *[McLeman and McNicholas, 2011] Assume that H is a graph on n vertices whose adjacency matrix is such that, when considered as a matrix over the field of a rational function $\mathscr{C}(\lambda)$, the characteristic matrix $\lambda I - B$ has determinant* $\det(\lambda I - B) = f_H(\lambda) \neq 0$*, indicating it as inevitable. The coronal $\chi_H(\lambda) \in \mathscr{C}(\lambda)$ of H is defined to be the sum of the entries of the matrix $(\lambda I - B)^{-1}$. Therefore, this can be obtained as*

$$\chi_H(\lambda) = \mathbf{1}^T(\lambda I - B)^{-1}\mathbf{1}$$

where $\mathbf{1}$ denotes the all-one vector of compatible dimension, that is n.

Now I introduce the notion of isomorphism for signed marked graphs. Let $G^s_\mu = (V_\mu, E^s, \sigma)$ be a signed marked graph with the marking scheme $\mu : V_\mu \to \{-1, +1\}$, and the signature function $\sigma : E^s \to \{-1, +1\}$. As mentioned before, the marking scheme μ is considered the canonical marking scheme for the signed graph $G^s_\mu = (V, E^s, \sigma)$.

Definition 3.2.2. *Given two signed marked graphs $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1)$ and $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2)$, I call a bijective map $\phi : V_{\mu_1} \to V_{\mu_2}$ isomorphism if*

1. *$(i, j) \in E^s_1$ if and only if $(\phi(i), \phi(j)) \in E^s_2$, and $\sigma_1((i, j)) = \sigma_2(\phi(i), \phi(j))$ for all $(i, j) \in E^s_1$*

2. $\mu_1[V_{\mu_1}] = \mu_2[\phi(V_{\mu_1})]$, *where*

$$\begin{aligned}\mu_1[V_{\mu_1}] &= [\mu_1(v_1), \mu_1(v_2), \ldots, \mu_1(v_k)]^T, V_{\mu_1} = \{v_1, \ldots, v_k\} \text{ and} \\ \mu_2[\phi(V_{\mu_1})] &= [\mu_2(\phi(v_1)), \mu_2(\phi(v_2)), \ldots, \mu_2(\phi(v_k))]^T.\end{aligned}$$

Obviously, the isomorphic signed graphs $G^s_{\mu_j} = (V_{\mu_j}, E^s, \sigma_j)$, $j = 1,2$ have the same characteristic polynomials. Indeed, if ϕ denotes the isomorphism then $A(G^s_{\mu_1}) = P_\phi A(G^s_{\mu_2}) P_\phi^T$ and $\mu[V_{\mu_1}] = P_\phi \mu[V_{\mu_2}]$, where $A(G^s_{\mu_j})$ denotes the adjacency matrix of $G^s_{\mu_j}$ and P_ϕ is the permutation matrix corresponding to the bijective function ϕ.

Let $G^s_\mu = (V_\mu, E^s, \sigma)$ be a signed graph with vertex set $V_\mu = \{v_1, v_2, \ldots, v_n\}$ and adjacency matrix $A(G^s_\mu)$. Suppose for $l = 1,2,\ldots,n$, $H^s_{\mu_l} = (V_{\mu_l}, E^s_l, \sigma_l)$ be a signed graph with $|V_{\mu_l}| = t_l$, adjacency matrix B_l, and vertex set $V\mu_l = \{v_{k_l,1}, v_{k_l,2}, \ldots, v_{k_l,t_l}\}$, where $\{(k_1,1),(k_l,2),\ldots,(k_l,t_l)\} = \{n + \sum_{k=1}^{l-1} t_k + 1, n + \sum_{k=1}^{l-1} t_k + 2, \ldots, n + \sum_{k=1}^{l} t_k\}$. The adjacency matrix of $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ is presented with this labeling as follows:

$$A\left(G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}\right) = \begin{bmatrix} A(G^s_\mu) & PQ \\ (PQ)^T & D \end{bmatrix}$$

where

$$P = \begin{bmatrix} \mu(v_1) & 0 & 0 & \ldots & 0 \\ 0 & \mu(v_2) & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & \mu(v_n) \end{bmatrix}, Q = \begin{bmatrix} \mu_1[V_{\mu_1}]^T & 0 & 0 & \ldots & 0 \\ 0 & \mu_2[V_{\mu_2}]^T & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & \mu_n[V_{\mu_n}]^T \end{bmatrix}, \tag{3.3}$$

and

$$D = \begin{array}{c} \\ H^s_{\mu_1} \\ H^s_{\mu_2} \\ \vdots \\ H^s_{\mu_{n-1}} \\ H^s_{\mu_n} \end{array} \begin{array}{c} \begin{array}{ccccc} H^s_{\mu_1} & H^s_{\mu_2} & H^s_{\mu_3} & \ldots & H^s_{\mu_n} \end{array} \\ \begin{bmatrix} B_1 & 0 & 0 & \ldots & 0 \\ 0 & B_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & B_{n-1} & 0 \\ 0 & 0 & \ldots & 0 & B_n \end{bmatrix} \end{array} \tag{3.4}$$

Consequently $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{n\times(\sum_{l=1}^n t_l)}$ and $D \in R^{(\sum_{l=1}^n t_l)\times(\sum_{l=1}^n t_l)}$. Now, I recall the following, which will be used in the sequel.

Lemma 3.2.1. *(Schur Complement). (See [Horn and Johnson, 2012].) Suppose A is a $n \times n$ matrix with partitioned form $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ where A_{11} and A_{22} are inevitable matrices. Then*

$$\begin{aligned}\det \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} &= \det(A_{22}) \det(A_{11} - A_{12} {A_{22}}^{-1} A_{21}) \\ &= \det(A_{11}) \det(A_{22} - A_{21} {A_{11}}^{-1} A_{12}).\end{aligned}$$

For a signed graph $H^s_\mu = (V_\mu, E^s, \sigma)$, I define signed coronal and L-signed coronal to be the scalars

$$\chi_{H^s_\mu}(\lambda) = \mu\left[V_\mu\right]^T (\lambda I - A(H^s_\mu))^{-1} \mu\left[V_\mu\right]$$

and

$$\chi_{L(H_\mu^s)}(\alpha) = \mu\left[V_\mu\right]^T((\alpha-1)I - L(H_\mu^s))^{-1}\mu\left[V_\mu\right],$$

respectively. Similarly, I define Q-signed coronal of the signed graph H_μ^s as the scalar

$$\chi_{Q(H_\mu^s)}(\beta) = \mu[V_\mu]^T((\beta-1)I - Q(H_\mu^s))^{-1}\mu[V_\mu].$$

In the following proposition, I show that the coronal of a marked signed graph is preserved under isomorphism.

Proposition 3.2.2. *If $H_{\mu_1}^s = (V_{\mu_1}, E^s, \sigma_1)$ and $Z_{\mu_2}^s = (V_{\mu_2}, E^s, \sigma_2)$ are two isomorphic signed marked graphs then*

$$\chi_{H_{\mu_1}^s}(\lambda) = \chi_{H_{\mu_2}^s}(\lambda).$$

Proof. Since $H_{\mu_1}^s$ and $Z_{\mu_2}^s$ are isomorphic, therefore there exist a permutation matrix P_ϕ such that

$$A(Z_{\mu_2}^s) = P_\phi A(H_{\mu_1}^s)P_\phi^T \quad \text{and} \quad \mu_2[V_{\mu_2}] = P_\phi\mu_1[V_{\mu_1}].$$

Now,

$$\begin{aligned}
\chi_{Z_{\mu_2}^s}(\lambda) &= \mu_2\left[V_{\mu_2}\right]^T\left(\lambda I - A(Z_{\mu_2}^s)\right)^{-1}\mu_2\left[V_{\mu_2}\right] \\
&= \mu_1\left[V_{\mu_1}\right]^T P_\phi^T\left(\lambda I - P_\phi A(H_{\mu_1}^s)P_\phi^T\right)^{-1}P_\phi\mu_1\left[V_{\mu_1}\right] \\
&= \mu_1\left[V_{\mu_1}\right]^T P_\phi^T P_\phi^T\left(\lambda I - A(H_{\mu_1}^s)\right)^{-1}P_\phi P_\phi\mu_1\left[V_{\mu_1}\right] \\
&= \mu_1\left[V_{\mu_1}\right]^T\left(\lambda I - A(H_{\mu_1}^s)\right)^{-1}\mu_1\left[V_{\mu_1}\right] \\
&= \chi_{H_{\mu_1}^s}(\lambda). \quad \square
\end{aligned}$$

A signed graph $H_{\mu_2}^s = (V_{\mu_2}, E_2^s, \sigma_2)$ is called k-net regular if $d^+(v) - d^-(v) = k$, for each $v \in V_{\mu_2}$, where $d^+(v)$ and $d^-(v)$ are the positive and negative degree of node v, respectively [Zaslavsky, 2010].

A signed graph is defined as co-regular if it is net-regular with net-degree k for some integer k, and its underlying unsigned graph is r-regular for some positive integer r. In this case, I define the ordered pair (r,k) as the co-regularity pair [Hameed and Imran, 2012].

In the subsequent propositions, I establish explicit formulas to determine the signed coronal, L-signed coronal, and Q-signed coronal, focusing on particular signed graph instances.

Proposition 3.2.3. *Let H_μ^s be a k-net regular signed graph of order n. Then*

$$\chi_{H_\mu^s}(\lambda) = \frac{n}{\lambda - k}.$$

Proof. If H_μ^s is a co-regular signed graph then $\mu(v_i) = \mu(v_j)$, $\forall\, 1 \le i < j \le n$ (Indeed if $d^-(v_i) = m_1$ and $d^-(v_j) = m_2$, where m_1 and m_2 are non negative integers then $d^+(v_i) = r - m_1$ and $d^+(v_j) =$

$r-m_2$ and hence by k-regularity, I have, $(r-m_1)-m_1=(r-m_2)-m_2$ this implies $m_1=m_2$). Thus $\mu[V_\mu]=\mathbf{1}_n$ or $-\mathbf{1}_n$, this leads to $A(H_\mu^s)\mu[V_\mu]=k\mu[V_\mu]$ (for arbitrary $\mu[V_\mu]$ the condition doesnot hold). This implies that $(\lambda I-A(H_\mu^s))\mu[V_\mu]=(\lambda-k)\mu[V_\mu]$. Therefore,,

$$\chi_{H_\mu^s}(\lambda)=\mu[V_\mu]^T\left(\lambda I-A(H_\mu^s)\right)^{-1}\mu[V_\mu]=\frac{\mu[V_\mu]^T\mu[V_\mu]}{\lambda-k}=\frac{n}{\lambda-k}$$ □

Proposition 3.2.4. *Let H_μ^s be a co-regular signed graph of co-regularity pair (r,k). Then*

$$\chi_{L(H_\mu^s)}(\alpha)=\frac{n}{\alpha-1-2d^-},$$

where d^- is the negative degree of any vertex of H_μ^s.

Proof. Since H_μ^s is a co-regular signed graph, each node has the same net degree, and the sum of the entries in each row of its Laplacian matrix is equal to two times the negative degree of any vertex of H_μ^s so I have $L(H_\mu^s)\mu[V_\mu]=2d^-\mu[V_\mu]$, where $\mu[V_\mu]=\mathbf{1}_n$ or $-\mathbf{1}_n$ and in either case

$$\begin{aligned}\left((\alpha-1)I-L(H_\mu^s)\right).\mu[V_\mu]&=\left((\alpha-1)-2d^-\right)I\mu[V_\mu]\\&=\left(\alpha-1-2d^-\right)\mu[V_\mu].\end{aligned}$$

This implies $\mu[V_\mu]^T\left((\alpha-1)I-L(H_\mu^s)\right)^{-1}\mu[V_\mu]=\frac{\mu[V_\mu]^T\mu[V_\mu]}{\alpha-1-2d^-}$. Therefore the desired result follows. □

Proposition 3.2.5. *Let H_μ^s be a co-regular signed graph of order n and co-regularity pair (r,k). Then*

$$\chi_{Q(H_\mu^s)}(\beta)=\frac{n}{\beta-1-2d^+},$$

where d^+ is the positive degree of any vertex of H_μ^s.

Proof. The proof is similar to Proposition 3.2.4. □

Notation 3.2.1. *Consider a signed graph G_μ^s with n vertices. Additionally, let $H_{\mu_1}^s,H_{\mu_2}^s,\ldots,H_{\mu_n}^s$ be signed graphs with $t_1,t_2,\ldots,t_n$ vertices, respectively. In order to get the desired expression, I introduce the following three polynomials involving n variables each*

$$g(\chi_{H_{\mu_1}^s}(\lambda),\ldots,\chi_{H_{\mu_n}^s}(\lambda);G_\mu^s)=\det\left(\begin{bmatrix}\lambda-\chi_{H_{\mu_1}^s}(\lambda) & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \lambda-\chi_{H_{\mu_n}^s}(\lambda)\end{bmatrix}-A(G_\mu^s)\right),$$

$$L(\chi_{L(H_{\mu_1}^s)},\ldots,\chi_{L(H_{\mu_n}^s)};G_\mu^s)=\det\left(\begin{bmatrix}\alpha-t_1-\chi_{L(H_{\mu_1}^s)}(\alpha) & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \alpha-t_n-\chi_{L(H_{\mu_n}^s)}(\alpha)\end{bmatrix}-L(G_\mu^s)\right),$$

$$Q(\chi_{Q(H_{\mu_1}^s)}(\beta),\ldots,\chi_{Q(H_{\mu_n}^s)}(\beta);G_\mu^s)=\det\left(\begin{bmatrix}\beta-t_1-\chi_{Q(H_{\mu_1}^s)}(\beta) & 0 & 0\\ 0 & \ddots & 0\\ 0 & 0 & \beta-t_n-\chi_{Q(H_{\mu_n}^s)}(\beta)\end{bmatrix}-Q(G_\mu^s)\right).$$

Remark 3.2.1. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs. If $\chi_{H^s_{\mu_1}}(\lambda) = \chi_{H^s_{\mu_2}}(\lambda) \cdots = \chi_{H^s_{\mu_n}}(\lambda) = \chi_{H^s_\mu}(\lambda)$ then*

$$g(\chi_{H^s_{\mu_1}}(\lambda), \ldots, \chi_{H^s_{\mu_n}}(\lambda); G^s_\mu) = f_{G^s_\mu}\left(\lambda - \chi_{H^s_\mu}(\lambda)\right)$$

where $f_{G^s_\mu}$ is the signed characteristic polynomial of G^s_μ.

Remark 3.2.2. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs each of them has order m. If $\chi_{L(H^s_{\mu_1})}(\alpha) = \chi_{L(H^s_{\mu_2})}(\alpha) \cdots = \chi_{L(H^s_{\mu_n})}(\alpha) = \chi_{L(H^s_\mu)}(\alpha)$, then*

$$L_g(\chi_{L(H^s_{\mu_1})}(\alpha), \ldots \chi_{L(H^s_{\mu_n})}(\alpha); G^s_\mu) = f_{L(G^s_\mu)}\left(\alpha - m - \chi_{L(H^s_\mu)}(\alpha)\right)$$

where $f_{L(G^s_\mu)}(\alpha)$ is the signed Laplacian polynomial of G^s_μ.

Remark 3.2.3. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs each of them has order m. If $\chi_{Q(H^s_{\mu_1})}(\beta) = \chi_{Q(H^s_{\mu_2})}(\beta) \cdots = \chi_{Q(H^s_{\mu_n})}(\beta) = \chi_{Q(H^s_\mu)}(\beta)$ then*

$$Q_g(\chi_{Q(H^s_{\mu_1})}(\beta), \ldots, \chi_{Q(H^s_{\mu_n})}(\beta); G^s_\mu) = f_{Q(G^s_\mu)}\left(\beta - m - \chi_{Q(H^s_\mu)}(\beta)\right)$$

where $f_{Q(G^s_\mu)}(\beta)$ is the signed signless Laplacian polynomial of G^s_μ.

3.2.1 Characteristic polynomials of $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$

I shall obtain the characteristic polynomial of the generalized corona product for signed graphs which will enable us to identify certain cospectral signed graphs.

Theorem 3.2.6. *Let G^s_μ be a signed graph with n vertices and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs (not necessarily non-isomorphic) with orders $t_1, \ldots, t_n$, respectively. Then*

$$f_{G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}}(\lambda) = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot g(\chi_{H^s_{\mu_1}}(\lambda), \ldots, \chi_{H^s_{\mu_n}}(\lambda); G^s_\mu).$$

Proof. Let A and B_l be the adjacency matrices of graphs G^s_μ and $H^s_{\mu_l}$, respectively, for $l = 1, \ldots, n$. In light of the equations (3.3)-(3.4), it can be inferred that

$$f_{G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}}(\lambda) = \det \begin{bmatrix} \lambda I - A(G^s_\mu) & PQ \\ (PQ)^T & \lambda I - D \end{bmatrix},$$

By applying Lemma 3.2.1, I have

$$
\begin{aligned}
& f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\lambda I - A(G^s_\mu) - PQ \begin{bmatrix} \lambda I - B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda I - B_n \end{bmatrix}^{-1} (PQ)^T\right) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\lambda I - A(G^s_\mu) - PQ \begin{bmatrix} (\lambda I - B_1)^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\lambda I - B_n)^{-1} \end{bmatrix} Q^T P^T\right) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\lambda I - A(G^s_\mu) - P \begin{bmatrix} \mu_1[V_{\mu_1}]^T(\lambda I - B_1)^{-1}\mu_1[V_{\mu_1}] & 0 & 0 \\ & \ddots & \\ 0 & 0 & \mu_n[V_{\mu_n}]^T(\lambda I - B_n)^{-1}\mu_n[V_{\mu_n}] \end{bmatrix} P^T\right) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\lambda I - A(G^s_\mu) - \begin{bmatrix} \chi_{H^s_{\mu_1}}(\lambda) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \chi_{H^s_{\mu_n}}(\lambda) \end{bmatrix}\right) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\begin{bmatrix} \lambda - \chi_{H^s_{\mu_1}}(\lambda) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda - \chi_{H^s_{\mu_n}}(\lambda) \end{bmatrix} - A(G^s_\mu)\right) \\
= & \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot g(\chi_{H^s_{\mu_1}}(\lambda), \ldots, \chi_{H^s_{\mu_n}}(\lambda); G^s_\mu)
\end{aligned}
$$

This validates the proof. □

The following corollary is a direct consequence of Theorem 3.2.6.

Corollary 3.2.6.1. *If $G^s_{\mu_1}$ and $H^s_{\mu'}$ are two signed graphs having orders n and m respectively, then the following holds:*

$$f_{G^s_\mu \circ H^s_{\mu'}}(\lambda) = \left(f_{H^s_{\mu'}}(\lambda)\right)^n \cdot f_{G^s_{\mu_1}}\left(\lambda - \chi_{H^s_{\mu'}}(\lambda)\right).$$

Proof. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_n} \cong H^s_{\mu'}$. Then utilizing Theorem 3.2.6 and Proposition 3.2.2, I deduce

$$f_{G^s_\mu \circ H^s_{\mu'}}(\lambda) = \left(f_{H^s_{\mu'}}(\lambda)\right)^n \cdot f_{G^s_{\mu_1}}\left(\lambda - \chi_{H^s_{\mu'}}(\lambda)\right)$$ □

Corollary 3.2.6.2. *Suppose G^s_μ is a signed graph of order n. Let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs such that $\chi_{H^s_{\mu_1}}(\lambda) = \cdots = \chi_{H^s_{\mu_n}}(\lambda) = \chi_{H^s_{\mu'}}(\lambda)$. Then*

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot f_{G^s_\mu}\left(\lambda - \chi_{H^s_{\mu'}}(\lambda)\right)$$

Proof. It is a straight forward implication of Theorem 3.2.6 and Remark 3.2.1. □

Corollary 3.2.6.3. *Suppose G^s_μ is a signed graph of order n. Let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be a collection of k-net regular signed graphs and each of them consists of m vertices. Then*

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot f_{G^s_\mu}\left(\lambda - \frac{m}{\lambda - k}\right)$$

Proof. It directly follows from Proposition 3.2.3 and Corollary 3.2.6.2. □

Corollary 3.2.6.4. *Let $G^s_{\mu'}$ and $G^s_{\mu''}$ be two co-spectral signed graphs each of order n, and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs. If $\chi_{H^s_{\mu_1}}(\lambda) = \chi_{H^s_{\mu_2}}(\lambda) \cdots = \chi_{H^s_{\mu_n}}(\lambda) = \chi_{H^s_\mu}(\lambda)$ then $G^s_{\mu'} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}(\lambda)$ and $G^s_{\mu''} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}(\lambda)$ are co-spectral.*

Proof. From Theorem 3.2.6 I have

$$\begin{aligned}
&f_{G^s_{\mu'} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) \\
&\quad = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot \det\left(\begin{bmatrix} \lambda - \chi_{H^s_\mu}(\lambda) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda - \chi_{H^s_\mu}(\lambda) \end{bmatrix} - A(G^s_{\mu'})\right) \\
&\quad = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot f_{G^s_{\mu'}}\left(\lambda - \chi_{H^s_\mu}(\lambda)\right) \\
&\quad = \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda)\right) \cdot f_{G^s_{\mu''}}\left(\lambda - \chi_{H^s_\mu}(\lambda)\right) \\
&\quad = f_{G^s_{\mu''} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) \quad \square
\end{aligned}$$

The corollary below is readily provable using Theorem 3.2.6.

Corollary 3.2.6.5. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_{2n}}$ be a family of cospectral signed graphs such that $\chi_{H^s_{\mu_1}}(\lambda) = \chi_{H^s_{\mu_2}}(\lambda) \cdots = \chi_{H^s_{\mu_{2n}}}(\lambda)$ then $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}(\lambda)$ and $G^s_\mu \circ \overset{2n}{\underset{l=n+1}{\Lambda}} H^s_{\mu_l}(\lambda)$ are cospectral.*

Definition 3.2.3. *[Pirzada and Aijaz, 2008] A signed graph G is called a signed bipartite graph if its set of vertices can be divided into two disjoint sets, M and N, such that each edge connects a vertex from M to a vertex from N. If the signed graph is bipartite, I will denote it by $G(M,N)$ where M and N are the disjoint partitions of the vertex set of G.*

Notation 3.2.2. *Suppose $G^s_\mu(M,N)$ is a signed bipartite graph of order n, where $|M| = h$ and $|N| = n-h$. Also, assume that W is a matrix of order $h \times (n-h)$ such that*

$$A(G^s_\mu) = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}$$

Therefore,

$$f_{G^s_\mu}(\lambda) = \det(\lambda I - A(G)) = \det \begin{bmatrix} \lambda I_h & -W \\ -W^T & \lambda I_{n-h} \end{bmatrix}$$

and with two methods of Schur complement (Lemma 3.2.1), I have:

$$f_{G^s_\mu}(\lambda) = \begin{cases} \lambda^{n-2h} \det(\lambda^2 I_h - WW^T) = \lambda^{n-2h} g_{G^s_\mu}(\lambda). \\ \\ \lambda^{2h-n} \det(\lambda^2 I_{n-h} - W^T W) = \lambda^{2h-n} q_{G^s_\mu}(\lambda). \end{cases}$$

If $n = 2h$ then it is clear that $f_{G^s_\mu}(\lambda) = g_{G^s_\mu}(\lambda) = q_{G^s_\mu}(\lambda)$.

Theorem 3.2.7. *Let $G^s_\mu(M,N)$ be a signed bipartite graph of order n, where $|M| = i$ and $|N| = n - i$. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_i} \cong Z^s_{\mu'}$ and $H^s_{\mu_{i+1}} \cong \ldots \cong H^s_{\mu_n} \cong Z^s_{\mu''}$. Then the following hold:*
Then

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda)$$
$$= \begin{cases} (f_{Z^s_{\mu'}}(\lambda))^i \cdot (f_{Z^s_{\mu''}}(\lambda))^{n-i} (\lambda - \chi_{Z^s_{\mu''}}(\lambda))^{n-2i} g_G(S) & if \;\; n \geq 2i \\ (f_{Z^s_{\mu'}}(\lambda))^i \cdot (f_{Z^s_{\mu''}}(\lambda))^{n-i} (\lambda - \chi_{Z^s_{\mu'}}(\lambda))^{2i-n} q_G(S) & if \;\; n \leq 2i \end{cases}$$

where $S = \sqrt{(\lambda - \chi_{Z^s_{\mu'}(\lambda)})(\lambda - \chi_{Z^s_{\mu''}(\lambda)})}$.

Proof. Based on the identical argument presented in the proof of Theorem 3.2.6, it follows that.

$$\begin{aligned} f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) &= \left(\prod_{l=1}^{n} f_{H^s_{\mu_l}}(\lambda) \right) \cdot \det \left(\begin{bmatrix} \lambda - \chi_{H^s_{\mu_1}}(\lambda) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda - \chi_{H^s_{\mu_n}}(\lambda) \end{bmatrix} - A(G) \right) \\ &= \left(f_{Z^s_{\mu'}}(\lambda) \right)^i \cdot \left(f_{Z^s_{\mu''}}(\lambda) \right)^{n-i} \cdot \det \left(\begin{bmatrix} \lambda - \chi_{Z^s_{\mu'}}(\lambda) & 0 \\ 0 & \lambda - \chi_{Z^s_{\mu''}}(\lambda) \end{bmatrix} - A(G) \right). \end{aligned}$$

Since G^s_μ is bipartite, there is a matrix W of order $i \times (n-i)$ such that

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) = \left(f_{Z^s_{\mu'}}(\lambda) \right)^i \cdot (f_{Z^s_{\mu''}}(\lambda))^{n-i} \cdot \det \begin{bmatrix} \lambda - \chi_{Z^s_{\mu'}}(\lambda) & -W \\ -W^T & \lambda - \chi_{Z^s_{\mu''}}(\lambda) \end{bmatrix}.$$

Let $S = \sqrt{(\lambda - \chi_{Z^s_{\mu'}(\lambda)})(\lambda - \chi_{Z^s_{\mu''}(\lambda)})}$, then by utilizing Lemma 3.2.1 and Notation 3.2.2, I can derive

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda)$$
$$= \begin{cases} (f_{Z^s_{\mu'}}(\lambda))^i \cdot (f_{Z^s_{\mu''}}(\lambda))^{n-i} (\lambda - \chi_{Z^s_{\mu''}}(\lambda))^{n-2i} g_G(S) & if \;\; n \geq 2i \\ (f_{Z^s_{\mu'}}(\lambda))^i \cdot (f_{Z^s_{\mu''}}(\lambda))^{n-i} (\lambda - \chi_{Z^s_{\mu'}}(\lambda))^{2i-n} q_G(S) & if \;\; n \leq 2i \end{cases}.$$

□

To supply the subsequent corollaries, I will then introduce the concept of the complement of a connected marked signed graph.

Definition 3.2.4. *Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1)$ be a connected marked signed graph. The graph $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2)$ is called complement of $G^s_{\mu_1}$ if $V(G^s_{\mu_1}) = V(G^s_{\mu_2})$ with $\mu_1 = \mu_2 = \mu(say)$ and $E(G^s_{\mu_2}) = \{uv : u, v \in V(G^s_{\mu_1})$and u and v are not adjacent in$G^s_{\mu_1}\}$ such that $\sigma_2(uv) = \mu(u)\mu(v)$.*

It is easy to see that $\chi_{\bar{K}^s_n}(\lambda) = \frac{n}{\lambda}$, where $\bar{K}^s_n$ is the complement of the complete signed graph K^s_n on n vertices. Therefore, the subsequent corollaries can be easily deduced from Theorem 3.2.7.

Corollary 3.2.7.1. *Let $G^s_\mu(M,N)$ be a signed bipartite graph of order n, where $|M| = i$ and $|N| = n - i$. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_i} \cong \bar{K}^s_m$ and $H^s_{\mu_{i+1}} \cong \ldots \cong H^s_{\mu_n} \cong \bar{K}^s_t$. Then I have*

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) = \begin{cases} \lambda^{mi+(n-i)t} \cdot (\lambda - \frac{t}{\lambda})^{n-2i} g_G\left(\sqrt{(\lambda - \frac{m}{\lambda})(\lambda - \frac{t}{\lambda})}\right) & if \quad n \geq 2i \\ \lambda^{mi+(n-i)t} \cdot (\lambda - \frac{m}{\lambda})^{2i-n} q_G\left(\sqrt{(\lambda - \frac{m}{\lambda})(\lambda - \frac{t}{\lambda})}\right) & if \quad n \leq 2i \end{cases}$$

Corollary 3.2.7.2. *Let $G^s_\mu(M,N)$ be a signed bipartite graph of order n, where $|M| = i$ and $|N| = n - i$. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_i} \cong \bar{K}^s_m$ and $H^s_{\mu_{i+1}} \cong \ldots \cong H^s_{\mu_n} \cong \emptyset$. Then I have*

$$f_{G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}}(\lambda) = \begin{cases} \lambda^{mi+n-2i} g_G\left(\sqrt{(\lambda - \frac{m}{\lambda})\lambda}\right) & if \;\; n \geq 2i \\ \\ \lambda^{mi}(\lambda - \frac{m}{\lambda})^{2i-n} q_G\left(\sqrt{(\lambda - \frac{m}{\lambda})\lambda}\right) & if \;\; n \leq 2i. \end{cases}$$

3.2.2 Laplacian polynomials of $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$

In the proof of the subsequent theorem, I utilize a notation involving a set of graphs denoted as $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$, with each individual graph having orders $t_1, t_2, \ldots, t_n$ respectively. The matrix V is defined to be diagonal with its diagonal entries being t_1 through t_n. $\Delta(G^s_\mu)$ denotes the diagonal matrix with its entries as the degree sequence of G^s_μ. Similarly, $\Delta(H^s_{\mu_l})$ denotes the diagonal matrix with its entries as the degree sequence of $H^s_{\mu_l}$ $(1 \leq l \leq n)$. Δ is the diagonal matrix with its entries being $\Delta(H^s_{\mu_1})$ through $\Delta(H^s_{\mu_n})$.

Theorem 3.2.8. *Suppose G^s_μ is a signed graph with n vertices and let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ represent n signed graphs of orders $t_1, t_2, \ldots, t_n$ respectively, not necessarily non-isomorphic. Then*

$$f_{L(G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l})}(\alpha) = \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha - 1)\right) \cdot L_g(\chi_{L(H^s_{\mu_1})}(\alpha), \ldots \chi_{L(H^s_{\mu_n})}(\alpha); G^s_\mu).$$

Proof. Consider $A(G^s_\mu)$ as the adjacency matrix of graph G and B_l as the adjacency matrix of graph $H^s_{\mu_l}$ for each $l = 1, \ldots, n$. Utilizing the equations (3.3)-(3.4), it can be deduced that

$$
\begin{aligned}
&f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) \\
&\qquad = \det \begin{bmatrix} \alpha I_n - V - \Delta(G^s_\mu) + A(G^s_\mu) & PQ \\ (PQ)^T & \alpha I_{(\sum_{l=1}^n t_l)} - \Delta - I_{(\sum_{l=1}^n t_l)} + D \end{bmatrix} \\
&= \det \begin{bmatrix} \alpha I - V - \Delta(G^s_\mu) + A(G^s_\mu) & PQ \\ (PQ)^T & \begin{bmatrix} (\alpha-1)I - \Delta(H^s_{\mu_1}) + B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\alpha-1)I - \Delta(H^s_{\mu_n}) + B_n \end{bmatrix} \end{bmatrix} \\
&\qquad = \det \begin{bmatrix} \alpha I - V - L(G^s_\mu) & PQ \\ (PQ)^T & \begin{bmatrix} (\alpha-1)I - L(H^s_{\mu_1}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\alpha-1)I - L(H^s_{\mu_n}) \end{bmatrix} \end{bmatrix}
\end{aligned}
$$

By applying Lemma 3.2.1, I arrive at

$$
\begin{aligned}
&= \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \\
&\quad \cdot \det \left(\alpha I - V - L(G^s_\mu) - (PQ) \begin{bmatrix} (\alpha-1)I - L(H^s_{\mu_1}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & (\alpha-1)I - L(H^s_{\mu_n}) \end{bmatrix}^{-1} (PQ)^T \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \\
&\quad \cdot \det \left(\alpha I - V - L(G^s_\mu) - PQ \begin{bmatrix} ((\alpha-1)I - L(H^s_{\mu_1}))^{-1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & ((\alpha-1)I - L(H^s_{\mu_n}))^{-1} \end{bmatrix} (PQ)^T \right) \\
&= \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \cdot \det \big(\alpha I - V - L(G^s_\mu) - \\
&\quad P \begin{bmatrix} \mu_1[V_1]^T I_{t_1} ((\alpha-1)I - L(H^s_{\mu_1}))^{-1} \mu_1[V_1] I_{t_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mu_1[V_n]^T I_{t_n} ((\alpha-1)I - L(H^s_{\mu_n}))^{-1} \mu_1[V_n] I_{t_n} \end{bmatrix} P^T \Bigg) \\
&= \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \cdot \det \left(\alpha I - V - L(G^s_\mu) - \begin{bmatrix} \chi_{L(H^s_{\mu_1})}(\alpha) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \chi_{L(H^s_{\mu_n})}(\alpha) \end{bmatrix} \right) \\
&= \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \cdot \det \left(\begin{bmatrix} \alpha - t_1 - \chi_{L(H^s_{\mu_1})}(\alpha) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \alpha - t_n - \chi_{L(H^s_{\mu_n})}(\alpha) \end{bmatrix} - L(G^s_\mu) \right)
\end{aligned}
$$

Under the notation given in Notation 3.2.1, the result is that

$$
f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1) \right) \cdot L_g(\chi_{L(H^s_{\mu_1})}(\alpha), \ldots, \chi_{L(H^s_{\mu_n})}(\alpha); G^s_\mu) \qquad \square
$$

Corollary 3.2.8.1. *Suppose G^s_μ and $H^s_{\mu'}$ represent two signed graphs with n and m vertices, respectively. Then*

$$f_{L(G^s_\mu \circ H^s_{\mu'})}(\alpha) = \left(f_{L(H^s_{\mu'})}(\alpha-1)\right)^n \cdot f_{L(G^s_\mu)}\left(\alpha - m - \chi_{L(H^s_{\mu'})}(\alpha)\right)$$

Proof. Assuming $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_n} \cong H^s_{\mu'}$, the application of Theorem 3.2.8 along with the consideration of Remark 3.2.2 leads to the desired result. □

Corollary 3.2.8.2. *Let G^s_μ be a signed graph of order n and $H_1, \ldots, H_n$ be n signed graphs, each of them has order m such that $\chi_{L(H^s_{\mu_1})}(\alpha) = \chi_{L(H^s_{\mu_2})}(\alpha) \cdots = \chi_{L(H^s_{\mu_n})}(\alpha) = \chi_{L(H^s_{\mu'})}(\alpha)$. Then*

$$f_{L(G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l})}(\alpha) = \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1)\right) \cdot f_{L(G^s_\mu)}\left(\alpha - m - \chi_{L(H^s_{\mu'})}(\alpha)\right).$$

Proof. It is clear from Theorem 3.2.8 and Remark 3.2.2. □

Corollary 3.2.8.3. *Suppose G^s_μ represents a signed graph consisting of n vertices. Let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n co-regular signed graphs each having order m and a common co-regularity pair (r,k). Then*

$$f_{L(G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l})}(\alpha) = \left(\prod_{l=1}^{n} f_{L(H^s_{\mu_l})}(\alpha-1)\right) \cdot f_{L(G^s_\mu)}\left(\alpha - m - \frac{m}{\alpha - 1 - 2d^-}\right)$$

where d^- is the negative degree of any vertex of $H^s_{\mu_l}$ $(1 \leq l \leq n)$.

Proof. It is clear from Theorem 3.2.8 and Proposition 3.2.4. □

The implications of Theorem 3.2.8 directly validate the assertions made in the subsequent Corollaries.

Corollary 3.2.8.4. *Suppose $G^s_{\mu'}$ and $G^s_{\mu''}$ represent two L-cospectral signed graphs of order n and let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ denote a collection of n signed graphs each with order m. Then $G^s_{\mu'} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ and $G^s_{\mu''} \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ are L-cospectral.*

Corollary 3.2.8.5. *Suppose G^s_μ is a signed graphs of size n and let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_{2n}}$ be a collection of L-cospectral signed graphs each with order m. Then $G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l}$ and $G^s_\mu \circ \overset{2n}{\underset{l=n+1}{\Lambda}} H^s_{\mu_l}$ are L-cospectral.*

Theorem 3.2.9. *Let G^s_μ be a signed bipartite graph with $|V(G)| = n = 2k$ where k denotes the size of each part. Let the underlying graph of G be r-regular. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong Z^s_{\mu^1}$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong Z^s_{\mu^2}$. If $|V(Z^s_{\mu^1})| = m$ and $|V(Z^s_{\mu^2})| = s$ then*

$$f_{L(G^s_\mu \circ \overset{n}{\underset{l=1}{\Lambda}} H^s_{\mu_l})}(\alpha) = \prod_{j=1}^{2} (f_{L(Z^s_{\mu^j})}(\alpha-1))^k \cdot f_{G^s_\mu}\left(\sqrt{\left(\alpha - m - r - \chi_{L(Z^s_{\mu^1})}(\alpha)\right)\left(\alpha - s - r - \chi_{L(Z^s_{\mu^2})}(\alpha)\right)}\right)$$

Proof. Through a parallel line of reasoning as demonstrated in the Theorem 3.2.8 proof, I have

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = \prod_{j=1}^{2} (f_{L(Z^s_{\mu^j})}(\alpha-1))^k \cdot det\left(\begin{bmatrix} (\alpha-m-r-\chi_{L(Z^s_{\mu^1})}(\alpha))I_k & 0 \\ 0 & (\alpha-s-r-\chi_{L(Z^s_{\mu^2})}(\alpha))I_k \end{bmatrix} + A(G^s_\mu)\right)$$

Due to the bipartite nature of G^s_μ, the existence of a k-order matrix W is guaranteed such that

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = \prod_{j=1}^{2} (f_{L(Z^s_{\mu^j})}(\alpha-1))^k \cdot det\left(\begin{bmatrix} (\alpha-m-r-\chi_{L(Z^s_{\mu^1})}(\alpha))I_k & W \\ W^T & (\alpha-s-r-\chi_{L(Z^s_{\mu^2})}(\alpha))I_k \end{bmatrix}\right)$$

Using both Lemma 3.2.1 and Notation 3.2.2 I obtain that

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = \prod_{j=1}^{2} (f_{L(Z^s_{\mu^j})}(\alpha-1))^k \cdot f_{G^s_\mu}\left(\sqrt{\left(\alpha-m-r-\chi_{L(Z^s_{\mu^1})}(\alpha)\right)\left(\alpha-s-r-\chi_{L(Z^s_{\mu^2})}(\alpha)\right)}\right) \square$$

Corollary 3.2.9.1. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k denotes the size of each part. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong Z^s_{\mu^1}$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong Z^s_{\mu^2}$. If $|V(Z^s_{\mu^1})| = m$ and $|V(Z^s_{\mu^2})| = s$ then*

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = \prod_{j=1}^{2} (f_{L(Z^s_{\mu^j})}(\alpha-1))^k \cdot f_{G^s_\mu}\left(\sqrt{\left(\alpha-m-r-\frac{m}{\alpha-1-2d_1^-}\right)\left(\alpha-s-r-\frac{s}{\alpha-1-2d_2^-}\right)}\right)$$

where d_1^- and d_2^- are the negative degree of any vertex of $Z^s_{\mu^1}$ and $Z^s_{\mu^2}$ respectively.

Proof. It is straightforward from Theorem 3.2.9 and Proposition 3.2.4. □

Corollary 3.2.9.2. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k denotes the size of each part. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong \bar{K}_m$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong \bar{K}_s$. Then the following holds:*

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = (\alpha-1)^{k(m+s)} \cdot f_{G^s_\mu}\left(\sqrt{\left(\alpha-m-r-\frac{m}{\alpha-1}\right)\left(\alpha-s-r-\frac{s}{\alpha-1}\right)}\right).$$

Proof. It is a direct consequence of Corollary 3.2.9.1 and Definition 3.2.4. □

Corollary 3.2.9.3. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k is the size of each part. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong \bar{K}_m$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong \emptyset$. Then the following holds:*

$$f_{L(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\alpha) = (\alpha-1))^{km} \cdot f_{G^s_\mu}\left(\sqrt{\left(\alpha-m-r-\frac{m}{\alpha-1}\right)(\alpha-r)}\right).$$

Proof. It can be easily obtained by Corollary 3.2.9.2 for $s = 0$. □

3.2.3 Signless Laplacian polynomials of $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$

Theorem 3.2.10. *Suppose G^s_μ is a signed graph with n vertices and let $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ represent n signed graphs of orders $t_1, t_2, \ldots, t_n$ respectively, not necessarily non-isomorphic. Then*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = (\prod_{l=1}^{n} f_{Q(H^s_{\mu_l})}(\beta-1)) \cdot Q_g(\chi_{Q(H^s_{\mu_1})}(\beta), \ldots, \chi_{Q(H^s_{\mu_n})}(\beta); G^s_\mu).$$

Proof. The proof is similar to Theorem 3.2.8. □

Corollary 3.2.10.1. *Suppose G^s_μ and $H^s_{\mu'}$ represent two signed graphs with n and m vertices, respectively. Then*

$$f_{Q(G^s_\mu \circ H^s_{\mu'})}(\beta) = \left(f_{Q(H^s_{\mu'})}(\beta-1)\right)^n \cdot f_{Q(G^s_\mu)}\left(\beta - m - \chi_{Q(H^s_{\mu'})}(\beta)\right)$$

Proof. If I set $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_n} \cong H^s_{\mu'}$ and subsequently utilizing Theorem 3.2.10 along with Remark 3.2.3 I obtain that

$$f_{Q(G^s_\mu \circ H^s_{\mu'})}(\beta) = \left(f_{Q(H^s_{\mu'})}(\beta-1)\right)^n \cdot f_{Q(G^s_\mu)}\left(\beta - m - \chi_{Q(H^s_{\mu'})}(\beta)\right)$$ □

Corollary 3.2.10.2. *Let G^s_μ be a signed graph of order n and $H_1, \ldots, H_n$ be n signed graphs, each of them has order m such that $\chi_{Q(H^s_{\mu_1})}(\beta) = \chi_{Q(H^s_{\mu_2})}(\beta) \cdots = \chi_{Q(H^s_{\mu_n})}(\beta) = \chi_{Q(H^s_{\mu'})}(\beta)$. Then*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = \left(\prod_{l=1}^{n} f_{Q(H^s_{\mu_l})}(\beta-1)\right) \cdot f_{Q(G^s_\mu)}\left(\beta - m - \chi_{Q(H^s_{\mu'})}(\beta)\right).$$

Proof. It is clear from Theorem 3.2.10 and Remark 3.2.3. □

Corollary 3.2.10.3. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n co-regular signed graphs each with order m and common co-regularity pair (r,k). Then*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = (\prod_{l=1}^{n} f_{Q(H^s_{\mu_l})}(\beta-1)) \cdot f_{Q(G^s_\mu)}\left(\beta - m - \frac{m}{\beta - 1 - 2d^+}\right)$$

where d^+ is the positive degree of any vertex of $H^s_{\mu_l}$ $(1 \leq l \leq n)$.

Proof. It is straightforward from Proposition 3.2.5 and Corollary 3.2.10.2. □

The implications of Theorem 3.2.10 directly validate the assertions made in the subsequent Corollaries.

Corollary 3.2.10.4. *Let $G^s_{\mu'}$ and $G^s_{\mu''}$ be two Q-cospectral signed graphs, each of them has order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_n}$ be n signed graphs each of them has order m. Then $G^s_{\mu'} \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ and $G^s_{\mu''} \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ are Q-cospectral.*

Corollary 3.2.10.5. *Let G^s_μ be a signed graph of order n and $H^s_{\mu_1}, H^s_{\mu_2}, \ldots, H^s_{\mu_{2n}}$ be a family of Q-cospectral graphs each of them has order m. Then $G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$ and $G^s_\mu \circ \bigwedge_{l=n+1}^{2n} H^s_{\mu_l}$ are Q-cospectral.*

Theorem 3.2.11. *Let G^s_μ be a signed bipartite graph with $|V(G)| = n = 2k$ where k denotes the size of each part. Let the underlying graph of G be r-regular. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong Z^s_{\mu^1}$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong Z^s_{\mu^2}$. If $|V(Z^s_{\mu^1})| = m$ and $|V(Z^s_{\mu^2})| = s$ then*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = \prod_{j=1}^{2} (f_{Q(Z^s_{\mu^j})}(\beta-1))^k \cdot f_{G^s_\mu}\left(\sqrt{\left(\beta - m - r - \chi_{Q(Z^s_{\mu^1})}(\beta)\right)\left(\beta - s - r - \chi_{Q(Z^s_{\mu^2})}(\beta)\right)}\right)$$

Proof. The proof is equivalent to the one presented in Theorem 3.2.9. □

Corollary 3.2.11.1. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k denotes the size of each part. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong Z^s_{\mu^1}$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong Z^s_{\mu^2}$. If $|V(Z^s_{\mu^1})| = m$ and $|V(Z^s_{\mu^2})| = s$ then*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = \prod_{j=1}^{2} (f_{Q(Z^s_{\mu^j})}(\beta-1))^k \cdot f_{G^s_\mu}\left(\sqrt{\left(\beta - m - r - \frac{m}{\beta - 1 - 2d_1^+}\right)\left(\beta - s - r - \frac{s}{\beta - 1 - 2d_2^+}\right)}\right)$$

where d_1^+ and d_2^+ are the positive degree of any vertex of $Z^s_{\mu^1}$ and $Z^s_{\mu^2}$ respectively.

Proof. It is straightforward from Theorem 3.2.11 and Proposition 3.2.5. □

Corollary 3.2.11.2. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k denotes the size of each part. Suppose $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong \bar{K}_m$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong \bar{K}_s$. Then the following holds:*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = (\beta-1)^{k(m+s)} \cdot f_{G^s_\mu}\left(\sqrt{\left(\beta - m - r - \frac{m}{\beta - 1}\right)\left(\beta - s - r - \frac{s}{\beta - 1}\right)}\right)$$

Proof. It is a direct consequence of Corollary 3.2.11.1 and Definition 3.2.4. □

Corollary 3.2.11.3. *Let G^s_μ be a signed co-regular bipartite graph of co-regularity pair (r,k) and with $|V(G)| = n = 2k$ where k is the size of each part. Let $H^s_{\mu_1} \cong \ldots \cong H^s_{\mu_k} \cong \bar{K}_m$ and $H^s_{\mu_{k+1}} \cong \ldots \cong H^s_{\mu_n} \cong \phi$. Then the following holds:*

$$f_{Q(G^s_\mu \circ \bigwedge_{l=1}^{n} H^s_{\mu_l})}(\beta) = (\beta-1)^{km} \cdot f_{G^s_\mu}\left(\sqrt{\left(\beta - m - r - \frac{m}{\beta - 1}\right)(\beta - r)}\right).$$

Proof. It can be easily obtained by Corollary 3.2.11.2 for $s = 0$. □

3.3 CONCLUSION

In conclusion, this chapter has significantly expanded the scope of the generalized corona product, initially defined for unsigned graphs, to encompass signed graphs. This extension has allowed us to delve into the structural and spectral characteristics of signed graphs. Specifically, I have formulated statistical properties, such as edge counts and triangle categorizations, tailored for the generalized corona product of signed graphs. These statistical properties have enabled us to establish sufficient conditions for the generalised corona product of signed graphs to be unbalanced.

Furthermore, our investigation has yielded explicit expressions for the signed coronal, L-signed coronal, and Q-signed coronal, with a particular focus on specific structured signed graphs. Additionally, I have obtained the characteristic polynomial, Laplacian polynomial, and Signless Laplacian polynomial for the generalized corona product of signed graphs, all expressed in terms of their respective signed coronal, L-signed coronal, and Q-signed coronal.

Moreover, I have introduced crucial criteria to identify situations where generalised corona product signed graphs may exhibit cospectral properties, including L-cospectrality and Q-cospectrality.