Chapter 8

On survival of coherent systems subject to random shocks

We consider coherent systems subject to random shocks that can damage a random number of components of a system. Based on the distribution of the number of failed components, we discuss three models, namely, (i) a shock can damage any number of components (including zero) with the same probability, (ii) each shock damages, at least, one component, and (iii) a shock can damage, at most, one component. Shocks arrival times are modeled using three important counting processes, namely, the PGGP, the PPHP and the RPMML. For the defined shock models, we discuss relevant reliability properties of coherent systems. An optimal replacement policy for repairable systems is considered as an application of the proposed modeling.

8.1 Introduction

Shock models were studied not only for single unit systems but also for multi-component systems, namely, k-out-of-n systems, series-parallel system, parallel-series system and general coherent systems [see Barlow and Proschan, 1975, for the definitions]. Some follow-up articles review a few of these studies. For instance, Sheu and Liou [1992] have considered the optimal replacement policy for a k-out-of-n system subject to random shocks occurred according to the NHPP. Skoulakis [2000] has discussed reliability characteristics of systems subject to random shocks described by the renewal process. Sheu and Chang [2001] have studied the optimal replacement policy for a k-out-of-n system based on the NHPP. Levitin and Finkelstein [2017] have studied the optimal backup in non-repairable 1-out-of-N heterogeneous warm standby systems by considering two different causes of failures of components, namely, internal failures and external shocks. Levitin and Finkelstein [2017] developed a new general approach to obtain the performance characteristics of complex non-repairable systems in the presence of shocks affecting individual element as well as groups of elements. Eryilmaz and Devrim [2019] have considered reliability of a k-out-of-n system subject to random shocks that occur at random times and cause the failure of random number of components. Moreover, it was assumed in their model that shocks occur according to the PH renewal process. Huang *et al.* [2019] have performed the reliability analysis of

coherent systems subject to internal failures and external shocks. Bian *et al.* [2021] have studied multi-component systems subject to dependent competing failure processes under the extreme and the cumulative shock models. Wang *et al.* [2022] have introduced a novel mixed shock model for a multi-state weighted k-out-of-n system by considering system's resistance against shocks. Lorvand and Kelkinnama [2022] have studied some reliability properties of k-out-of-n systems subject to random shocks under the δ-shock model.

A brief literature review of multi-component systems subject to random shocks shows that the NHPP and the renewal processes are mostly used to model the process of shocks that affect these systems. However, the NHPP has some basic limitations, e.g., the independent increments property which is a very restrictive in many real-life applications. On the other hand, the renewal processes, considered so far in the literature, are not tailored to describe the heavy-tailed property for inter-arrival times. However, there are many real-life scenarios (e.g., shocks generated by earthquakes, etc.) where the inter-arrival time between two shocks may have the heavy-tailed behavior. Note that there are many more general counting processes that are free from these limitations. For instance, the PGGP and the PPHP are two rather general counting processes that possess the dependent increments property, whereas the RPMML is a renewal process with inter-arrival times having the heavy-tailed behavior.

The contribution of this chapter can be summarized as follows: We suggest and describe three new models for the effect of shocks on performance of coherent systems. Specifically,

(i) A shock can damage any number of components (including zero) with the same probability;
(ii) Each shock damages at least one component;
(iii) A shock can damage at most one component.

For the defined models, we obtain survival probabilities for coherent systems. These results hold for any counting process. Furthermore, we discuss in detail some special cases by considering three important counting processes, namely, the PPHP, the PGGP and the RPMML. The specific case of the k-out-of-n system is also considered. Our results are presented in the closed form that allow for straightforward computations.

The rest of the chapter is organized as follows. In Section 8.2, we formulate our model. In Section 8.3, we discuss some general results for the defined models. In Section 8.4, we derive some results for the defined models for three specific counting processes, namely, the PGGP, the PPHP and the RPMML. Lastly, in Section 8.5, as an application, we discuss the optimal replacement policy.

8.2 Model Description

Let L be a random variable representing the lifetime of a coherent system with n iid components that started operating at time $t = 0$. Assume that the system is subject to

external shocks that arrive at random times being the only cause of system's failure. Let $\{N(t) : t \geq 0\}$ be an orderly counting process where $N(t)$ represents the number of shocks arrived by the time t. Assume that each shock can affect a random number of components. Let Z_i be the random variable representing the number of components failed due to the i-th shock, $i \in \mathbb{N}$. In what follows, we give a list of model assumptions.

Assumptions:

- Each shock is harmless with probability p_0 and harmful with probability $1 - p_0$, i.e., $P(Z_i = 0) = p_0$ and $P(Z_i \neq 0) = 1 - p_0$, for $i \in \mathbb{N}$.
- If the i-th shock is harmful then it can damage j components with probability p_j, i.e., $P(Z_i = j) = p_j$, for all $j = 1, 2, \ldots, n$, $i \in \mathbb{N}$. Clearly, $p_1 + p_2 + \cdots + p_n = 1 - p_0$.
- $\{Z_i : i \in \mathbb{N}\}$ is a sequence of iid random variables.
- The shock process $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent.

In what follows in this chapter, we consider three special cases (with respect to distribution of Z_i):

$\mathbf{M_1}$: $P(Z_i = j) = 1/(n+1)$, $j = 0, 1, 2, \ldots, n$, i.e., Z_i follows the uniform distribution over $\{0, 1, 2, \ldots, n\}$. In other words, a shock can damage any number of components (including zero) with the same probability. This model may be appropriate to use in a situation where the distribution of Z_i's is in question.

$\mathbf{M_2}$: Each shock is assumed to be harmful, and at least one component is affected by the shock, i.e., $p_0 = 0$. Consequently, the range of Z_i is given by $\{1, 2, \ldots, n\}$, for all $i \in \mathbb{N}$.

$\mathbf{M_3}$: Each shock can harm at most one component, i.e., $p_2 = p_3 = \cdots = p_n = 0$. Consequently, the range of Z_i is given by $\{0, 1\}$, for all $i \in \mathbb{N}$.

Some real-world settings that comply with the described models are given below:

- Consider an apartment lighting system where n lamps are used to illuminate the apartment. The apartment's lighting is suitable if at least k $(1 \leq k \leq n)$ lamps are on. If the magnitude of electricity is above a fixed threshold, then some of these lamps may be damaged. Consequently, the brightness of the apartment will be reduced. If each lamp is considered a component and electric impulses with high magnitudes are considered as shocks, then one can associate this problem with the model introduced in this chapter. This example is borrowed from Lorvand and Kelkinnama [2022].
- The Proton Exchange Membrane Fuel Cells power systems, described in detail in Eryilmaz and Devrim [2019], is another example of the proposed model.
- In a cricket match, the batting team can be considered as a 2-out-of-11 system. Each ball to the batsman can be considered a shock for the team (system). This setting can be described by the M_3 model.

8.3 General Model

In this section, we obtain some general results. We derive the expression for the survival function of a coherent system for the defined model that is valid for an arbitrary orderly counting process $\{N(t) : t \geq 0\}$. We first consider the k-out-of-n systems, and then generalize the result to coherent systems.

Let $M(t)$ be a random variable representing the number of failed components of a system by time t. Then

$$M(t) = \sum_{i=1}^{N(t)} Z_i. \tag{8.3.1}$$

Further, let $L_{k:n}$ be a random variable representing the lifetime of a k-out-of-n system with n iid components. Note that this system functions until the $(n-k+1)$-th component fails. Then

$$\bar{F}_{L_{k:n}}(t) = P(M(t) < n-k+1) = \sum_{i=0}^{n-k} P(M(t) = i). \tag{8.3.2}$$

Further, let L be a random variable representing the lifetime of a coherent system with n iid components, described by the system's signature vector $(s_1, s_2, \ldots, s_n)$ [see Samaniego, 2007]). Then

$$\bar{F}_L(t) = \sum_{k=1}^{n} P(L > t | L = L_{k:n}) P(L = L_{k:n}) = \sum_{k=1}^{n} s_k \bar{F}_{L_{k:n}}(t), \tag{8.3.3}$$

where $s_k = P(L = L_{k:n})$, for $k = 1, 2, \ldots, n$.

To obtain the survival functions of L and $L_{k:n}$, we first need to find the probabilities $P(M(t) = i)$, for $i = 0, 1, 2, \ldots, n-k$. In the following lemma, we derive these probabilities.

Lemma 8.3.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$, and $\{N(t) : t \geq 0\}$ be a counting process. Then*

$$P(M(t) = 0) = \sum_{j=0}^{\infty} p_0^j P(N(t) = j)$$

and

$$P(M(t) = i) = \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{\boldsymbol{z} \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t) = j), \quad i = 1, 2, \ldots, n-1,$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i < n$.

Proof: Note that the event "$M(t) = 0$" happens if and only if each shock, occurred till time t, is harmless for the system. Then,

$$\begin{aligned} P(M(t)=0) &= P(N(t)=0) + \sum_{j=1}^{\infty} P(N(t)=j, Z_1=0, Z_2=0, Z_3=0, \ldots, Z_j=0) \\ &= \sum_{j=0}^{\infty} p_0^j P(N(t)=j). \end{aligned}$$

Now, for $1 \leq i < n$, the event "$M(t) = i$" means that exactly i components of the system failed till time t. Thus, if $N(t) = j$, then there are exactly m harmful shocks out of j shocks, where $j \geq m, 1 \leq m \leq i$. Consequently, we can write

$$\begin{aligned} P(M(t)=i) &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \mathrm{P}(i \text{ components fail due to exactly } m \text{ harmful shocks when } N(t)=j) \\ &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \sum_{\boldsymbol{z} \in \Omega_{m,i}} P(N(t)=j, Z_1=z_1, \ldots, Z_m=z_m, Z_{m+1}=0, \ldots, Z_j=0) \\ &= \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{\boldsymbol{z} \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t)=j), \end{aligned}$$

where the last equality holds because Z_i's are iid, and the process $\{N(t) : t \geq 0\}$ and $\{Z_i : i \in \mathbb{N}\}$ are independent. Hence the result is proved. □

The next theorem immediately follows from Lemma 8.3.1 and Eq. (8.3.2).

Theorem 8.3.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to a counting process $\{N(t) : t \geq 0\}$. Then*

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} p_0^j P(N(t)=j) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \left(\sum_{\boldsymbol{z} \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) p_0^{j-m} P(N(t)=j),$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$ and $1 \leq m \leq i \leq n-k$. □

The succeeding corollary follows from Theorem 8.3.1.

Corollary 8.3.1 *The following results hold.*

(i) For model M_1:

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} \left(\frac{1}{n+1} \right)^j P(N(t)=j) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \sum_{j=m}^{\infty} \binom{j}{m} \binom{i-1}{m-1} \left(\frac{1}{n+1} \right)^j P(N(t)=j);$$

(ii) For model M_2:

$$\bar{F}_{L_{k:n}}(t) = P(N(t)=0) + \sum_{i=1}^{n-k} \sum_{m=1}^{i} \left(\sum_{\boldsymbol{z} \in \Omega_{m,i}} \prod_{r=1}^{m} p_{z_r} \right) P(N(t)=m);$$

(*iii*) *For model* M_3*:*

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty} p_0^j P(N(t)=j) + \sum_{i=1}^{n-k}\sum_{j=i}^{\infty} \binom{j}{i} p_1^i p_0^{j-i} P(N(t)=j).$$

Proof:

(*i*) If $p_j = 1/(n+1)$, for all $j = 1, 2, \ldots, n$, then

$$\sum_{\boldsymbol{z}\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \left(\frac{1}{n+1}\right)^m \sum_{\boldsymbol{z}\in\Omega_{m,i}} 1 = \left(\frac{1}{n+1}\right)^m |\Omega_{m,i}|.$$

By using Bose-Einstien value, one can show that $|\Omega_{m,i}| = \binom{i-1}{m-1}$. Thus

$$\sum_{\boldsymbol{z}\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \left(\frac{1}{n+1}\right)^m \binom{i-1}{m-1}.$$

By using the above equality in Theorem 8.3.1, we get the required result.

(*ii*) The result immediately follows from Theorem 8.3.1 by substituting $p_0 = 0$.

(*iii*) Note that

$$\sum_{\boldsymbol{z}\in\Omega_{m,i}} \prod_{r=1}^{m} P_{z_r} = \begin{cases} 0 & if\ m < i \\ p_1^m & if\ m = i. \end{cases}$$

By using the above equality in Theorem 8.3.1, we get the required result. □

In the next theorem we derive the survival function of a coherent system for the defined model. The proof is straightforward by using Eq. (8.3.3) and Theortem 8.3.1 and hence, omitted.

Theorem 8.3.2 *Let* $P(Z_i = j) = p_j$*, for all* $j = 0, 1, 2, \ldots, n$*,* $i \in \mathbb{N}$*. Assume that shocks occur according to a counting process* $\{N(t) : t \geq 0\}$*. Then*

$$\bar{F}_L(t) = \sum_{j=0}^{\infty} p_0^j P(N(t)=j) + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty} s_k \binom{j}{m} \left(\sum_{\boldsymbol{z}\in\Omega_{m,i}} \prod_{r=1}^{m} p_{z_r}\right) p_0^{j-m} P(N(t)=j),$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \leq z_l \leq n, 1 \leq l \leq m\}$*,* $s_k = P(L = L_{k:n})$*,* $1 \leq k \leq n-1$ *and* $1 \leq m \leq i \leq n-k$*.*

Remark 8.3.1 *By proceeding in the same line as in Corollary 8.3.1, the survival function of a coherent system for models* M_1, M_2 *and* M_3 *can be obtained.*

8.4 Some Special Cases

In this section, we derive survival functions for the defined model for some specific shock processes, namely, the PGGP, the PPHP and the RPMML. As these processes are quite 'sophisticated', each case needs specific technique and reasoning.

8.4.1 PGGP

Theorem 8.4.1 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, $i \in \mathbb{N}$. Assume that shocks occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, \mu, \alpha, l\}$. Then*

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \left(\frac{\alpha}{\alpha+(1-p_0)\Lambda(t)}\right)^{\mu-\nu}\left[\frac{\Gamma_\nu(\mu,(\alpha+(1-p_0)\Lambda(t))l)}{\Gamma_\nu(\mu,\alpha l)}\right.\\ &\left.+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\left(\frac{\Lambda(t)}{\alpha+(1-p_0)\Lambda(t)}\right)^m\frac{\Gamma_\nu(m+\mu,(\alpha+(1-p_0)\Lambda(t))l)}{m!\Gamma_\nu(\mu,\alpha l)}\right],\end{aligned}$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$ and $1 \le m \le i \le n-k$.

Proof: From Theorem 8.3.1, we have

$$\bar{F}_{L_{k:n}}(t) = \sum_{j=0}^{\infty}p_0^j P(N(t)=j) + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)p_0^{j-m}P(N(t)=j). \quad (8.4.1)$$

Let Q be the mixing distribution of the PGGP. Then, for $m \ge 0$, we have

$$\begin{aligned}\sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}P(N(t)=j) &= \sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}\int_0^\infty \exp\{-q\Lambda(t)\}\frac{(q\Lambda(t))^j}{j!}dF_Q(q)dq\\ &= \int_0^\infty \exp\{-q\Lambda(t)\}\frac{(q\Lambda(t))^m}{m!}\left(\sum_{j=m}^{\infty}\frac{(qp_0\Lambda(t))^{j-m}}{(j-m)!}\right)dF_Q(q)\\ &= \int_0^\infty \exp\{-q(1-p_0)\Lambda(t)\}\frac{(q\Lambda(t))^m}{m!}dF_Q(q)\\ &= \left(\frac{\alpha}{\alpha+(1-p_0)\Lambda(t)}\right)^{\mu-\nu}\left(\frac{\Lambda(t)}{\alpha+(1-p_0)\Lambda(t)}\right)^m\\ &\quad\times\frac{\Gamma_\nu(m+\mu,(\alpha+(1-p_0)\Lambda(t))l)}{m!\Gamma_\nu(\mu,\alpha l)}, \quad (8.4.2)\end{aligned}$$

where the first equality follows from Definition 1.2.13; the second equality holds due to the *Dominated Convergence Theorem*, and the last equality follows from Eq. (1.2.2). Now, by using Eq. (8.4.2) in Eq. (8.4.1), we get the required result. □

The next corollary immediately follows from Theorem 8.4.1. Here, we obtain the survival function of the k-out-of-n system for models M_1, M_2 and M_3.

Corollary 8.4.1 *The following results hold.*

(*i*) *For model M_1:*

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha+(n/(n+1))\Lambda(t)}\right)^{\mu-\nu}\left[\frac{\Gamma_\nu(\mu,(\alpha+(n/(n+1))\Lambda(t))l)}{\Gamma_\nu(\mu,\alpha l)}+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1}\right.$$

$$\times \left(\frac{1}{n+1}\right)^m \left(\frac{\Lambda(t)}{\alpha+(n/(n+1))\Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu,(\alpha+(n/(n+1))\Lambda(t))l)}{m!\Gamma_\nu(\mu,\alpha l)}\Bigg];$$

(*ii*) *For model* M_2*:*

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha+\Lambda(t)}\right)^{\mu-\nu}\Bigg[\frac{\Gamma_\nu(\mu,(\alpha+\Lambda(t))l)}{\Gamma_\nu(\mu,\alpha l)} + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\left(\frac{\Lambda(t)}{\alpha+\Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu,(\alpha+\Lambda(t))l)}{m!\Gamma_\nu(\mu,\alpha l)}\Bigg];$$

(*iii*) *For model* M_3*:*

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{\alpha}{\alpha+p_1\Lambda(t)}\right)^{\mu-\nu}\left[\frac{\Gamma_\nu(\mu,(\alpha+p_1\Lambda(t))l)}{\Gamma_\nu(\mu,\alpha l)} + \sum_{i=1}^{n-k}\left(\frac{p_1\Lambda(t)}{\alpha+p_1\Lambda(t)}\right)^i \frac{\Gamma_\nu(i+\mu,(\alpha+p_1\Lambda(t))l)}{i!\Gamma_\nu(\mu,\alpha l)}\right]. \quad \square$$

The next corollary immediately follows from Theorem 8.4.1, and Remark 1.2.4 (*b*) and (*c*).

Corollary 8.4.2 *For the general model, the following results hold true for the k-out-of-n system.*

(*i*) *Assume that shocks occur according to the NHPP with intensity* $\lambda(t)$*. Then*

$$\bar{F}_{L_{k:n}}(t) = \exp\{-(1-p_0)\Lambda(t)\}\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\frac{\Lambda(t)^m}{m!}\right].$$

(*ii*) *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$*. Then*

$$\bar{F}_{L_{k:n}}(t) = \left(\frac{b}{(1-p_0)t+b}\right)^\beta\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!}\left(\frac{t}{(1-p_0)t+b}\right)^m\right].$$

Remark 8.4.1 *From the above corollary, one can obtain the survival functions of the system for models* M_1*,* M_2 *and* M_3 *for the NHPP and the Pólya shock processes.* $\square$

In the succeeding theorem, we derive the survival function of a coherent system for the defined model. The proof follows from Eq. (8.3.3) and Theorem 8.4.1.

Theorem 8.4.2 *Let* $P(Z_i = j) = p_j$*, for all* $j = 0, 1, 2, \ldots, n$*,* $i \in \mathbb{N}$*. Assume that shocks occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, \mu, \alpha, l\}$*. Then*

$$\bar{F}_L(t) = \left(\frac{\alpha}{\alpha+(1-p_0)\Lambda(t)}\right)^{\mu-\nu}\Bigg[\frac{\Gamma_\nu(\mu,(\alpha+(1-p_0)\Lambda(t))l)}{\Gamma_\nu(\mu,\alpha l)} + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i}s_k\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right) \times \left(\frac{\Lambda(t)}{\alpha+(1-p_0)\Lambda(t)}\right)^m \frac{\Gamma_\nu(m+\mu,(\alpha+(1-p_0)\Lambda(t))l)}{m!\Gamma_\nu(\mu,\alpha l)}\Bigg],$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$*,* $s_k = P(L = L_{k:n})$*, and* $1 \le m \le i \le n-k$*.* $\square$

The next corollary follows from Theorem 8.4.2, and Remark 1.2.4 (b) and (c).

Corollary 8.4.3 *For the general model, the following results hold true.*

(i) *Assume that shocks occur according to the NHPP with intensity* $\lambda(t)$. *Then*

$$\bar{F}_L(t) = \exp\{-(1-p_0)\Lambda(t)\}\left[1+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Lambda(t)^m}{m!}\right];$$

(ii) *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$. *Then*

$$\bar{F}_L(t) = \left(\frac{b}{(1-p_0)t+b}\right)^{\beta}\left[1+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!}\left(\frac{t}{(1-p_0)t+b}\right)^{m}\right]. \quad \square$$

In the succeeding example, we illustrate the result given in Corollary 8.4.2 (ii).

Example 8.4.1 *Consider a 3-out-of-4 system, i.e.,* $k=3$ *and* $n=4$. *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$. *Then, from Corollary 8.4.2* (ii), *we get*

$$\bar{F}_{L_{3:4}}(t) = \left(\frac{b}{(1-p_0)t+b}\right)^{\beta}\left[1+\beta p_1\left(\frac{t}{(1-p_0)t+b}\right)\right].$$

Let $\beta=4$ *and* $b=2$. *In Figure 8.1a, we plot the above survival function against* $t\in[0,15]$, *for different values of* p_0 *and for fixed* $p_1=0.2$. *This figure shows that the system's survival probability increases as* p_0 *increases. In Figure 8.1b, we plot the system's survival function against* $t\in[0,15]$, *for model* M_1, *with* $p_0=1/5$ *and* $p_2=1/5$. *In Figure 8.1c, the system's survival function against* $t\in[0,10]$, *for different values of* p_1, *is plotted for model* M_2. *From this figure, we conclude that an increment in* p_1 *increases the system's survivability. In Figure 8.1d, we plot the system's survival function against* $t\in[0,30]$, *for different values of* p_1, *for model* M_3. *This figure shows that an increment in* p_1 *decreases the system's survivability.* $\square$

In the next example, we illustrate the result given in Corollary 8.4.3 (ii).

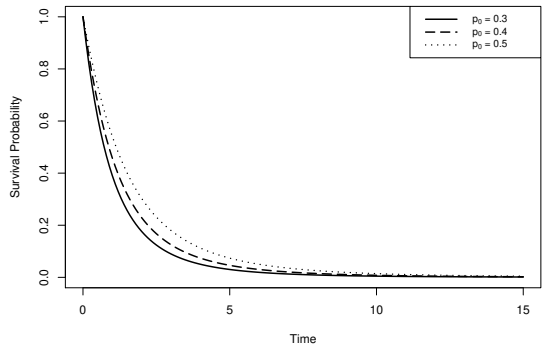
Example 8.4.2 *Consider a coherent system with the structure function [see Barlow and Proschan, 1975] given by*

$$\phi(x_1,x_2,x_3) = min(x_1, max(x_2,x_3)).$$

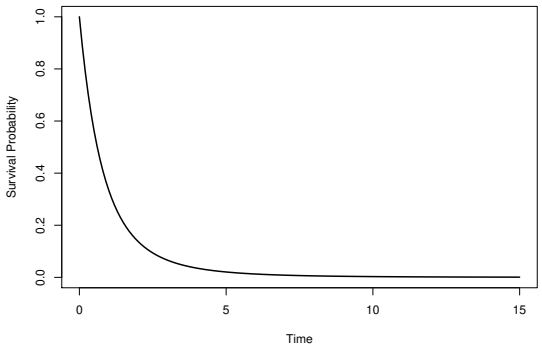
Now, it can easily be shown that $(s_1,s_2,s_3)=(1/3,2/3,0)$. *Then, from Corollary 8.4.3* (ii), *we get*

$$\bar{F}_L(t) = \left(\frac{b}{(1-p_0)t+b}\right)^{\beta}\left[1+p_1\left(\frac{\beta t}{(1-p_0)t+b}\right)+\frac{1}{3}p_2\left(\frac{\beta t}{(1-p_0)t+b}\right)\right.$$
$$\left.+\frac{1}{3}\left(\frac{\beta(\beta+1)}{2}\right)\left(\frac{p_1 t}{(1-p_0)t+b}\right)^2\right].$$

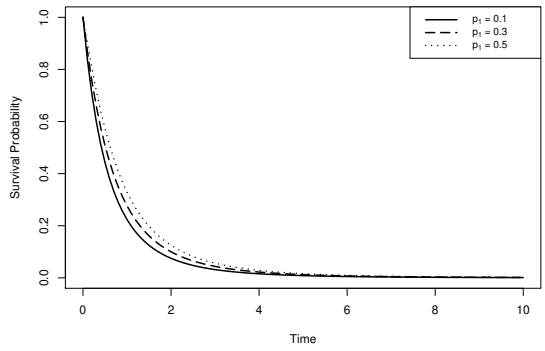
For fixed $p_0=0.4, p_1=0.3, p_2=0.2, b=2$ *and* $\beta=4$, *we plot the survival function of the above coherent system in Figure 8.2.* $\square$



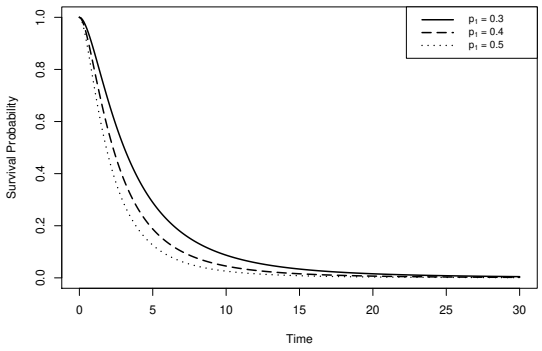
(a) General Model



(b) Model M_1



(c) Model M_2



(d) Model M_3

Figure 8.1: Plot of the survival function of a 3-out-of-4 system against time.

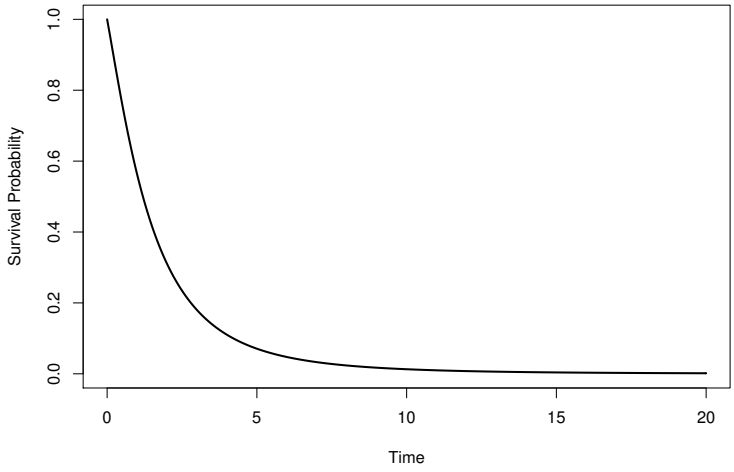


Figure 8.2: Plot of the survival function of a coherent system with the signature vector $(s_1, s_2, s_3) = (1/3, 2/3, 0)$ against time.

In the succeeding theorem, the mean lifetime of a k-out-of-n system is derived under the assumption that shocks occur according to the Pólya process or the HPP. We only give the proof of part (i). Part (ii) can be proved in the same line.

Theorem 8.4.3 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$.*

(i) *Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$. Then the mean lifetime of the k-out-of-n system is given by*

$$E(L_{k:n}) = \left(\frac{b}{\beta - 1}\right)\left(\frac{1}{1-p_0}\right)\left[1 + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^m\right];$$

(ii) *Assume that shocks occur according to the HPP with the intensity λ. Then the mean lifetime of the k-out-of-n system is given by*

$$E(L_{k:n}) = \frac{1}{\lambda}\left(\frac{1}{1-p_0}\right)\left[1 + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^m\right];$$

here $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$ and $1 \le m \le i \le n - k$.

Proof (i): Now, from Corollary 8.4.2 (ii), we have

$$E(L_{k:n}) = \int_0^\infty \bar{F}_{L_{k:n}}(t)dt$$

$$
\begin{aligned}
&= \int_0^\infty \left(\frac{b}{(1-p_0)t+b}\right)^\beta dt + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!} \\
&\qquad \times \int_0^\infty \left(\frac{b}{(1-p_0)t+b}\right)^\beta \left(\frac{t}{(1-p_0)t+b}\right)^m dt \\
&= \left(\frac{b}{\beta-1}\right)\left(\frac{1}{1-p_0}\right) + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\frac{\Gamma(\beta+m)}{\Gamma(\beta)m!} \\
&\qquad \times \left(\frac{1}{1-p_0}\right)^{m+1} \int_0^\infty \left(\frac{b}{z+b}\right)^\beta \left(\frac{z}{z+b}\right)^m dz. \qquad (8.4.3)
\end{aligned}
$$

Again, we have

$$
\int_0^\infty \left(\frac{b}{z+b}\right)^\beta \left(\frac{z}{z+b}\right)^m = \frac{b\Gamma(m+1)\Gamma(\beta-1)}{\Gamma(m+\beta)}.
$$

By using the above equality in Eq. (8.4.3), we get the required result. □

The next corollary follows from the above theorem. Here we discuss the results for models M_1, M_2 and M_3. We only give the proof of part (i). Other parts can be proved in the same line.

Corollary 8.4.4 *The following results hold.*

(i) *For model M_1: If the shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$, then*

$$
E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left(1+\frac{1}{n}\right)^{n-k+1},
$$

and if shocks occur according to the HPP with the intensity λ, then

$$
E(L_{k:n}) = \frac{1}{\lambda}\left(1+\frac{1}{n}\right)^{n-k+1};
$$

(ii) *For model M_2: If shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$, then*

$$
E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\right],
$$

and if shocks occur according to the HPP with the intensity λ, then

$$
E(L_{k:n}) = \frac{1}{\lambda}\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\right];
$$

(*iii*) *For model M_3: If shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$, then*

$$E(L_{k:n}) = \left(\frac{b}{\beta-1}\right)\left(\frac{1+n-k}{p_1}\right),$$

and if shocks occur according to the HPP with the intensity λ, then

$$E(L_{k:n}) = \frac{1}{\lambda}\left(\frac{1+n-k}{p_1}\right).$$

Proof (*i*): Now, from Theorem 8.4.3 (*i*), we have, for model M_1,

$$\begin{aligned} E(L_{k:n}) &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left[1+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1}\left(\frac{1}{n}\right)^m\right] \\ &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left[1+\left(\frac{1}{n}\right)\sum_{i=1}^{n-k}\left(1+\frac{1}{n}\right)^{i-1}\right] \\ &= \left(\frac{b}{\beta-1}\right)\left(\frac{n+1}{n}\right)\left(1+\frac{1}{n}\right)^{n-k} = \left(\frac{b}{\beta-1}\right)\left(1+\frac{1}{n}\right)^{n-k+1}. \end{aligned}$$

Hence the result is proved. □

In the following theorem we derive the mean lifetime of a coherent system for the general model.

Theorem 8.4.4 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$.*

(*i*) *Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta > 1$. Then the mean lifetime of the coherent system is given by*

$$E(L) = \left(\frac{b}{\beta-1}\right)\left(\frac{1}{1-p_0}\right)\left[1+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^k\right];$$

(*ii*) *Assume that shocks occur according to the HPP with the intensity λ. Then the mean lifetime of the coherent system is given by*

$$E(L) = \left(\frac{1}{\lambda}\right)\left(\frac{1}{1-p_0}\right)\left[1+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\left(\frac{1}{1-p_0}\right)^k\right];$$

here $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$ and $1 \le m \le i \le n-k$.

Proof: We have $P(L > t) = \sum_{i=1}^{n} s_i P(L_{i:n} > t)$, which implies $E(L) = \sum_{i=1}^{n} s_i E(L_{i:n})$. Consequently, the result follows from Theorem 8.4.3.

8.4.2 PPHP

In the following theorem we derive the expression of the reliability function of a k-out-of-n system for the defined model under the assumption that shock process is the PPHP.

Theorem 8.4.5 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\pi}, T\}$. Then*

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-(m+1)}\boldsymbol{T^0},\end{aligned}$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$ and $1 \le m \le i \le n - k$.

Proof: From Theorem 8.3.1 and Theorem 5.3.3(ii), we get

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \sum_{j=0}^{\infty}p_0^j\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)p_0^{j-m}\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0}.\end{aligned} \tag{8.4.4}$$

Further, we can write

$$\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} = \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(j+1)}\boldsymbol{T^0}, \quad j \in \mathbb{N}\cup\{0\}. \tag{8.4.5}$$

Now, for $m \ge 0$, we have

$$\begin{aligned}&\sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I - T)^{-(j+1)}\boldsymbol{T^0} \\ &= \Lambda(t)^{-1}\sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(j+1)}\boldsymbol{T^0} \\ &= \Lambda(t)^{-1}\boldsymbol{\pi}\left(\sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}(I - \Lambda(t)^{-1}T)^{-(j+1)}\right)\boldsymbol{T^0} \\ &= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)}\left(\sum_{j=m}^{\infty}\binom{j}{m}p_0^{j-m}(I - \Lambda(t)^{-1}T)^{-(j-m)}\right)\boldsymbol{T^0} \\ &= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)}\left(\sum_{l=0}^{\infty}\binom{m+l}{m}p_0^{l}(I - \Lambda(t)^{-1}T)^{-l}\right)\boldsymbol{T^0} \\ &= \Lambda(t)^{-1}\boldsymbol{\pi}(I - \Lambda(t)^{-1}T)^{-(m+1)}\left(\sum_{l=0}^{\infty}\binom{m+l}{l}p_0^{l}(I - \Lambda(t)^{-1}T)^{-l}\right)\boldsymbol{T^0}\end{aligned}$$

$$
\begin{aligned}
&= \Lambda(t)^{-1}\boldsymbol{\pi}(I-\Lambda(t)^{-1}T)^{-(m+1)}(I-p_0(I-\Lambda(t)^{-1}T)^{-1})^{-(m+1)}\boldsymbol{T^0}\\
&= \Lambda(t)^{-1}\boldsymbol{\pi}((1-p_0)I-\Lambda(t)^{-1}T)^{-(m+1)}\boldsymbol{T^0}\\
&= \Lambda(t)^{m}\boldsymbol{\pi}((1-p_0)\Lambda(t)I-T)^{-(m+1)}\boldsymbol{T^0}, \qquad (8.4.6)
\end{aligned}
$$

where the first equality follows from Eq. (8.4.5); the fifth equality holds because $\binom{m+l}{m} = \binom{m+l}{l}$, and the sixth equality holds because $\sum_{l=0}^{\infty}\binom{m+l}{l}T^l = (I-T)^{-(m+1)}$. Now, by using Eq. (8.4.6), we can write

$$
\begin{aligned}
&\sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)p_0^{j-m}\Lambda(t)^j\boldsymbol{\pi}(\Lambda(t)I-T)^{-(j+1)}\boldsymbol{T^0}\\
&=\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}((1-p_0)\Lambda(t)I-T)^{-(m+1)}\boldsymbol{T^0}. \qquad (8.4.7)
\end{aligned}
$$

Finally, by using Eq. (8.4.7) in Eq. (8.4.4), we get the required result. □

The following corollary immediately follows from Theorem 8.4.5. Here we discuss the results for specific models M_1, M_2 and M_3.

Corollary 8.4.5 *The following results hold.*

(*i*) *For model* M_1*:*

$$
\begin{aligned}
\bar{F}_{L_{k:n}}(t) &= \boldsymbol{\pi}\left(\Lambda(t)\left(\frac{n}{n+1}\right)I-T\right)^{-1}\boldsymbol{T^0}\\
&\quad+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\binom{i-1}{m-1}\left(\frac{\Lambda(t)}{n+1}\right)^m\boldsymbol{\pi}\left(\Lambda(t)\left(\frac{n}{n+1}\right)I-T\right)^{-(m+1)}\boldsymbol{T^0};
\end{aligned}
$$

(*ii*) *For model* M_2*:*

$$
\begin{aligned}
\bar{F}_{L_{k:n}}(t) &= \boldsymbol{\pi}(\Lambda(t)I-T)^{-1}\boldsymbol{T^0}\\
&\quad+\sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m}p_{z_r}\right)\Lambda(t)^m\boldsymbol{\pi}(\Lambda(t)I-T)^{-(m+1)}\boldsymbol{T^0};
\end{aligned}
$$

(*iii*) *For model* M_3*:*

$$
\bar{F}_{L_{k:n}}(t) = \boldsymbol{\pi}(\Lambda(t)p_1I-T)^{-1}\boldsymbol{T^0}+\sum_{i=1}^{n-k}p_1^i\Lambda(t)^i\boldsymbol{\pi}(\Lambda(t)p_1I-T)^{-(i+1)}\boldsymbol{T^0}. \qquad \square
$$

In the next theorem, we derive the survival function of a coherent system for the general model.

Theorem 8.4.6 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the PPHP with the set of parameters $\{\lambda(t), \boldsymbol{\pi}, T\}$. Then*

$$\begin{aligned}\bar{F}_L(t) &= \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} \\ &\quad + \sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i} s_k \left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)\Lambda(t)^m \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-(m+1)}\boldsymbol{T^0},\end{aligned}$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$ and $1 \le m \le i \le n-k$. □

The results given in above theorems and corollary are illustrated by the following two examples.

Example 8.4.3 *Consider a 3-out-of-4 system. Then, from Theorem 8.4.5, we have*

$$\bar{F}_{L_{3:4}}(t) = \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} + p_1\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0}.$$

Let the parameters be

$$\boldsymbol{\pi} = (0.2, 0.8), \quad T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix} \text{ and } \lambda = 2.$$

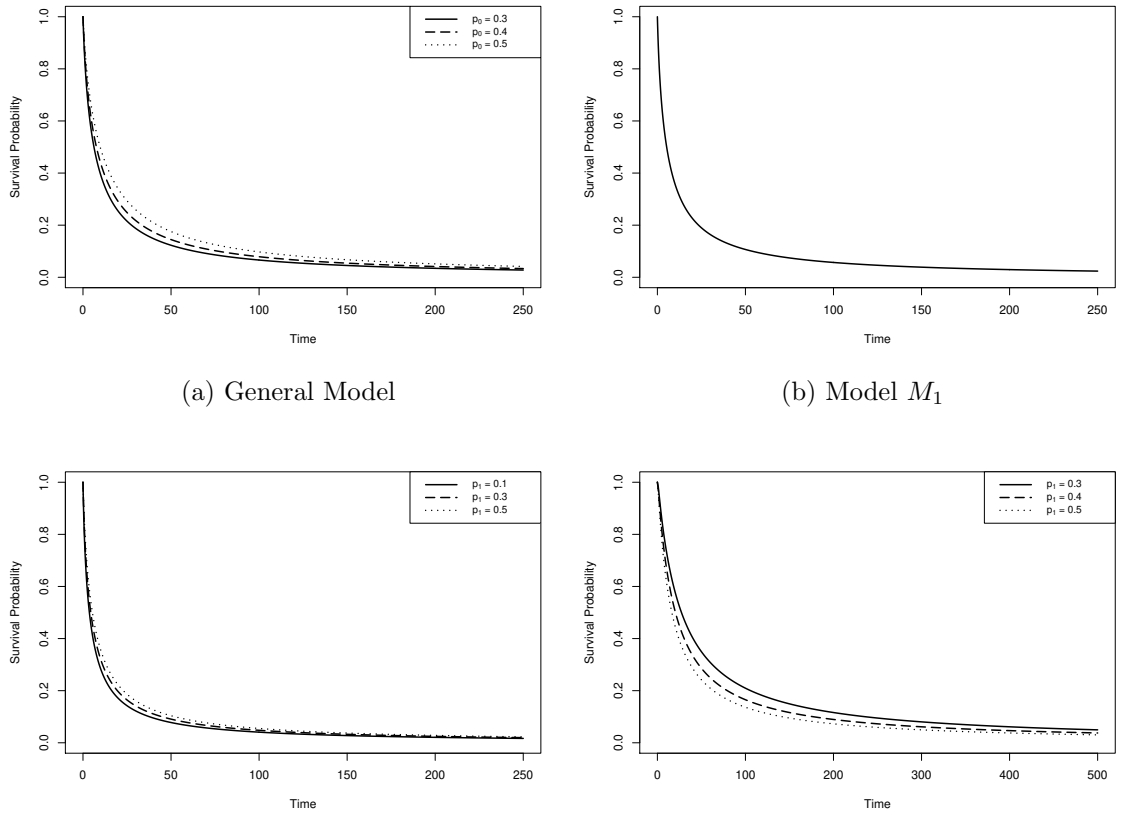
In Figure 8.3a, we plot the survival function of the system (given in Theorem 8.4.5) against $t \in [0, 250]$, for different values of p_0 and for fixed $p_1 = 0.2$. In Figure 8.3b, we plot the system's survival function (given in Corollary 8.4.5 (i)) against $t \in [0, 250]$, for fixed $p_0 = 1/5$ and $p_2 = 1/5$. In Figure 8.3c, we plot the system's survival function (given in Corollary 8.4.5 (ii)) against $t \in [0, 250]$, for different values of p_1. Lastly, in Figure 8.3d, we plot the system's survival function (given in Corollary 8.4.5 (iii)) against $t \in [0, 500]$, for different values of p_1. From all these figures, we can make the same observations as in Example 8.4.1. □

Example 8.4.4 *Consider the coherent system mentioned in Example 8.4.2. Then, from Theorem 8.4.6, we have*

$$\begin{aligned}\bar{F}_L(t) &= \boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-1}\boldsymbol{T^0} + p_1\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0} \\ &\quad + s_1p_2\Lambda(t)\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-2}\boldsymbol{T^0} + s_1p_1^2\Lambda(t)^2\boldsymbol{\pi}(\Lambda(t)(1-p_0)I - T)^{-3}\boldsymbol{T^0}.\end{aligned}$$

In Figure 8.4, we plot this against $t \in [0, 1000]$ for fixed parameters values given by

$$\boldsymbol{\pi} = (0.2, 0.8),\ T = \begin{pmatrix} -2 & 1 \\ 0.5 & -10 \end{pmatrix}, \lambda = 1,\ p_0 = 0.4,\ p_1 = 0.3 \text{ and } p_2 = 0.2.$$



(a) General Model

(b) Model M_1

(c) Model M_2

(d) Model M_3

Figure 8.3: Plot of the survival function of a 3-out-of-4 system against time, for different models.

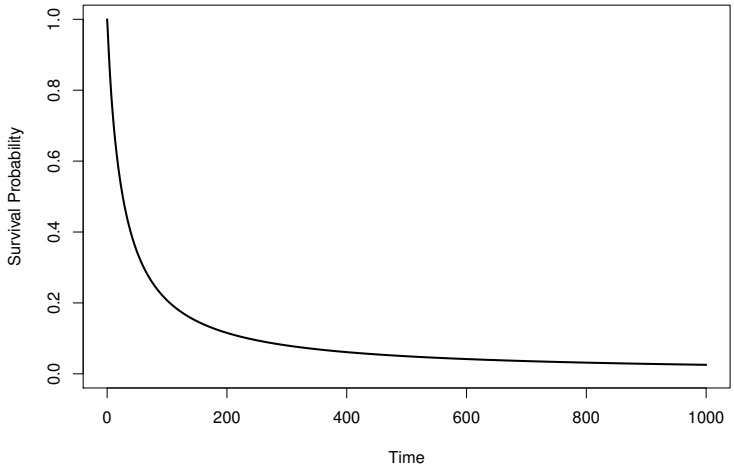


Figure 8.4: Plot of $\bar{F}_L(t)$ against $t \in [0, 1000]$.

8.4.3 RPMML

We begin this subsection with the following lemma. Here we derive the pmf of the random variable $N(t)$ following RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$.

Lemma 8.4.1 *Let $\{N(t) : t \geq 0\}$ be the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$. Then the pmf of $N(t)$ is given by*

$$P(N(t) = n) = \boldsymbol{\pi_{n+1}} E_{\gamma,1}(T_{n+1} t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_n} E_{\gamma,1}(T_n t^{\gamma})\boldsymbol{e}, \quad t > 0,\ n = 0, 1, 2, \ldots,$$

where $\boldsymbol{\pi_n} = (\boldsymbol{\pi},\ 0, \ldots, 0)$ and

$$T_n = \begin{pmatrix} T & -T\boldsymbol{e}\boldsymbol{\pi_{n-1}} \\ 0 & T_{n-1} \end{pmatrix},$$

for $n \geq 2$, with $\boldsymbol{\pi_1} = \boldsymbol{\pi}$, $T_1 = T$ and $\boldsymbol{\pi_0} = T_0 = 0$.

Proof: Let $S_1, S_2, \ldots$ be a sequence of arrival times. Then

$$P(N(t) = n) = P(S_{n+1} > t) - P(S_n > t). \tag{8.4.8}$$

Note that $S_n = X_1 + X_2 + \cdots + X_n \sim MML(\gamma, \boldsymbol{\pi_n}, T_n)$, where X_i's are inter-arrival times. Then, from Lemma 1.2.1(ii), we have $P(S_n > t) = \boldsymbol{\pi_n} E_{\gamma,1}(T_n t^{\gamma})\boldsymbol{e}$. Consequently, the result follows from Eq. (8.4.8). □

In the following theorem we derive the survival function of a k-out-of-n system for the defined model by assuming the shock process as the RPMML. The proof immediately follows from Theorem 8.3.1 and Lemma 8.4.1 and is therefore omitted.

Theorem 8.4.7 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$. Then*

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty} p_0^j(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}) \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right) p_0^{j-m}(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}),\end{aligned}$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $1 \le m \le i \le n-k$, and $\boldsymbol{\pi_j}$'s and T_j's are the same as in Lemma 8.4.1. □

The next corollary immediately follows from Theorem 8.4.7.

Corollary 8.4.6 *The following results hold.*

(i) For model M_1:

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \sum_{j=0}^{\infty}(1/(n+1))^j\,(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}) \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty}\binom{j}{m}\binom{i-1}{m-1}(1/(n+1))^j(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e});\end{aligned}$$

(ii) For model M_2:

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} \\ &\quad + \sum_{i=1}^{n-k}\sum_{m=1}^{i}\left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right)(\boldsymbol{\pi_{m+1}} E_{\gamma,1}(T_{m+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_m} E_{\gamma,1}(T_m t^{\gamma})\boldsymbol{e});\end{aligned}$$

(iii) For model M_3:

$$\begin{aligned}\bar{F}_{L_{k:n}}(t) &= \sum_{j=0}^{\infty} p_0^j(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}) \\ &\quad + \sum_{i=1}^{n-k}\sum_{j=i}^{\infty}\binom{j}{i} p_1^i p_0^{j-i}(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e}).\end{aligned}$$ □

The survival function of a coherent system for the defined model is given in the following theorem. The proof is obvious and therefore, omitted.

Theorem 8.4.8 *Let $P(Z_i = j) = p_j$, for all $j = 0, 1, 2, \ldots, n$, and $i \in \mathbb{N}$. Assume that shocks occur according to the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$. Then*

$$\bar{F}_L(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^{\gamma})\boldsymbol{e} + \sum_{j=1}^{\infty} p_0^j(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^{\gamma})\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^{\gamma})\boldsymbol{e})$$

$$+\sum_{k=1}^{n-1}\sum_{i=1}^{n-k}\sum_{m=1}^{i}\sum_{j=m}^{\infty} s_k \binom{j}{m} \left(\sum_{\boldsymbol{z}\in\Omega_{m,i}}\prod_{r=1}^{m} p_{z_r}\right) p_0^{j-m}(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}),$$

where $\Omega_{m,i} = \{\boldsymbol{z} = (z_1, z_2, \ldots, z_m) | z_1 + z_2 + \cdots + z_m = i, 1 \le z_l \le n, 1 \le l \le m\}$, $s_k = P(L = L_{k:n})$, $1 \le k \le n-1$, $1 \le m \le i \le n-k$, *and* $\boldsymbol{\pi_j}$*'s and* T_j*'s are the same as in Lemma 8.4.1.*

Remark 8.4.2 *Since* $\int_0^\infty \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e}dt = \infty$ *for* $\gamma < 1$*, one can conclude from Theorem 8.4.8 that* $E(L) = \infty$*, for* $0 < \gamma < 1$. □

In the next two examples, we illustrate the foregoing results.

Example 8.4.5 *Consider a 3-out-of-4 system. Then, from Theorem 8.4.7, we have*

$$\bar{F}_{L_{3:4}}(t) = \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty}(p_0^j + jp_0^{j-1}p_1)(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}).$$

Let the parameters be $\boldsymbol{\pi_1} = 1$, $T = -\lambda$ *and* $\lambda = 2$*, i.e., the shock process be the FHPP with the set of parameters* $\{\gamma, 2\}$*. In Figure 8.5a, we plot the survival function of the system (given in Theorem 8.4.7) against* $t \in [0, 10]$*, for different values of* p_0*, and for fixed* $\gamma = 0.9$ *and* $p_1 = 0.2$*. This figure show that the system survivability increases as* p_0 *increases. In Figure 8.5b, we plot the survival function of the system against* $t \in [0, 10]$ *for the same model, for different values of* γ*, and for fixed* $p_0 = 0.5$ *and* $p_1 = 0.1$*.*

Example 8.4.6 *Consider the coherent system given in Example 8.4.2. Then, from Theorem 8.4.8, we have*

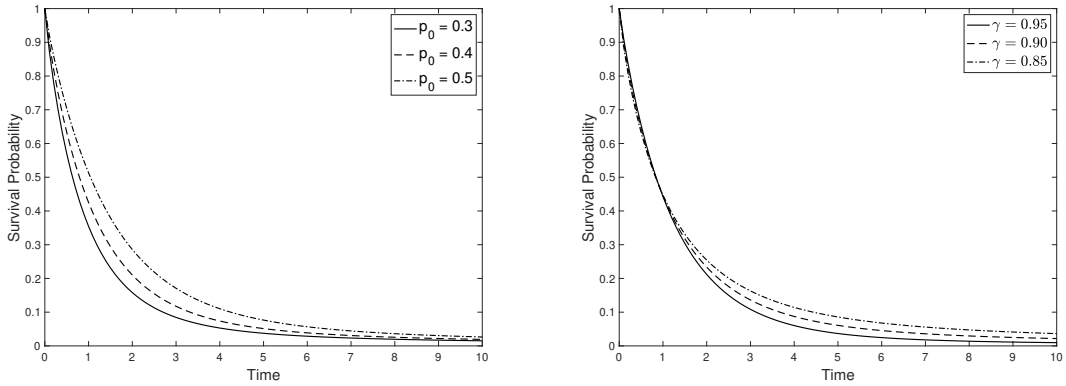
$$\begin{aligned}\bar{F}_L(t) &= \boldsymbol{\pi} E_{\gamma,1}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1 + s_1 p_2) j p_0^{j-1}(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}) \\ &\quad + s_1 \sum_{j=2}^{\infty} p_1^2 \binom{j}{2} p_0^{j-2}(\boldsymbol{\pi_{j+1}} E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j} E_{\gamma,1}(T_j t^\gamma)\boldsymbol{e}).\end{aligned}$$

Let the shock process be the FHPP with the set of parameters $\{\gamma, \lambda\}$*, where* $\gamma = 0.9$ *and* $\lambda = 2$*. In Figure 8.6, we plot the survival function of the coherent system against time, for fixed* $p_1 = 0.4$*,* $p_2 = 0.3$ *and* $p_3 = 0.2$*.*

8.5 Application: Optimal Replacement Policy

The study of the optimal replacement policies is one of the important applications of reliability theory. In this section, we consider the problem of finding the optimal replacement time for a coherent system that minimizes the long-run average cost per unit time. According to the classical age replacement policy, a system should be replaced upon failure or reaching the predetermined age t, whichever occurs first.

Below we give a list of assumptions which are similar to those given in Eryilmaz and Devrim [2019].



(a) Plot of system's survival function against time, for different values of p_0.

(b) Plot of system's survival function against time, for different values of γ.

Figure 8.5: Plot of the survival function of a 3-out-of-4 system against time.

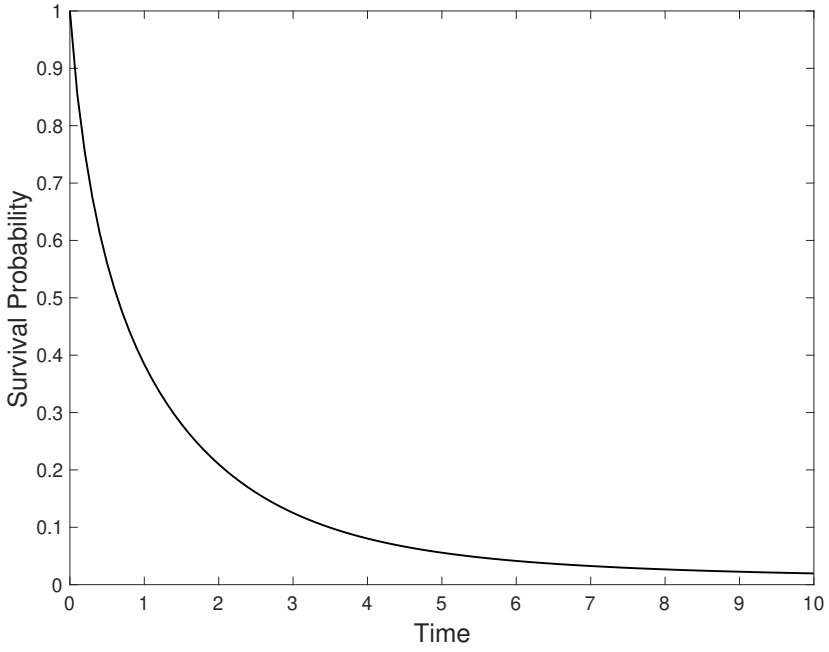


Figure 8.6: Plot of $\bar{F}_L(t)$ against $t \in [0, 10]$.

Assumptions:

(a) A new coherent system with the lifetime L described by the defined shock model is incepted into operation at time zero.

(b) The external shocks are the only reason for the system's failure, and occur according to the RPMML with the set of parameters $\{\gamma, \boldsymbol{\pi}, T\}$

(c) The system is replaced by a new one either upon failure or after reaching the predetermined age t, whichever occurs first.

(d) Let c_1 and c_2 denote the cost of replacement of the system before failure and the cost of replacement of the system after failure, respectively.

(e) If the system is replaced before failure, then all failed components are replaced by new components and all non-failed components are repaired.

(f) A component is as good as new after repair and hence, the system after replacement becomes new.

(g) Let c_a denote the acquisition cost of one component and c_r be the repair cost for a non-failed component. Further, let c_f denote the additional cost of the system's failure.

Note that the expected number of components to be replaced before failure is equal to $E(M(t)|L > t)$, and the expected number of components to be repaired before failure is equal to $E(n - M(t)|L > t)$. Then

$$c_1 = c_a E(M(t)|L > t) + c_r E(n - M(t)|L > t) = nc_r + (c_a - c_r)E(M(t)|L > t).$$

On the other hand, the cost of replacement of the system after failure is given by $c_2 = nc_a + c_f$. Consequently, the total long-run average cost per unit time is given by

$$\begin{aligned} C_L(t) &= \frac{[nc_r + (c_a - c_r)E(M(t)|L > t)]\bar{F}_L(t) + (nc_a + c_f)F_L(t)}{E(min(L, t))} \\ &= \frac{nc_a + c_f + [nc_r + (c_a - c_r)E(M(t)|L > t) - nc_a - c_f]\bar{F}_L(t)}{\int_0^t \bar{F}_L(u)du}. \end{aligned} \tag{8.5.1}$$

Note that the denominator of the above equation tends to ∞ as $t \to \infty$ (see Remark 8.4.2) and hence, the finite optimal replacement time cannot be determined on the infinite horizon. In real life, due to specifications and internal degradation process, an item might be replaced at some large time t^u (say) even if it is not an optimal decision. Thus, we study the optimal replacement policy under the updated Assumption (c) as given below.

Updated Assumption (c)**:** The system is replaced by a new one either upon its failure or after reaching its age to a predetermined threshold value $t \leq t^u$, whichever occurs first.

Consequently, our goal is to find the optimal t^* ($\leq t^u$) such that $C_L(t^*) < C_L(t^u)$ and

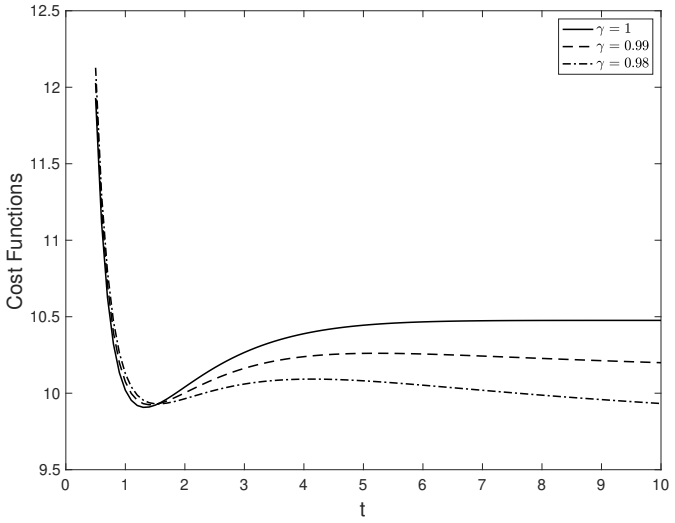


Figure 8.7: Plot of $C_L(t)$ against time

$C_L(t^*) = min_{t\in(0,t^u]}C_L(t)$. Now, to evaluate $C_L(t)$, we need the expressions of $E(M(t)|L > t)$ and $\bar{F}_L(t)$. The first expression is given by

$$E(M(t)|L > t) = \frac{1}{\bar{F}_L(t)} \sum_{k=1}^{n-1} s_k \sum_{i=1}^{n-k} iP(M(t) = i),$$

where the pmf of $M(t)$ is the same as in Lemma 8.3.1. Further, the expression of $\bar{F}_L(t)$ is obtained from Theorem 8.4.8.

Due to mathematical complexity, the aforementioned problem cannot be analytically solved. Thus, we consider a numerical example as given below.

Example 8.5.1 *Consider the coherent system mentioned in Example 8.4.2. Assume that shocks occur according to the RPMML with the parameter set* $\{\gamma, \boldsymbol{\pi}, T\}$, *where* $\boldsymbol{\pi} = (1, 0)$, $T = \begin{pmatrix} -\lambda & \lambda \\ 0 & -\lambda \end{pmatrix}$ *and* $\lambda = 1.2$. *Then*

$$E(M(t)|L > t) = \frac{1}{\bar{F}_L(t)}(P(M(t) = 1) + 2s_1P(M(t) = 2)), \tag{8.5.2}$$

where $P(M(t) = 1)$ *and* $P(M(t) = 2)$ *are the same as in Lemma 8.3.1, and*

$$\begin{aligned}\bar{F}_L(t) &= \boldsymbol{\pi}E_{\gamma,1}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1 + s_1p_2)jp_0^{j-1}(\boldsymbol{\pi_{j+1}}E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j}E_{\gamma,1}(T_jt^\gamma)\boldsymbol{e}) \\ &\quad + s_1\sum_{j=2}^{\infty}p_1^2\binom{j}{2}p_0^{j-2}(\boldsymbol{\pi_{j+1}}E_{\gamma,1}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j}E_{\gamma,1}(T_jt^\gamma)\boldsymbol{e}).\end{aligned} \tag{8.5.3}$$

Note that $\int_0^t \boldsymbol{\pi} E_{\gamma,1}(Tu^\gamma)\boldsymbol{e}du = t\boldsymbol{\pi} E_{\gamma,2}(Tt^\gamma)\boldsymbol{e}$. *Consequently,*

$$\begin{aligned}\int_0^t \bar{F}_L(u)du &= t\Bigg[\boldsymbol{\pi} E_{\gamma,2}(Tt^\gamma)\boldsymbol{e} + \sum_{j=1}^{\infty}(p_1+s_1p_2)jp_0^{j-1}(\boldsymbol{\pi_{j+1}}E_{\gamma,2}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j}E_{\gamma,2}(T_jt^\gamma)\boldsymbol{e}) \\ &\quad + s_1\sum_{j=2}^{\infty}p_1^2\binom{j}{2}p_0^{j-2}(\boldsymbol{\pi_{j+1}}E_{\gamma,2}(T_{j+1}t^\gamma)\boldsymbol{e} - \boldsymbol{\pi_j}E_{\gamma,2}(T_jt^\gamma)\boldsymbol{e})\Bigg]. \qquad (8.5.4)\end{aligned}$$

By using Eq. (8.5.2), *Eq.* (8.5.3) *and Eq.* (8.5.4) *in Eq.* (8.5.1), *we get the expression of* $C_L(t)$. *Now, in Figure 8.7, we plot this* $C_L(t)$ *against* $t \in [0, 10]$, *for different values of* γ, *and for fixed* $c_a = 3$, $c_f = 10$, $c_r = 1$, $p_0 = 0$, $p_1 = 0.2$ *and* $p_2 = 0.1$. *From this figure, we see that the the cost function* $C_L(t)$ *is in U-shaped in initial time period. Let* $t^u = 10$. *Then, for* $\gamma = 1$, $t^* = 1.3$ *and* $C_L(t^*) = 9.9075$; *for* $\gamma = 0.99$, $t^* = 1.4$ *and* $C_L(t^*) = 9.9243$, *and, for* $\gamma = 0.98$, $t^* = 1.5$ *and* $C_L(t^*) = 9.9308$. *Moreover,* $C_L(t)$ *tends to* 0 *when* $t \to \infty$ *because* $E(L) = \infty$ *for* $\gamma < 1$.