

2D Homographic Ricker Map

4.1 INTRODUCTION

We apply the Holling type - II functional response as a growth function in the classical two dimensional Ricker map and propose a discrete-time competition model, namely the two-dimensional (2D) Homographic Ricker map. We investigate the non-linear behavior of the proposed model. This includes the boundedness of the solutions, existence, and uniqueness of the coexisting fixed point. We discuss the geometry of the 2D Homographic Ricker map with the help of critical curves and singular points. Further, with the help of the centre manifold theorem and bifurcation theory, we obtain the specific conditions of the existence of the flip bifurcations in the 2D Homographic Ricker map. Finally, we employ numerical simulations to illustrate the stability and the bifurcation analysis of the competing species.

The chapter is organized as follows: In Section 4.2, we discuss the construction of the 2D Homographic Ricker map and show the boundedness of the solutions. Section 4.3 contains the geometry of the 2D Homographic Ricker map with the help of critical curves and singular points. Section 4.4 discusses the existence and uniqueness of the coexisting fixed point and the local asymptotic stability of the trivial, exclusion, and coexisting fixed points of the 2D Homographic Ricker map. Section 4.5 explains the various conditions under which the map undergoes the flip bifurcation. Further, we discuss the stability of the various fixed points of the considered map with the help of their fold and flip bifurcation curves using the concept of contour lines. Section 4.6 describes the stability of the positive coexisting fixed point and the bifurcation analysis of the x and y species using numerical simulations.

4.2 THE 2D HOMOGRAPHIC RICKER COMPETITION MODEL

A discrete-time classical two-dimensional (2D in short) competition model after n generation, with $n \in \mathbb{N}$ is given as

$$\begin{aligned} x_{n+1} &= p(x_n)x_n^{\gamma-1}S_1(x_n, y_n), \\ y_{n+1} &= p(y_n)y_n^{\gamma-1}S_2(x_n, y_n), \end{aligned} \tag{4.1}$$

where $p(x_n)$ and $p(y_n)$ are the per capita growth rates of the species x and y respectively and γ is the cooperation or Allee's effect parameter. Here, $S_1(x_n, y_n)$ is the survival function of x species with respect to y species, and $S_2(x_n, y_n)$ is the survival function of y species with respect to x species at n^{th} generation. The mathematical expressions of the $S_1(x_n, y_n)$ and $S_2(x_n, y_n)$ are given as follows

$$\begin{aligned} S_1(x_n, y_n) &= e^{r-x_n-ay_n}, \\ S_2(x_n, y_n) &= e^{s-y_n-bx_n}. \end{aligned}$$

The parameters r, s, a, b are assumed to be positive real constants. In terms of population dynamics, the parameters r and s represent the environment's carrying capacity for species x and y , respectively,

while a and b are the competition parameters. We introduce Holling type II functional response as a per capita growth rate of both x and y species in the system (4.1), and the resulting map is given as

$$\begin{aligned} x_{n+1} &= \left(\frac{x_n^\gamma}{x_n + \beta} \right) e^{r - x_n - ay_n}, \\ y_{n+1} &= \left(\frac{y_n^\gamma}{y_n + \beta} \right) e^{s - y_n - bx_n}. \end{aligned} \quad (4.2)$$

The parameter β represents the half-saturation constant of the Holling type II functional response as shown in Figure 4.1. The existence of a Holling type II functional response and six real parameters make the dynamical analysis of the system (4.2) extremely complicated. Because of this, we limit our investigation to the specific scenario where $\gamma = 1$. Throughout this work, the model obtained from Eq.(4.2) with $\gamma = 1$ is designated by 2D Homographic Ricker map (in short 2D Homographic map) which is defined as

$$\begin{aligned} x_{n+1} &= \left(\frac{x_n}{x_n + \beta} \right) e^{r - x_n - ay_n}, \\ y_{n+1} &= \left(\frac{y_n}{y_n + \beta} \right) e^{s - y_n - bx_n}. \end{aligned} \quad (4.3)$$

We denote the solution of the system (4.3) by (x_n, y_n) , $n = 0, 1, 2, 3, \dots$. In population dynamics, we

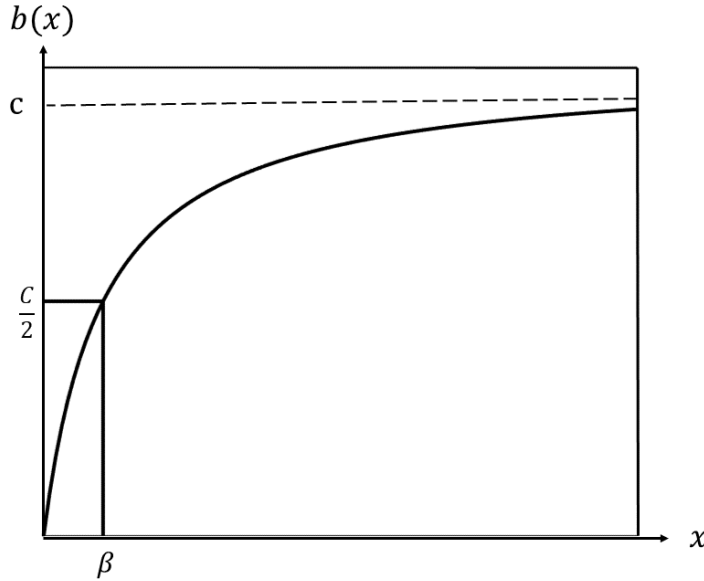


Figure 4.1 : Holling type II functional response.

are interested only in the solution having the non-negative components, i.e., $x_n \geq 0$, $y_n \geq 0$. The Eq.(4.3) can be represented by the map

$$F(x, y) = (g_1(x, y), g_2(x, y)), \quad (4.4)$$

where $g_1(x, y) = \frac{x}{x+\beta} e^{r-x-ay}$ and $g_2(x, y) = \frac{y}{y+\beta} e^{s-y-bx}$. Let $\mathbb{R}_+^2 = [0, +\infty) \times [0, +\infty)$ and $\mathbb{R}_+^2 \setminus \{(0, 0)\} = (0, +\infty) \times (0, +\infty)$, then the map F takes the $\mathbb{R}_+^2$ into itself.

Lemma 4.2.1. *Let $F(x, y)$ be the 2D Homographic map then the following statements hold true:*

(i) The x -axis and y -axis are invariant sets.

(ii) $F(x, y) \rightarrow (0, 0)$, whenever $\|(x, y)\| \rightarrow \infty$.

Proof. (i) Let x -axis and y -axis are denoted as $\hat{X} = \{(x, 0) : x \geq 0\}$ and $\hat{Y} = \{(0, y) : y \geq 0\}$ respectively.

Let $q = (x, 0) \in \hat{X}$ then we have $F(q) = \left(\frac{x}{x+\beta}e^{r-x}, 0\right) \in \hat{X}$. Also, if $F(q) \in \hat{X}$ then $y = 0$. Therefore, the x -axis is invariant under F . Similarly, the y -axis is also invariant.

(ii) Assume that $\|(x, y)\| \rightarrow \infty$ then we have either $x \rightarrow \infty$ or $y \rightarrow \infty$. Suppose $x \rightarrow \infty$ then

$$\lim_{\|(x, y)\| \rightarrow \infty} F(x, y) = \lim_{\|(x, y)\| \rightarrow \infty} \left(\frac{x}{x+\beta}e^{r-x-ay}, \frac{y}{y+\beta}e^{s-y-bx} \right) = (0, 0).$$

□

Lemma 4.2.2. *Let $F(x, y)$ be the 2D Homographic map. Then the image of $\mathbb{R}_+^2$ under $F(x, y)$, i.e., $F(\mathbb{R}_+^2)$ is compact.*

Proof. Let $\tilde{\mathbb{R}}_+^2 = \mathbb{R}_+^2 \cup \{\infty\}$ be the one point compactification of $\mathbb{R}_+^2$. Let $\tilde{F} : \tilde{\mathbb{R}}_+^2 \rightarrow \tilde{\mathbb{R}}_+^2$ be the map given as

$$\tilde{F}(x, y) = \begin{cases} F(x, y) & (x, y) \in \mathbb{R}_+^2, \\ (0, 0) & (x, y) = \infty. \end{cases}$$

Since,

$$\lim_{\|(x, y)\| \rightarrow \infty} F(x, y) = (0, 0),$$

hence, the map $\tilde{F}(x, y)$ is continuous. We know that the continuous image of a compact set is compact; therefore, $\tilde{F}(\tilde{\mathbb{R}}_+^2)$ is compact.

For $(x, y) = \infty$, $\tilde{F}(\infty) = (0, 0)$ and $(0, 0) \in F(\mathbb{R}_+^2)$ which implies $\tilde{F}(\tilde{\mathbb{R}}_+^2) \subseteq F(\mathbb{R}_+^2)$. Hence, $\tilde{F}(\tilde{\mathbb{R}}_+^2) = F(\mathbb{R}_+^2)$. □

Remark 4.2.3. *We recall that a map is said to be a proper map if the inverse image of a compact set is compact. Hence, the continuous extension $\tilde{F}(x, y)$ is a proper map.*

Let $x_0 > 0$, $y_0 > 0$ be the initial condition. The solutions (x_n, y_n) of the 2D Homographic map with positive initial condition (x_0, y_0) are always positive. Moreover, all the solutions in $\mathbb{R}_+^2$ are forward bounded. The specific bounds of the 2D Homographic map are obtained in the below-mentioned proposition.

Proposition 4.2.4. *Every positive solution $\{(x_n, y_n)\}$ of the 2D Homographic map $F(x, y)$ is uniformly bounded.*

Proof. Assume that $\{(x_n, y_n)\}$ be any positive solution of system (4.3). The x coordinate of the map $F(x_n, y_n)$ follows:

$$x_{n+1} \leq \frac{x_n}{x_n + \beta} e^{r-x_n}, \quad \text{for all } n = 0, 1, 2, \dots \quad (4.5)$$

Assume that $x_0 > 0$, then we have

$$\frac{x_n}{x_n + \beta} \leq \frac{x_n}{\beta} \quad \text{for all } n = 0, 1, 2, \dots$$

Therefore, the inequality (4.5) is written as

$$x_{n+1} \leq \frac{x_n}{\beta} e^{r-x_n} \quad \text{for all } n = 0, 1, 2, \dots$$

We know that

$$\max_{x \in \mathbb{R}} x e^{r-x} = e^{r-1}, \quad r > 0,$$

hence

$$\limsup_{n \rightarrow \infty} x_n \leq M_1,$$

where $M_1 = \frac{e^{r-1}}{\beta}$.

Similarly, from the y coordinate of the map $F(x_n, y_n)$ with the assumption $y_0 > 0$, we obtain

$$y_{n+1} \leq \frac{y_n}{y_n + \beta} e^{s-y_n}, \quad \text{for all } n = 0, 1, 2, \dots$$

Since

$$\max_{y \in \mathbb{R}} y e^{s-y} = e^{s-1}, \quad s > 0,$$

hence,

$$\limsup_{n \rightarrow \infty} y_n \leq M_2,$$

where $M_2 = \frac{e^{s-1}}{\beta}$. Take $M = \max\{M_1, M_2\}$ and it follows that

$$\limsup_{n \rightarrow \infty} (x_n, y_n) \leq M.$$

Hence, we conclude that the solutions of the system (4.3) are bounded. $\square$

4.3 GEOMETRY OF 2D HOMOGRAPHIC MAP

In this section, we first recall some definitions from the work of Whitney [Whitney, 1955] on the singularities of the Euclidean plane. Let f be a continuously differentiable map defined on the open set $U \subset \mathbb{R}^2$ and V be any vector in $\mathbb{R}^2$. For any point $p \in U$ and for any small real number t , the $p + tV \in U$, and $f(p + tV) - f(p)$ is a vector in $\mathbb{R}^2$. The derivative of f with respect to V at a point p is given as

$$\nabla_V f(p) = \lim_{t \rightarrow 0} \frac{1}{t} (f(p + tV) - f(p)). \quad (4.6)$$

The mapping $\nabla f(p)$ of vectors in $\mathbb{R}^2$ into vectors in $\mathbb{R}^2$ is linear. If V_1 and V_2 are the components of vector V and we use the coordinate (x, y) and (u, v) around p and $f(p)$ respectively, then we have

$$\nabla_V f(p) = V_1 f_x(p) + V_2 f_y(p),$$

and

$$\nabla_V \nabla_V f(p) = V_1 (V_1 f_x(p) + V_2 f_y(p))_x + V_2 (V_1 f_x(p) + V_2 f_y(p))_y.$$

Let f be a continuously differential map defined on the open set $U \subset \mathbb{R}^2$ into $\mathbb{R}^2$. The map f is said to be *regular* at a point p if

$$\nabla_V f(p) \neq 0, \text{ whenever } V \neq 0.$$

Otherwise, f is *singular* at p . The above definition of regular and singular points can also be given in terms of the determinant of the Jacobian of f , i.e., $\det(Jf)$. The map f is said to be regular at p if $\det(Jf(p)) \neq 0$, and singular if $\det(Jf(p)) = 0$. The smooth curve formed by the singular points of f is called the critical curve denoted by LC_{-1} . We note that Mira [Mira, 1996] defined the critical curve of a two-dimensional continuous map f as the collection of points for which the determinant of Jacobian of f vanishes or for which the map f is not differentiable. Further, we discuss the theory of singularities of singular points of the map f . Let $\phi(t)$ be a C^2 parametrization of the critical curve LC_{-1} through the point p with the property $\phi(0) = p$. The point p is a *fold point* of f if

$$\frac{d}{dt}(f \circ \phi) \neq 0 \quad (4.7)$$

and p is a *cusp point* of f if

$$\frac{d}{dt}(f \circ \phi) = 0, \text{ and } \frac{d^2}{dt^2}(f \circ \phi) \neq 0. \quad (4.8)$$

Remark 4.3.1. Let $f : U \rightarrow \mathbb{R}^2$ be a continuously differentiable map. If $p \in U$ is a fold point; (x, y) and (u, v) are the coordinates around p and $f(p)$ respectively. Then, the normal form of the fold point is given as

$$u = x, \quad v = y^2.$$

Similarly, if p is a cusp point, then the normal form of the cusp point is

$$u = x \text{ and } v = y^3 - xy.$$

Let us recall the equation of the 2D Homographic map $F(x, y)$, which is given as follows

$$F(x, y) = \left(\frac{x}{x+\beta} e^{r-x-ay}, \frac{y}{y+\beta} e^{s-y-bx} \right),$$

where $r, s, a, b, \beta > 0$. The Jacobian $JF(x, y)$ and its determinant $\det(JF(x, y))$ of the 2D Homographic map $F(x, y)$ are as follows:

$$JF(x, y) = \begin{pmatrix} \left(\frac{(\beta-x(x+\beta))}{(x+\beta)^2} \right) e^{r-x-ay} & \left(\frac{-ax}{x+\beta} \right) e^{r-x-ay} \\ \left(\frac{-by}{y+\beta} \right) e^{s-y-bx} & \left(\frac{(\beta-y(y+\beta))}{(y+\beta)^2} \right) e^{s-y-bx} \end{pmatrix}, \quad (4.9)$$

and

$$\det(JF(x, y)) = \frac{(\beta^2 - \beta x(x+\beta) - \beta y(y+\beta) + (1-ab)xy(x+\beta)(y+\beta))e^{(r+s-(1+b)x-(1+a)y)}}{(x+\beta)^2(y+\beta)^2}. \quad (4.10)$$

The $\det(JF(x, y)) = 0$ if and only if

$$y = \frac{-\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta x(x+\beta))}{\beta - (1-ab)x(x+\beta)}}}{2}, \quad x \neq \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2}.$$

The critical curve LC_{-1} of a 2D Homographic map is given as follows:

$$LC_{-1} = \left\{ (x, y) \in \mathbb{R}_+^2 : y = \frac{-\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta x(x+\beta))}{\beta - (1-ab)x(x+\beta)}}}{2}, x \neq \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\}. \quad (4.11)$$

Whenever $ab < 1$, the critical curve LC_{-1} have the two branches LC_{-1}^1 and LC_{-1}^2 , which divided the plane $\mathbb{R}_+^2$ into three regions R_1 , R_2 and R_3 as shown in Figure 4.2, where

$$R_1 = \left\{ (x, y) \in \mathbb{R}_+^2 : y \leq \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta(\beta - x(x+\beta))}{\beta - (1-ab)x(x+\beta)}}}{2} \text{ and } x < \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\},$$

$$R_3 = \left\{ (x, y) \in \mathbb{R}_+^2 : y \geq \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta(\beta - x(x+\beta))}{\beta - (1-ab)x(x+\beta)}}}{2} \text{ and } x > \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\},$$

$$R_2 = \{\mathbb{R}_+^2 \setminus (R_1 \cup R_3)\}.$$

The critical curve LC of the 2D Homographic map $F(x, y)$ is defined as the image of the critical curve LC_{-1} , i.e., $LC = F(LC_{-1})$. Corresponding to the two branches LC_{-1}^1 and LC_{-1}^2 of the LC_{-1} curve, we have two branches LC^1 and LC^2 of the LC curve. The critical curve LC of the map $F(x, y)$ is shown in Figure 4.3. It shows the geometric information of the image of $F(x, y)$ under $\mathbb{R}_+^2$.

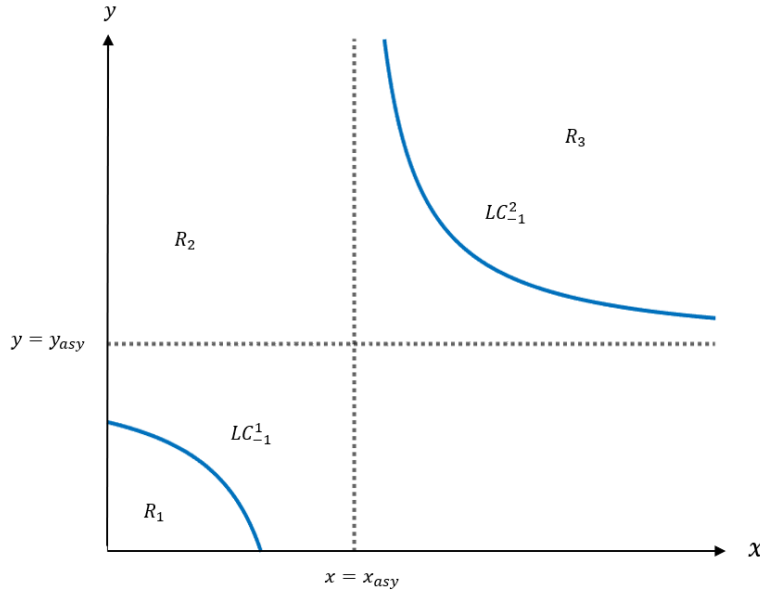


Figure 4.2 : Branches LC_{-1}^1 and LC_{-1}^2 of the critical curve LC_{-1} of the map $F(x, y)$, where

$$x_{asy} = \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \text{ and } y_{asy} = \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2}.$$

Lemma 4.3.2. *For the 2D Homographic map $F(x, y)$, the following statements hold true:*

- (i) *The map $F(x, y)$ is orientation preserving in the region R_1 .*
- (ii) *The map $F(x, y)$ is orientation reversing in the region R_2 .*

(iii) The map $F(x, y)$ is orientation preserving in the region R_3 .

Proof. A map $F(x, y)$ is said to be orientation preserving (or reversing) if its determinant of Jacobian is positive (or negative).

(i) Let $(x, y) \in R_1 \setminus LC_{-1}^1$, then we have

$$y(y + \beta) < \frac{\beta(\beta - x(x + \beta))}{\beta - (1 - ab)x(x + \beta)}, \quad (4.12)$$

and

$$x < \frac{-\beta + \sqrt{\beta^2 + 4\beta}}{2}.$$

Since $0 < ab < 1$, therefore

$$0 < (1 - ab)x(x + \beta) < \beta. \quad (4.13)$$

From Eq. (4.12) and Eq. (4.13), it follows that

$$0 < \beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta), \quad (4.14)$$

The $e^{(r+s-(1+b)x-(1+a)y)}$ and $(x + \beta)^2(y + \beta)^2$ are the positive quantities, hence the sign of $\det(JF(x, y))$ is determined by $J = \beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta)$. From the inequality in Eq.(4.14), we have $\det(JF(x, y)) > 0$. This concludes that the map $F(x, y)$ is orientation preserving in R_1 .

(ii) Let us define the three sets $R_{2,1}$, $R_{2,2}$ and $R_{2,3}$ as follows:

$$\begin{aligned} R_{2,1} &= \left\{ (x, y) \in \mathbb{R}_+^2 : y(y + \beta) > \frac{\beta(\beta - x(x + \beta))}{\beta - (1 - ab)x(x + \beta)} \text{ and } x < \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\}, \\ R_{2,2} &= \left\{ (x, y) \in \mathbb{R}_+^2 : y \geq 0 \text{ and } x = \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\}, \\ R_{2,3} &= \left\{ (x, y) \in \mathbb{R}_+^2 : y(y + \beta) < \frac{\beta(\beta - x(x + \beta))}{\beta - (1 - ab)x(x + \beta)} \text{ and } x > \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{1-ab}}}{2} \right\}. \end{aligned}$$

Then the region $R_2 = \cup_{i=1}^3 R_{2,i} \cup LC_{-1}$.

(a) Let $(x, y) \in R_{2,1}$. Since

$$\beta y(y + \beta) - (1 - ab)xy(x + \beta)(y + \beta) > \beta(\beta - x(x + \beta)),$$

then from the $\det(JF(x, y))$, we have

$$\begin{aligned} J &= \beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta) \\ &< \beta^2 - \beta x(x + \beta) - \beta^2 + \beta x(x + \beta) \\ &< 0. \end{aligned}$$

Since $J < 0$, therefore $\det(JF(x, y)) < 0$.

(b) Let $(x, y) \in R_{2,2}$. In this case, the Jacobian is given as

$$\begin{aligned} J &= \beta^2 - \beta x(x + \beta) - y(y + \beta)(\beta - (1 - ab)x(x + \beta)) \\ &= \beta^2 - \beta \left(\frac{\beta}{1 - ab} \right) \\ &= \beta^2 \left(\frac{-ab}{1 - ab} \right) < 0, \end{aligned}$$

therefore $\det(JF(x, y)) < 0$.

(c) Let $(x, y) \in R_{2,3}$. Since

$$y(y + \beta) < \frac{\beta(\beta - x(x + \beta))}{\beta - (1 - ab)xy(x + \beta)},$$

in the region $R_{2,3}$, therefore $\det(JF(x, y)) < 0$.

(iii) This case can be solved similarly to the other cases.

□

Proposition 4.3.3. *Let $F(x, y)$ be the 2D Homographic map defined by Eq.(4.3), then for all $p = (x, y) \in \mathbb{R}_+^2$ either $\det(JF(p)) \neq 0$ or $\nabla \det(JF(p)) \neq 0$, whenever $ab \neq 0$.*

Proof. Let $p = (x, y) \in \mathbb{R}^2$, then there are two possibilities that either p is a regular point, or p is a singular point. If p is a regular point, then from the definition of a regular point, we have $\det(JF(p)) \neq 0$. If p is a singular point, then from the definition of the singular point, we have $\det(JF(p)) = 0$. To show $\nabla \det(JF(p)) \neq 0$, it is sufficient to show that $(\det JF)_x(p) \neq 0$. Consider the Eq.(4.10) and partially differentiating w.r.t. x , we get

$$\begin{aligned} (\det(JF))_x(p) &= \frac{e^{(r+s-(1+b)x-(1+a)y)}}{(x+\beta)^3(y+\beta)^2} \{ [\beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta)] (x + \beta) \\ &\quad - (\beta + 2x)[\beta - (1 - ab)y(y + \beta)](x + \beta) - 2[\beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + \\ &\quad (1 - ab)xy(x + \beta)(y + \beta)] \}. \end{aligned} \quad (4.15)$$

Since, p is singular hence

$$(\beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta)) = 0. \quad (4.16)$$

Substituting Eq.(4.16) in Eq.(4.15), we get

$$(\det(JF))_x(p) = \frac{e^{(r+s-(1+b)x-(1+a)y)}}{(x+\beta)^3(y+\beta)^2} \{ (\beta + 2x)(-\beta + (1 - ab)y(y + \beta)) \}.$$

Since, $\frac{e^{(r+s-(1+b)x-(1+a)y)}}{(x+\beta)^3(y+\beta)^2}$, $(\beta + 2x)$ are the non-zero quantities, so we need to show that

$$y(y + \beta) \neq \frac{\beta}{(1 - ab)}.$$

On the contrary, suppose that $y(y + \beta) = \frac{\beta}{(1 - ab)}$ then from Eq.(4.16) we obtain

$$\beta^2 \left(\frac{ab}{1 - ab} \right) = 0.$$

The above condition is satisfied only if $ab = 0$, which is not true. Therefore $y(y + \beta) \neq \frac{\beta}{(1 - ab)}$. Hence, for the singular point p , $\nabla \det(JF(p)) \neq 0$. □

With the help of parametrization of the critical curve and the singular points, we show that all the points in the domain of the 2D Homographic Ricker map are either regular, fold, or cusp.

Theorem 4.3.4. *Let $F(x, y)$ be the 2D Homographic map then all the points in $\mathbb{R}_+^2$ are either regular, fold, or cusp.*

Proof. Let $(x, y) \in \mathbb{R}_+^2 \setminus \{LC_{-1}\}$. Since $\det(JF(x, y)) \neq 0$, therefore (x, y) is a regular point. Next, we need to check the nature of points on the critical curve LC_{-1} . Since we have $LC_{-1} = LC_{-1}^1 \cup LC_{-1}^2$. First, we examine the nature of points on LC_{-1}^1 .

- (i) Let the parametrization of the curve LC_{-1}^1 is given by the curve ϕ_1 , where $\phi_1 : [0, \beta] \rightarrow \mathbb{R}^2$ is defined as

$$\phi_1(t) = \left(\frac{-\beta + \sqrt{\beta^2 + 4t}}{2}, \frac{-\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}}}{2} \right).$$

The fold points satisfy the equation below

$$\frac{d}{dt}(F \circ \phi_1)(0) \neq 0.$$

Consider $(F \circ \phi_1)(t) = (\gamma_1(t), \gamma_2(t))$. To show that the points on the curve LC_{-1}^1 are fold points, the points should satisfies $\gamma_1'(t) \neq 0$ and $\gamma_2'(t) \neq 0$ for $t \in [0, \beta]$. A direct computation gives

$$(\gamma_1'(t), \gamma_2'(t)) = (\psi_1(t)h_1(t), \psi_2(t)h_1(t))$$

where

$$\begin{aligned} \psi_1(t) &= \frac{e \left(r - \left(\frac{-\beta + \sqrt{\beta^2 + 4t}}{2} \right) - a \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}}}{2} \right) \right)}{\sqrt{\beta^2 + 4t}(\beta - (1 - ab)t)^2(\beta + \sqrt{\beta^2 + 4t})^2 \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}}}, \\ \psi_2(t) &= \frac{-b\beta e \left(s - b \left(\frac{-\beta + \sqrt{\beta^2 + 4t}}{2} \right) - \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}}}{2} \right) \right)}{\sqrt{\beta^2 + 4t}(\beta - (1 - ab)t)^3 \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}} \left(\beta + \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}} \right)^2}, \end{aligned}$$

and,

$$h_1(t) = 4 \left((\beta - t)(\beta - (1 - ab)t)^2 \sqrt{\beta^2 + \frac{4(\beta^2 - \beta t)}{\beta - (1 - ab)t}} + a^2 \beta^2 b \sqrt{\beta^2 + 4t} \right).$$

Since ψ_1 and ψ_2 do not vanish in the interval $[0, \beta]$, thus if $\gamma_1'(t)$ and $\gamma_2'(t)$ are equal to zero, then $h_1(t)$ should be equal to zero. The $h_1(t) > 0$ for all $t \in [0, \beta]$. Hence $h_1(t)$ do not vanish in the interval $[0, \beta]$. Therefore, all the points on the curve LC_{-1}^1 are the fold points.

- (ii) Now, we check the nature of points on the curve LC_{-1}^2 . Consider a parametrization ϕ_2 of the curve LC_{-1}^2 where $\phi_2 : (0, 1) \rightarrow \mathbb{R}^2$ is defined as

$$\phi_2(t) = \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{(1 - ab)t}}}{2}, \frac{-\beta + \sqrt{\beta^2 + \frac{4\beta((1 - ab)t - 1)}{(1 - ab)(t - 1)}}}{2} \right).$$

Let $F \circ \phi_2(t) = (\gamma_3(t), \gamma_4(t)) = \gamma(t)$. We have

$$\lim_{t \rightarrow 0} (\gamma_3(t), \gamma_4(t)) = \lim_{t \rightarrow 1} (\gamma_3(t), \gamma_4(t)) = (0, 0).$$

Since the map $F(x, y)$ is bounded, $\gamma(t)$ always lies in the first quadrant. Also, the curve starts and ends at the origin, and this implies the curve changes its direction at least once. Therefore there exist at least one t_0 such that $(\gamma'_3(t_0), \gamma'_4(t_0)) = (0, 0)$. A simple computation gives

$$(\gamma'_3(t), \gamma'_4(t)) = (\psi_3(t)h_2(t), \psi_4(t)h_2(t))$$

where,

$$\begin{aligned} \psi_3(t) &= \frac{-4\beta^2 e \left(r - \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}}}{2} \right) - a \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4\beta((1-ab)t-1)}}{(1-ab)(t-1)}}{2} \right) \right)}{t^3(t-1)^2(1-ab)^2 \left(\beta + \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}} \right)^2 \sqrt{\left(\beta^2 + \frac{4\beta((1-ab)t-1)}{(1-ab)(t-1)} \right)} \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}}}, \\ \psi_4(t) &= \frac{4\beta^2 b e \left(s - b \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}}}{2} \right) - \left(\frac{-\beta + \sqrt{\beta^2 + \frac{4\beta((1-ab)t-1)}}{(1-ab)(t-1)}}{2} \right) \right)}{t^2(t-1)^3(1-ab)^2 \left(\beta + \sqrt{\beta^2 + \frac{4\beta((1-ab)t-1)}{(1-ab)(t-1)}} \right)^2 \sqrt{\left(\beta^2 + \frac{4\beta((1-ab)t-1)}{(1-ab)(t-1)} \right)} \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}}}, \end{aligned}$$

and

$$h_2(t) = (t-1)^2((1-ab)t-1) \sqrt{\beta^2 + \frac{4\beta((1-ab)t-1)}{(1-ab)(t-1)}} + a^2 b t^2 \sqrt{\beta^2 + \frac{4\beta}{(1-ab)t}}.$$

Since the terms $\psi_3(t)$ and $\psi_4(t)$ do not vanish in the interval $(0, 1)$ for all t , we see that $(\gamma'_3(t), \gamma'_4(t))$ equal to $(0, 0)$ only when $h_2(t)$ becomes zero.

Clearly, $h_2(0) = -\sqrt{\beta^2 + \frac{4\beta}{(1-ab)}} < 0$ and $h_2(1) = a^2 b \sqrt{\beta^2 + \frac{4\beta}{(1-ab)}} > 0$, therefore there exists at least one root of $h_2(t)$ in the interval $(0, 1)$. A simple calculation shows that $h'_2(t) > 0$ i.e., $h_2(t)$ is monotonically increasing. Therefore it crosses t axis exactly once at some $t_0 \in (0, 1)$ where $h_2(t_0) = 0$ with $h'_2(t_0) \neq 0$. Hence for $t \neq t_0$

$$\frac{d}{dt}(F \circ \phi_2)(t) \neq 0,$$

and for $t = t_0$

$$\frac{d}{dt}(F \circ \phi_2)(t) = 0 \text{ and } \frac{d^2}{dt^2}(F \circ \phi_2)(t) \neq 0.$$

This shows that all the points on the curve LC_{-1}^2 are fold points except the point $p = \phi_2(t_0)$ which is a cusp point. □

The computation of the cusp point in the Theorem 4.3.4 gives the following interesting result.

Corollary 4.3.5. *The image of the cusp point p determines the position of the LC^2 curve, i.e.,*

$$LC^2 \subseteq [0, g_1(p)] \times [0, g_2(p)]. \quad (4.17)$$

Proof. From the above Theorem 4.3.4, we know that $h_2(0) < 0$, $h_2(1) > 0$ and $h_2(t) = 0$ at $t = t_0$, therefore

$$h_2(t) = \begin{cases} -\text{ve}, & t \in (0, t_0) \\ +\text{ve}, & t \in (t_0, 1). \end{cases}$$

Now, from the formulas of $\psi_3(t)$ and $\psi_4(t)$, for all $t \in (0, 1)$, we have

$$\psi_3(t) < 0 \text{ and } \psi_4(t) < 0.$$

From the above, we conclude that

$$\gamma_3'(t) = \gamma_4'(t) = \begin{cases} +\text{ve}, & t \in (0, t_0) \\ -\text{ve}, & t \in (t_0, 1). \end{cases}$$

Hence, $\gamma_3(t) < \gamma_3(t_0)$ and $\gamma_4(t) < \gamma_4(t_0)$ for all $t \in (0, 1)$. Consequently, we say that,

$$LC^2 \subseteq [0, g_1(p)] \times [0, g_2(p)]. \quad (4.18)$$

□

Remark 4.3.6. (i) The LC^1 curve is the image of the fold points, i.e., every $w \in LC^1$ has only one critical preimage.

(ii) The critical curve LC^1 and LC^2 cannot intersect with each other. On the contrary suppose that LC^1 and LC^2 intersect at some $t = t'$ then at $t = t'$

$$(\gamma_1(t'), \gamma_2(t')) = (\gamma_3(t'), \gamma_4(t')).$$

Corresponding to $(\gamma_1(t'), \gamma_2(t'))$ and $(\gamma_3(t'), \gamma_4(t'))$, we have $\phi_1(t') \in LC_{-1}^1$ and $\phi_2(t') \in LC_{-1}^2$ respectively, such that

$$F(\phi_1(t')) = (\gamma_1(t'), \gamma_2(t')) \text{ and } F(\phi_2(t')) = (\gamma_3(t'), \gamma_4(t')).$$

But $\phi_1(t') \neq \phi_2(t')$ because $LC_{-1}^1 \cap LC_{-1}^2 = \emptyset$. This contradicts that every point on LC^1 curve has a unique critical preimage. Hence, LC^1 and LC^2 curves cannot intersect.

(iii) The LC^2 curve obtain its maximum value at $F(p)$.

Lemma 4.3.7. Let $F(x, y)$ be the 2D Homographic map, then the boundary of $F(\mathbb{R}_+^2)$ lies in the union of $F(\partial\mathbb{R}_+^2)$, and LC , i.e.,

$$\partial F(\mathbb{R}_+^2) \subseteq F(\partial\mathbb{R}_+^2) \cup LC.$$

Proof. $F(\mathbb{R}_+^2)$ is a closed set; hence it contains all of its boundary points. Let $v \in \partial F(\mathbb{R}_+^2)$ then exists $u \in \mathbb{R}_+^2$ such that $F(u) = v$. The point u can be either a regular point or a singular point. Assume that u is a regular point then we have the following cases (i) $u \in \text{int}(\mathbb{R}_+^2)$ and (ii) $u \in \partial(\mathbb{R}_+^2)$.

(i) We know that for a compact Euclidean set, the image of regular points cannot lie on the boundary; therefore, $u \notin \text{int}(\mathbb{R}_+^2)$.

(ii) If $u \in \partial(\mathbb{R}_+^2)$ then $v \in F(\partial\mathbb{R}_+^2)$.

If u is singular then $u \in LC_{-1}$, then $v \in LC$. From the above, we conclude that

$$\partial F(\mathbb{R}_+^2) \subseteq F(\partial \mathbb{R}_+^2) \cup LC.$$

□

In this work, our goal is to understand the dynamics of the 2D Homographic map; therefore, we are concerned with the image of $\mathbb{R}_+^2$ under F . Thus, we have the following result.

Theorem 4.3.8. *Let $F(x,y)$ be the 2D Homographic Ricker map then we have*

$$F(\mathbb{R}_+^2) = X \cup Y \cup LC^1.$$

Proof. We know that the x and y axes are invariant under $F(x,y)$. Then $F(\partial \mathbb{R}_+^2) = X \cup Y$, where $X = \left\{ (x,0) : 0 \leq x \leq \frac{-\beta + \sqrt{\beta^2 + 4\beta}}{\beta + \sqrt{\beta^2 + 4\beta}} e^{\frac{2r + \beta - \sqrt{\beta^2 + 4\beta}}{2}} \right\}$ and $Y = \left\{ (0,y) : 0 \leq y \leq \frac{-\beta + \sqrt{\beta^2 + 4\beta}}{\beta + \sqrt{\beta^2 + 4\beta}} e^{\frac{2s + \beta - \sqrt{\beta^2 + 4\beta}}{2}} \right\}$.

From the analytical work discussed in the proof of Theorem 4.3.4, we conclude that LC^2 is a curve that starts and ends at the origin $(0,0)$ and attains its maximum value at $F(p)$. Also, the critical curve LC^1 and LC^2 cannot intersect each other; therefore the possibility is that either $LC^1 \subseteq LC^2$ or $LC^2 \subseteq LC^1$. Suppose $LC^1 \subseteq LC^2$ then we get a contradiction to the fact that $F(x,y)$ is a continuous function, therefore $LC^2 \subseteq LC^1$.

We claim that $F(p) \notin LC^1$. On the contrary suppose that $F(p) \in LC^1$, since p is a cusp point then there exist $t = t_1$ such that

$$(\gamma_1'(t_1), \gamma_2'(t_1)) = (0,0).$$

The above condition contradicts the fact that all the points on LC_{-1}^1 are fold points and $(\gamma_1'(t_1), \gamma_2'(t_1)) \neq (0,0)$ for all $t \in [0, \beta]$. Hence, $F(p) \notin LC^1$ and the curve LC^2 lies below LC^1 curve. The above analysis proves that

$$F(\mathbb{R}_+^2) = X \cup Y \cup LC^1.$$

□

We recall some topological results from [Saldanha and Tomei, 2002] to count the number of preimages of $\mathbb{R}_+^2 \setminus LC$.

Lemma 4.3.9. *(Saldanha and Tomei [2002]) Let $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ be a proper function with critical set LC_{-1} . Then the number of preimages in any connected component of $\mathbb{R}_+^2 \setminus LC$ is constant.*

Lemma 4.3.10. *([Saldanha and Tomei, 2002]) Let $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ be a proper function having only fold and cusp points as singular points. Then the number of preimages in adjacent connected components of $\mathbb{R}_+^2 \setminus LC$ is differ by two.*

From Figure 4.3, we observe that $\mathbb{R}_+^2 \setminus LC$ consists of three connected components Q_1 , Q_2 and Q_3 . From Lemma 4.3.9, each Q_i for $i = 1, 2, 3$ has a constant cardinality. Let Z_n be the zone where the number of preimages is n . Then Theorem 4.3.8 shows that none of the point in the domain $\mathbb{R}_+^2$ is mapped to the region Q_1 . Hence, the Q_1 region is in Z_0 zone, i.e., the points in Q_1 have zero preimages. Now, from Lemma 4.3.10, the difference in the number of preimages in the Q_2 and Q_1 is two. Therefore the region Q_2 is in the zone Z_2 , i.e., the points in Q_2 have two preimages. The possibility for the region Q_3 to be either in Z_0 zone or in the Z_4 zone. Let $q \in Q_3$ be a point close to the cusp point $F(p)$. Then, the local structure of the cusp point consists of at least three preimages of the point q . This discards the possibility of region Q_3 to be in the Z_0 zone, therefore the region Q_3 lies in Z_4 zone. We conclude that the critical curve LC divides the plane $\mathbb{R}_+^2$ into three different zones Z_0 , Z_2 , and Z_4 . Hence the 2D Homographic map $F(x,y)$ is belongs to $Z_0 - Z_2 - Z_4$ class.

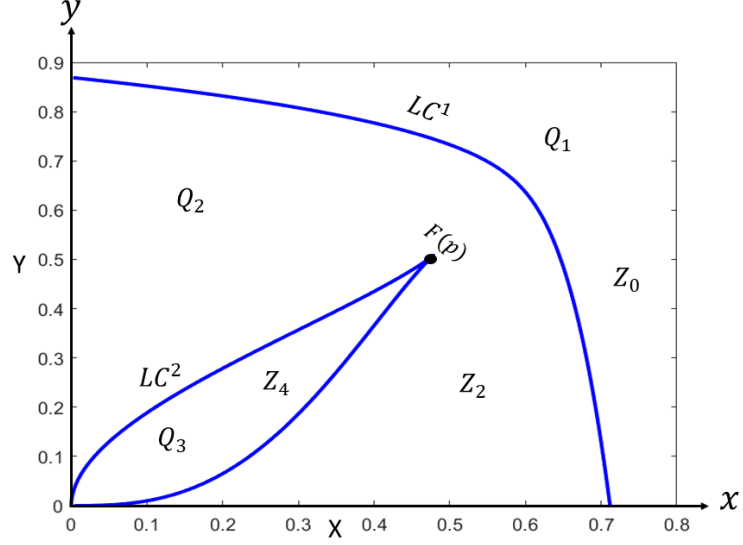


Figure 4.3 : The critical curve LC of the 2D Homographic map $F(x,y)$.

4.4 FIXED POINTS AND STABILITY

In this section, we calculate the fixed points of the 2D Homographic map $F(x,y)$ and discuss their local asymptotic stabilities. The fixed points of $F(x,y)$ are the points that are invariant under the map. These invariant points are obtained by solving the following equations

$$\frac{x}{x+\beta}e^{r-x-ay} = x, \text{ and } \frac{y}{y+\beta}e^{s-y-bx} = y. \quad (4.19)$$

The point $E_0 = (0,0)$ always satisfies the Eq.(4.19); hence it is a trivial fixed point of the map $F(x,y)$. If $y = 0$, then we have

$$\frac{x}{x+\beta}e^{r-x} = x,$$

$$e^{r-x} = x + \beta. \quad (4.20)$$

Eq. (4.20) is an implicit equation, and hence, we cannot obtain the analytic value of x , which satisfies this equation. Therefore, we use the Lambert W function to compute the fixed point. More about Lambert W function is included in [Corless et al., 1996; Rocha and Taha, 2023, 2020; Rocha et al., 2020]. Rearranging the Eq.(4.20), we obtain

$$x = W(X) - \beta, \quad (4.21)$$

where $X = e^{r+\beta}$ and $W(X)$ shows the Lambert W function. The exclusion fixed point of the 2D Homographic map is given as $E_1 = (W(X) - \beta, 0)$. Similarly, in the absence of x species, we have

$$\frac{y}{y+\beta}e^{s-y} = y,$$

$$e^{s-y} = y + \beta. \quad (4.22)$$

On rearranging the Eq.(4.22) we obtain

$$y = W(Y) - \beta,$$

where $Y = e^{s+\beta}$ and $W(Y)$ represents the Lambert W function. The other exclusion fixed point of the 2D Homographic map is given as $E_2 = (0, W(Y) - \beta)$. The exclusion fixed points E_1 and E_2 lies on the x -axis and y -axis respectively. The set of points whose x -coordinate is held fixed by the function $F(x, y)$ are called the x isocline, and its mathematical expression for the map $F(x, y)$ is given as

$$e^{r-x-ay} - x - \beta = 0, \quad (4.23)$$

we denote the resulting curve by L_1 . Similarly, the set of points on which the y -coordinate is held fixed by $F(x, y)$ are called the y isocline, and its equation is given as

$$e^{s-y-bx} - y - \beta = 0, \quad (4.24)$$

we denote the above curve by L_2 . The coexistence of the fixed points is obtained by the intersection of the curves L_1 and L_2 . If the two curves L_1 and L_2 intersect in the positive quadrant, we obtain the positive coexisting fixed point. Rearranging the Eq.(4.23) and (4.24), we obtain

$$y = \frac{1}{a}(r - x - \ln(x + \beta)) = f(x), \quad (4.25)$$

and

$$x = \frac{1}{b}(s - y - \ln(y + \beta)) = g(y). \quad (4.26)$$

The occurrence of coexisting fixed points can also be seen in the following way: Let $E_3 = (x^*, y^*)$ be the coexisting fixed point, then (x^*, y^*) can be obtained by solving the following equation:

$$G(x) = \frac{1}{b}(s - f(x) - \ln(f(x) + \beta)) - x. \quad (4.27)$$

Let x^* be the root of the equation $G(x) = 0$ and y^* is obtained by substituting x^* in Eq.(4.25), i.e., $y^* = f(x^*)$.

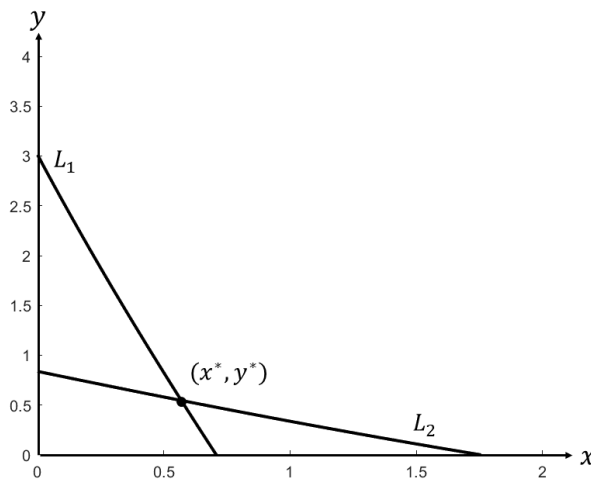


Figure 4.4 : Coexistence of fixed point by intersection of isoclines L_1 and L_2 .

Theorem 4.4.1. Assume that $ab < 1$. The 2D Homographic map $F(x, y)$ have a unique positive coexisting fixed point $(x^*, y^*) \in [0, M_1] \times [0, M_2]$ if the following inequalities holds:

$$s < f(0) + \ln(f(0) + \beta), \text{ and } \frac{1}{b}(s - \ln(\beta)) > \zeta$$

where ζ is the solution of $f(x) = 0$.

Proof. Assume that $s < f(0) + \ln(f(0) + \beta)$, then it follows that $G(0) < 0$ and let $\frac{1}{b}(s - \ln(\beta)) > \zeta$ which gives $G(\zeta) > 0$. Therefore, at least one root of $G(x)$ exists between 0 and ζ . Compute

$$G'(x) = \frac{(1-ab)}{ab} + \frac{1}{ab} \left(\frac{1}{\beta+x} + \frac{1}{\beta+f(x)} + \frac{1}{(x+\beta)(f(x)+\beta)} \right).$$

A simple calculation shows that $G'(x) > 0$ for $x \in \mathbb{R}$, i.e., $G(x)$ is a strictly monotonic increasing function on $\mathbb{R}$ and hence crosses the x -axis only once. This proves that the 2D Homographic map $F(x, y)$ has a unique positive coexisting fixed point. $\square$

4.4.1 Stability of the fixed points

The Jacobian of the 2D Homographic map $F(x, y)$ is

$$JF(x, y) = \begin{pmatrix} \left(\frac{(\beta-x(x+\beta))}{(x+\beta)^2} \right) e^{r-x-ay} & \left(\frac{-ax}{x+\beta} \right) e^{r-x-ay} \\ \left(\frac{-by}{y+\beta} \right) e^{s-y-bx} & \left(\frac{(\beta-y(y+\beta))}{(y+\beta)^2} \right) e^{s-y-bx} \end{pmatrix}. \quad (4.28)$$

The Jacobian evaluated at the fixed point $(0, 0)$ is given as:

$$J_0 = \begin{pmatrix} \frac{e^r}{\beta} & 0 \\ 0 & \frac{e^s}{\beta} \end{pmatrix}.$$

The eigenvalues of J_0 are $\lambda_1 = \frac{e^r}{\beta}$ and $\lambda_2 = \frac{e^s}{\beta}$. The fixed point $(0, 0)$ is locally asymptotically stable when $|\lambda_i| < 1$ for $i = 1, 2$, i.e., $r < \ln(\beta)$ and $s < \ln(\beta)$. We discuss the stability of the trivial fixed point $(0, 0)$ in two cases (i) when $r = \ln(\beta)$, $s < \ln(\beta)$ and (ii) when $r < \ln(\beta)$, $s = \ln(\beta)$. In the case (i) we have $\lambda_1 = 1$, $|\lambda_2| < 1$ and in case (ii) $|\lambda_1| < 1$, $\lambda_2 = 1$.

Case (i): We explain the stability of the fixed point with the help of centre manifolds. The centre manifold theory [Cabr  et al., 2003], [Carr and Muncaster, 1983], [Karydas and Schinas, 1992], [Vanderbauwhede, 1989] is quite interesting when the system does not have an unstable manifold. In this scenario, the dynamics of the centre manifold determine the dynamics of the original system. More about the centre manifold and its computation can be found in the Appendix of [Lu s et al., 2011]. The 2D Homographic map $F(x, y)$ can be written as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{e^s}{\beta} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \begin{pmatrix} \tilde{f}(x_n, y_n) \\ \tilde{g}(x_n, y_n) \end{pmatrix}, \quad (4.29)$$

where $\tilde{f}(x, y) = \frac{x\beta}{x+\beta}e^{-x-ay} - x$ and $\tilde{g}(x, y) = \frac{y}{y+\beta}e^{s-y-bx} - \frac{ye^s}{\beta}$. Using centre manifold theory, we have $y = h(x)$ where $h(x) = Ax^2 + Bx^3 + O(x^4)$, and $A, B \in \mathbb{R}^+$. The map h must satisfy the centre manifold equation

$$h(x + \tilde{f}(x, h(x))) - \frac{e^s h(x)}{\beta} - \tilde{g}(x, h(x)) = 0,$$

$$h \left(\frac{x\beta e^{-x-ah(x)}}{x+\beta} \right) - \frac{e^s h(x)}{\beta} - \frac{h(x)e^{s-h(x)-bx}}{h(x)+\beta} + \frac{h(x)e^s}{\beta} = 0.$$

By Taylor's series, this equation is equivalent to

$$A \left(\frac{e^s}{\beta} - 1 \right) x^2 + \left(\frac{2A + (-Ab + B)e^s + 2A\beta - B\beta}{\beta} \right) x^3 + O(x^4).$$

Solving the equation by keeping the coefficient of x^2 and x^3 equals zero, i.e.,

$$A \left(\frac{e^s}{\beta} - 1 \right) = 0,$$

$$\left(\frac{2A + (-Ab + B)e^s + 2A\beta - B\beta}{\beta} \right) = 0,$$

we get a unique solution $A = B = 0$. Hence $h(x) = 0$. The centre manifold of the system is $y = 0$, and on this centre manifold, the new map $\hat{f}$ is

$$\hat{f}(x) = x + \tilde{f}(x, 0) = \frac{x\beta e^{-x}}{x+\beta}.$$

The computations shows that $\hat{f}'(0) = 1$ and $\hat{f}''(0) \neq 0$. Hence by [Elaydi, 2007], the fixed point $(0, 0)$ is unstable. Case (ii) can be solved similarly to the case (i). In case (ii), the fixed point $(0, 0)$ is unstable. The above discussion will lead to the following proposition:

Proposition 4.4.2. *The trivial fixed point $(0, 0)$ of the map $F(x, y)$ is locally asymptotically stable when $r < \ln(\beta)$ and $s < \ln(\beta)$ and unstable when $r \geq \ln(\beta)$ or $s \geq \ln(\beta)$.*

Remark 4.4.3. *Note that under the condition $r < \ln(\beta)$, $s < \ln(\beta)$, the fixed point $(0, 0)$ is the only fixed point of the map $F(x, y)$. Therefore, whenever the fixed point $(0, 0)$ is locally asymptotically stable, it is also globally asymptotically stable.*

The Jacobian of the map $F(x, y)$ at the other two fixed points E_1 and E_2 are given as follows:

$$J_1 = \begin{pmatrix} \frac{\beta - W(X)(W(X) - \beta)}{W(X)} & -a(W(X) - \beta) \\ 0 & \frac{e^{s-b(W(X)-\beta)}}{\beta} \end{pmatrix},$$

$$J_2 = \begin{pmatrix} \frac{e^{r-a(W(Y)-\beta)}}{\beta} & 0 \\ -b(W(Y) - \beta) & \frac{\beta - W(Y)(W(Y) - \beta)}{W(Y)} \end{pmatrix}.$$

The eigenvalues of the Jacobians J_i 's are nothing but their diagonal entries. We discuss the stability of the exclusion fixed point E_1 . The fixed point E_1 is locally asymptotic stable when both the eigenvalues lie within the unit disc. The eigenvalues of the Jacobian J_1 are $\lambda_1 = \frac{\beta - W(X)(W(X) - \beta)}{W(X)}$ and $\lambda_2 = \frac{e^{s-b(W(X)-\beta)}}{\beta}$. The eigenvalue λ_1 lies within the unit disc if the following inequality satisfy

$$\beta < W(X) < \frac{(1 + \beta) + \sqrt{\beta^2 + 6\beta + 1}}{2}. \quad (4.30)$$

Similarly, the eigenvalue λ_2 lies within the unit disc if the below-mentioned inequalities hold

$$\frac{1}{b}(s + b\beta - \ln(\beta)) < W(X). \quad (4.31)$$

The Eq.(4.30) and Eq.(4.31) together give the condition in which λ_i , $i = 1, 2$ lies in the unit disc. Now, we discuss the stability of fixed point E_1 when the eigenvalue of J_1 lies on the unit disc. The eigenvalues lie on the unit disc in the following cases:

(i) When $W(X) = \frac{(1+\beta)+\sqrt{\beta^2+6\beta+1}}{2}$, we have $\lambda_1 = -1$ and $\lambda_2 < 1$.

(ii) When $W(X) = \frac{s+b\beta-\ln(\beta)}{b}$, we have $\lambda_1 < 1$ and $\lambda_2 = 1$.

Case (i): We shift the fixed point $E_1 = (x^*, 0)$ to $(0, 0)$ with the help of following change of variable

$$u = x - x^* \text{ and } v = y.$$

The 2D Homographic map under the above change of variable is obtained as

$$\begin{aligned} u_{n+1} &= \frac{(u_n + x^*)(x^* + \beta)}{u_n + x^* + \beta} e^{-u_n - av_n} \\ v_{n+1} &= \frac{v_n}{v_n + \beta} e^{s - bx^* - bu_n - v_n}. \end{aligned} \quad (4.32)$$

The system (4.32) can be written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & -ax^* \\ 0 & \frac{e^{s-bx^*}}{\beta} \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \tilde{f}(u_n, v_n) \\ \tilde{g}(u_n, v_n) \end{pmatrix} \quad (4.33)$$

where

$$\begin{aligned} \tilde{f}(u_n, v_n) &= \frac{(u_n + x^*)(x^* + \beta)}{u_n + x^* + \beta} e^{-u_n - av_n} + u_n + ax^* v_n - x^*, \\ \tilde{g}(u_n, v_n) &= \frac{v_n}{v_n + \beta} e^{s - bx^* - bu_n - v_n} - v_n \left(\frac{e^{s-bx^*}}{\beta} \right). \end{aligned}$$

Since $\lambda_1 = -1$, hence assume that $v = h(u)$ where $h: \mathbb{R} \rightarrow \mathbb{R}$ such that $h(0) = 0$ and $h'(0) = 0$. Let $h(u) = Au^2 + Bu^3 + O(u^4)$, where $A, B \in \mathbb{R}$. The map h must satisfy the centre manifold equation, i.e.,

$$h \left(\frac{(u + x^*)(x^* + \beta)}{u + x^* + \beta} e^{-u - ah(u)} - x^* \right) - \frac{h(u)}{h(u) + \beta} e^{s - bx^* - bu - h(u)} = 0. \quad (4.34)$$

Obtaining the Taylor series expansion of Eq.(4.34) and keeping the coefficient of u^2 and u^3 equal to zero, we get a unique solution $A = B = 0$. Consequently, the center manifold is $v = h(u) = 0$. The new center manifold map is

$$\hat{f}(u) = -u - ax^* h(u) + \tilde{f}(u, h(u)) = \frac{(u + x^*)(x^* + \beta)}{u + x^* + \beta} e^{-u} - x^*. \quad (4.35)$$

Simple calculations lead to the fact that the Schwarzian derivative of $\hat{f}(u)$ is negative. Hence by [Elaydi, 2007], the fixed point E_1 is asymptotically stable.

Case (ii): We consider $W(X) = \frac{1}{b}(s + b\beta - \ln(\beta))$ and in this case the 2D Homographic map $F(x, y)$ is written as

$$\begin{pmatrix} u_{n+1} \\ v_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta - x^*(x^* + \beta)}{x^* + \beta} & -ax^* \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_n \\ v_n \end{pmatrix} + \begin{pmatrix} \tilde{f}(u_n, v_n) \\ \tilde{g}(u_n, v_n) \end{pmatrix} \quad (4.36)$$

where

$$\begin{aligned} \tilde{f}(u_n, v_n) &= \frac{(u_n + x^*)}{u_n + x^* + \beta} e^{-u_n - av_n} - \left(\frac{\beta - x^*(x^* + \beta)}{x^* + \beta} \right) u_n + ax^* v_n - x^* \\ \tilde{g}(u_n, v_n) &= \frac{v_n \beta}{v_n + \beta} e^{-bu_n - v_n} - v_n. \end{aligned}$$

Since $\lambda_2 = 1$, let $u = h(v)$ be the center manifold of the above system. After simplification, the center manifold is given as

$$u = -a \left(\frac{x^* + \beta}{x^* + \beta + 1} \right) v + Av^2 + Bv^3,$$

where

$$A = \frac{-\left((x^* + \beta) \left(\frac{(Pa^2(-1+P) - (x^*a^2(-1+P)^2)/2)}{(x^* + \beta)} - \frac{(P^2x^*a^2)}{(x^* + \beta)^3} + \frac{(Pa(Pa - x^*a(-1+P)))}{(x^* + \beta)^2} \right) + Pa\beta \left(\frac{1}{\beta^2} - \frac{(-1+Pab)}{\beta} \right) \right)}{\left((x^* + \beta) \left(\frac{X}{(x^* + \beta)^2} + \frac{(-1+x^*)}{(x^* + \beta)} \right) + 1\right)}$$

$$B = \left[(Aa(P-1) - x^*p_0 + aPp_1) - (x^*(q_0 + q_1)) + (q_2(Pa - x^*a(P-1))) - \frac{aP(x^*p_1 - A + pa^2(P-1))}{x^* + \beta} \right]$$

$$+ A\beta^2q_3 - Pa \left(\frac{Ab - (Pab - 1)^2}{2} - \frac{1}{\beta^2} + \frac{(Pab - 1)}{\beta^2} \right),$$

where $P = \frac{x^* + \beta}{1 + \beta + x^*}$, $p_0 = a(P-1) \left(\frac{A}{2} - \frac{a^2(P-1)^2}{6} + \frac{Aa(P-1)}{2} \right)$, $p_1 = A - \frac{(a(P-1))^2}{2}$,
 $q_0 = \frac{aP}{x^* + \beta} \left(\frac{A}{x^* + \beta} - \frac{a^2P^2}{(x^* + \beta)^2} \right)$, $q_1 = \frac{APa}{(x^* + \beta)^2}$, $q_2 = \left(\frac{A}{x^* + \beta} - \frac{a^2P^2}{(x^* + \beta)^2} \right)$, $q_3 = \left(\frac{2}{\beta^3} - \frac{2(Pab-1)}{\beta^2} \right)$.

Thus, on the centre manifold, the new map is given as

$$\hat{f}(v) = v + \tilde{g}(h(v), v) = \left(\frac{v\beta}{v + \beta} \right) e^{-v - bh(v)}.$$

We have $\hat{f}'(0) = 1$ and $\hat{f}''(0) = \left(\frac{-2}{x^* + \beta + 1} \right) \left((x^* + \beta)(1 - ab) + \frac{2\beta + x^* + 1}{\beta} \right) \neq 0$. By [Elaydi, 2007], we conclude that the fixed point E_1 is unstable when $W(X) = \frac{1}{b}(s + b\beta - \ln(\beta))$. The above analysis leads to the following theorem:

Theorem 4.4.4. *Let $F(x, y)$ be the 2D Homographic map and E_1, E_2 are its fixed points. Then the following holds:*

- (i) E_1 is asymptotically stable, when $\beta < W(X) \leq \frac{1}{2}((1 + \beta) + \sqrt{\beta^2 + 6\beta + 1})$, $\frac{1}{b}(s + b\beta - \ln(\beta)) < W(X)$.
- (ii) E_2 is asymptotically stable, when $\beta < W(Y) \leq \frac{1}{2}((1 + \beta) + \sqrt{\beta^2 + 6\beta + 1})$, $\frac{1}{a}(r + a\beta - \ln(\beta)) < W(Y)$.

The intersection of the curves L_1 and L_2 gives the positive fixed point of the map $F(x, y)$ as shown in Figure 4.4. Since the equation of the curve L_1 and L_2 are the implicit equation, we cannot find the coexisting fixed point analytically. Let $E_3 = (x^*, y^*)$ be the unique coexisting fixed point. The Jacobian of the map $F(x, y)$ at E_3 is obtained as

$$JF(x^*, y^*) = \begin{pmatrix} \frac{(\beta - x^*(x^* + \beta))}{x^* + \beta} & -ax^* \\ -by^* & \frac{(\beta - y^*(y^* + \beta))}{y^* + \beta} \end{pmatrix}. \quad (4.37)$$

The trace and determinant of $JF(x^*, y^*)$ are given as follows:

$$tr(JF(x^*, y^*)) = \frac{(\beta - x^*(x^* + \beta))}{x^* + \beta} + \frac{(\beta - y^*(y^* + \beta))}{y^* + \beta},$$

and,

$$det(JF(x^*, y^*)) = \frac{(\beta^2 - \beta x^*(x^* + \beta) - \beta y^*(y^* + \beta) + (1 - ab)x^*y^*(x^* + \beta)(y^* + \beta))}{(x^* + \beta)(y^* + \beta)}.$$

Theorem 4.4.5. *The unique positive coexisting fixed point (x^*, y^*) is locally asymptotic stable if and only if*

$$|\beta(x^* + y^* + 2\beta) - (x^* + y^*)(x^* + \beta)(y^* + \beta)| < \beta^2 - \beta(x^*(x^* + \beta) + y^*(y^* + \beta)) + (1 + (1 - ab)x^*y^*)(x^* + \beta)(y^* + \beta),$$

and,

$$\beta^2 - \beta(x^*(x^* + \beta) + y^*(y^* + \beta)) < (1 + (1 - ab)x^*y^*)(x^* + \beta)(y^* + \beta).$$

Proof. The coexisting fixed point (x^*, y^*) is locally asymptotic stable if and only if the eigenvalues of $JF(x^*, y^*)$ lies within the unit disc, i.e., $\rho(JF(x^*, y^*)) < 1$ where ρ is the spectral radius of the matrix $JF(x^*, y^*)$. The condition $\rho(JF(x^*, y^*)) < 1$ is equivalent to the following

$$|tr(JF(x^*, y^*))| < 1 + \det(JF(x^*, y^*)) \text{ and } \det(JF(x^*, y^*)) < 1, \quad (4.38)$$

From the Eq.(4.37), substituting the value of trace and determinant in Eq.(4.38), we obtain the mentioned condition of local asymptotic stability of the positive coexisting fixed point (x^*, y^*) . $\square$

The above theoretical result of the local asymptotic stability of the positive coexisting fixed point (x^*, y^*) is explained, which is confirmed numerically in the numerical example presented in Section 4.6.

4.5 BIFURCATION ANALYSIS

In this section, we discuss the flip bifurcation of positive equilibrium (x^*, y^*) of the map $F(x, y)$. The Jacobian matrix of $F(x, y)$ at the equilibrium point (x^*, y^*) is given by Eq.(4.37), and the characteristic polynomial is given as follows:

$$\mathbf{C}(\lambda) = \lambda^2 - P(x^*, y^*)\lambda + Q(x^*, y^*), \quad (4.39)$$

where

$$P(x^*, y^*) = \frac{(\beta - x^*(x^* + \beta))}{x^* + \beta} + \frac{(\beta - y^*(y^* + \beta))}{y^* + \beta},$$

$$Q(x^*, y^*) = \frac{(\beta^2 - \beta x^*(x^* + \beta) - \beta y^*(y^* + \beta) + (1 - ab)x^*y^*(x^* + \beta)(y^* + \beta))}{(x^* + \beta)(y^* + \beta)}.$$

For the characteristics polynomial $\mathbf{C}(\lambda)$ to have two real roots, its discriminant should be positive, i.e., $P^2(x^*, y^*) - 4Q(x^*, y^*) > 0$. In the explicit form, the above condition can be written as

$$[\beta((x^* + \beta) + (y^* + \beta) - D(x^* + y^*))]^2 > 4D[\beta^2 - \beta(x^*(x^* + \beta) + y^*(y^* + \beta)) + (1 - ab)x^*y^*D] \quad (4.40)$$

where, $D = (x^* + \beta)(y^* + \beta)$. Also, assume that

$$P(x^*, y^*) + Q(x^*, y^*) = -1. \quad (4.41)$$

Considering the above conditions (Eq.(4.40) and Eq.(4.41)), the roots of Eq.(4.39) are $\lambda_1 = -1$ and $\lambda_2 = \beta(x^* + \beta) + \beta(y^* + \beta) + (1 - x^* - y^*)D$. Further, we assume that $|\lambda_2| \neq \pm 1$ i.e.,

$$\beta(x^* + \beta) + \beta(y^* + \beta) + (1 - x^* - y^*)D \neq \pm 1. \quad (4.42)$$

Consider a parameter region

$$\psi_{FB} = \{(r, s, a, b, \beta) \in \mathbb{R}^5 : \text{Eq.(4.40), Eq.(4.41), and Eq.(4.42) are satisfied}\}.$$

The positive equilibrium point (x^*, y^*) of the map $F(x, y)$ encounters a flip bifurcation when the parameter changes in the small neighborhood of ψ_{FB} . Solving the Eq.(4.41), we have

$$\begin{aligned} \beta^2[4 - 2x^2 - 2y^* + (1 - ab)x^*y^*] + \beta[2x^*(1 - x^*) + 2y^*(1 - y^*) - 2x^*y^* + (1 - ab)(x^* + y^*)x^*y^*] \\ + [1 - x^* - y^* + (1 - ab)x^*y^*]x^*y^* = 0. \end{aligned} \quad (4.43)$$

The positive root of the above equation is

$$\beta_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

where $A = 4 - 2x^2 - 2y^* + (1 - ab)x^*y^*$, $B = 2x^*(1 - x^*) + 2y^*(1 - y^*) - 2x^*y^* + (1 - ab)(x^* + y^*)x^*y^*$, and $C = (1 - x^* - y^* + (1 - ab)x^*y^*)x^*y^*$. Consider the parameters $(r, s, a, b, \beta_1) \in \psi_{FB}$ arbitrarily and describes the 2D Homographic map $F(x, y)$ in terms of these parameters then we get the following map

$$\begin{pmatrix} N \\ R \end{pmatrix} \rightarrow \begin{pmatrix} \frac{N}{N+\beta_1} e^{r-N-aR} \\ \frac{P}{P+\beta_1} e^{s-R-bN} \end{pmatrix}. \quad (4.44)$$

Let β be the bifurcation parameter and $\bar{\beta}$ be a small perturbation, then the system (4.44) can be described as

$$\begin{pmatrix} N \\ R \end{pmatrix} \rightarrow \begin{pmatrix} \frac{N}{N+\beta_1+\bar{\beta}} e^{r-N-aR} \\ \frac{P}{P+\beta_1+\bar{\beta}} e^{s-R-bN} \end{pmatrix} \quad (4.45)$$

where $|\bar{\beta}| \ll 1$. Taking $x = N - x^*$ and $y = R - y^*$ which transfer the equilibrium point (x^*, y^*) to the origin $(0, 0)$ and the system (4.45) is converted into the following form

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} g_1(x, y, \bar{\beta}) \\ g_2(x, y, \bar{\beta}) \end{pmatrix} \quad (4.46)$$

where

$$\begin{aligned} g_1(x, y, \bar{\beta}) = & A_{12}x^2 + A_{13}y^2 + A_{14}xy + A_{15}\bar{\beta}^2 + A_{16}y\bar{\beta} + A_{17}x\bar{\beta} + B_{12}x^3 + B_{13}y^3 \\ & + B_{14}\bar{\beta}^3 + B_{15}x^2\bar{\beta} + B_{16}x^2y + B_{17}y^2\bar{\beta} + B_{18}y^2x + B_{19}x\bar{\beta}^2 + B_{101}y\bar{\beta}^2 + B_{102}xy\bar{\beta}, \end{aligned}$$

$$\begin{aligned} g_2(x, y, \bar{\beta}) = & A_{21}x^2 + A_{23}y^2 + A_{24}xy + A_{25}\bar{\beta}^2 + A_{26}y\bar{\beta} + A_{27}x\bar{\beta} + B_{22}x^3 + B_{23}y^3 \\ & + B_{24}\bar{\beta}^3 + B_{25}x^2\bar{\beta} + B_{26}x^2y + B_{27}y^2\bar{\beta} + B_{28}y^2x + B_{29}x\bar{\beta}^2 + B_{201}y\bar{\beta}^2 + B_{202}xy\bar{\beta}, \end{aligned}$$

where,

$$\begin{aligned}
a_{11} &= \frac{(\beta_1 - x^*(x^* + \beta_1))}{x^* + \beta_1}, a_{12} = -ax^*, a_{21} = -by^*, a_{22} = \frac{(\beta_1 - y^*(y^* + \beta_1))}{y^* + \beta_1}, \\
A_{12} &= \frac{x^{*3} + 2\beta_1 x^{*2} + \beta_1 x^*(\beta_1 - 2) - 2\beta_1(1 + \beta_1)}{2(x^* + \beta_1)^2}, A_{13} = \frac{a^2 x^*}{2}, A_{14} = \frac{-a(\beta_1 - x^*(x^* + \beta_1))}{x^* + \beta_1} \\
A_{15} &= \frac{2x^*}{(x + \beta_1)^2}, A_{16} = \frac{ax^*}{2(x^* + \beta_1)}, A_{17} = \frac{-(\beta_1 - x^*(x^* + \beta_1 + 1))}{x^* + \beta_1}, \\
B_{12} &= \frac{x^* - 2}{6(x^* + \beta_1)} + \frac{8\beta_1 + 4x^*(2 - \beta_1) - 4x^{*2}}{6(x^* + \beta_1)^2} - \frac{6(x^{*2} + \beta_1 x^* - \beta_1)}{6(x^* + \beta_1)^3}, \\
B_{13} &= \frac{-a^3 x^*}{6}, B_{14} = \frac{-6x^*}{6(x^* + \beta_1)^3}, \\
B_{15} &= \frac{3\beta_1 + x^*(4 - \beta_1) - x^{*2}}{2(x^* + \beta_1)} + \frac{2x^*(3 - \beta_1 - x^*) + (4\beta_1 - 2)}{2(x^* + \beta_1)^2} - \frac{6(x^{*2} + \beta_1 x^* - \beta_1)}{2(x^* + \beta_1)^3}, \\
B_{16} &= \frac{-a(x^*(x^* + \beta_1 + 2) - 2\beta_1)}{2(x^* + \beta_1)} - \frac{2a(x^{*2} + \beta_1 x^* - \beta_1)}{2(x^* + \beta_1)^2}, B_{17} = \frac{-a^2 x^*}{2(x^* + \beta_1)}, \\
B_{18} &= \frac{a^2(\beta_1 - x^*(x^* + \beta_1))}{2(x^* + \beta_1)}, B_{19} = \frac{2(\beta_2 - x^*(x^* + \beta_1 + 2))}{2(x^* + \beta_1)^3}, \\
B_{101} &= \frac{-2ax^*}{2(x^* + \beta_1)^2}, B_{102} = \frac{-a(x^*(x^* + \beta_1 + 1) - \beta_1)}{(x^* + \beta_1)^2} \\
A_{21} &= \frac{b^2 y^*}{2}, A_{23} = \frac{y^{*3} + 2\beta_1 y^{*2} + \beta_1 y^*(\beta_1 - 2) - 2\beta_1(1 + \beta_1)}{2(y^* + \beta_1)^2}, A_{24} = \frac{-b(\beta_1 - y^*(y^* + \beta_1))}{y^* + \beta_1} \\
A_{25} &= \frac{2y^*}{2(y + \beta_1)^2}, A_{26} = \frac{y^*(y^* + \beta_1 - 1) + (2 - \beta_1)}{(y^* + \beta_1)^2}, A_{27} = \frac{by^*}{(y^* + \beta_1)^2}, \\
B_{22} &= \frac{-b^3 y^*}{6}, B_{23} = \frac{(y^* - 3)\beta_1^3 + (3y^{*2} - 6y^* - 6)\beta_1^2 + (3y^{*3} - 3y^{*2} - 6y^* - 6)\beta_1 + y^{*4}}{6(y^* + \beta_1)^3}, \\
B_{24} &= \frac{-6x^*}{6(x^* + \beta_1)^3}, B_{25} = \frac{-b^2 y^*}{2(y^* + \beta_1)}, B_{26} = \frac{b^2(\beta_1 - y^*(y^* + \beta_1))}{2(x^* + \beta_1)}, B_{17} = \frac{-a^2 x^*}{2(x^* + \beta_1)}, \\
B_{28} &= \frac{\beta_1^3 + (y^* - 2)\beta_1^2 + 2\beta_1(2y^{*2} - 2y^* - 2)}{2(y^* + \beta_1)^2}, B_{29} = \frac{-2by^*}{2(y^* + \beta_1)^2}, \\
B_{201} &= \frac{-2(y^{*2} + y^*(2 + \beta_1) - \beta_1)}{2(y^* + \beta_1)^3}, B_{202} = \frac{-b(y^*(y^* + \beta_1 + 1) - \beta_1)}{(y^* + \beta_1)^2}.
\end{aligned}$$

Now, we consider the following translation

$$\begin{pmatrix} x \\ y \end{pmatrix} = T \begin{pmatrix} u \\ v \end{pmatrix}, \quad (4.47)$$

where $T = \begin{pmatrix} a_{12} & a_{12} \\ -1 - a_{11} & \lambda_2 - a_{11} \end{pmatrix}$ be a non-singular matrix, then we have

$$T^{-1} = \frac{1}{a_{12}(1 + \lambda_2)} \begin{pmatrix} \lambda_2 - a_{11} & -a_{12} \\ 1 + a_{11} & a_{12} \end{pmatrix}.$$

Under the translation (4.47), the map (4.46) can be written as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g_3(u, v, \bar{\beta}) \\ g_4(u, v, \bar{\beta}) \end{pmatrix}, \quad (4.48)$$

where,

$$\begin{aligned}
g_3(u, v, \bar{\beta}) = & \left(\frac{(\lambda_2 - a_{11})A_{12} - a_{12}A_{22}}{a_{12}(1 + \lambda_2)} \right) x^2 + \left(\frac{(\lambda_2 - a_{11})A_{13} - a_{12}A_{23}}{a_{12}(1 + \lambda_2)} \right) y^2 \\
& + \left(\frac{(\lambda_2 - a_{11})A_{14} - a_{12}A_{24}}{a_{12}(1 + \lambda_2)} \right) xy + \left(\frac{(\lambda_2 - a_{11})A_{15} - a_{12}A_{25}}{a_{12}(1 + \lambda_2)} \right) \bar{\beta}^2 \\
& + \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{a_{12}(1 + \lambda_2)} \right) y\bar{\beta} + \left(\frac{(\lambda_2 - a_{11})A_{17} - a_{12}A_{27}}{a_{12}(1 + \lambda_2)} \right) x\bar{\beta} \\
& + \left(\frac{(\lambda_2 - a_{11})B_{12} - a_{12}B_{22}}{a_{12}(1 + \lambda_2)} \right) x^3 + \left(\frac{(\lambda_2 - a_{11})B_{13} - a_{12}B_{23}}{a_{12}(1 + \lambda_2)} \right) y^3 \\
& + \left(\frac{(\lambda_2 - a_{11})B_{14} - a_{12}B_{24}}{a_{12}(1 + \lambda_2)} \right) \bar{\beta}^3 + \left(\frac{(\lambda_2 - a_{11})B_{15} - a_{12}B_{25}}{a_{12}(1 + \lambda_2)} \right) x^2\bar{\beta} \\
& + \left(\frac{(\lambda_2 - a_{11})B_{16} - a_{12}B_{26}}{a_{12}(1 + \lambda_2)} \right) x^2y + \left(\frac{(\lambda_2 - a_{11})B_{17} - a_{12}B_{27}}{a_{12}(1 + \lambda_2)} \right) y^2\bar{\beta} \\
& + \left(\frac{(\lambda_2 - a_{11})B_{18} - a_{12}B_{28}}{a_{12}(1 + \lambda_2)} \right) y^2x + \left(\frac{(\lambda_2 - a_{11})B_{19} - a_{12}B_{29}}{a_{12}(1 + \lambda_2)} \right) x\bar{\beta}^2 \\
& + \left(\frac{(\lambda_2 - a_{11})B_{101} - a_{12}B_{201}}{a_{12}(1 + \lambda_2)} \right) y\bar{\beta}^2 + \left(\frac{(\lambda_2 - a_{11})B_{102} - a_{12}B_{202}}{a_{12}(1 + \lambda_2)} \right) xy\bar{\beta}.
\end{aligned}$$

$$\begin{aligned}
g_4(u, v, \bar{\beta}) = & \left(\frac{(1 + a_{11})A_{12} + a_{12}A_{22}}{a_{12}(1 + \lambda_2)} \right) x^2 + \left(\frac{(1 + a_{11})A_{13} + a_{12}A_{23}}{a_{12}(1 + \lambda_2)} \right) y^2 \\
& + \left(\frac{(1 + a_{11})A_{14} + a_{12}A_{24}}{a_{12}(1 + \lambda_2)} \right) xy + \left(\frac{(1 + a_{11})A_{15} + a_{12}A_{25}}{a_{12}(1 + \lambda_2)} \right) \bar{\beta}^2 \\
& + \left(\frac{(1 + a_{11})A_{16} + a_{12}A_{26}}{a_{12}(1 + \lambda_2)} \right) y\bar{\beta} + \left(\frac{(1 + a_{11})A_{17} + a_{12}A_{27}}{a_{12}(1 + \lambda_2)} \right) x\bar{\beta} \\
& + \left(\frac{(1 + a_{11})B_{12} + a_{12}B_{22}}{a_{12}(1 + \lambda_2)} \right) x^3 + \left(\frac{(1 + a_{11})B_{13} + a_{12}B_{23}}{a_{12}(1 + \lambda_2)} \right) y^3 \\
& + \left(\frac{(1 + a_{11})B_{14} + a_{12}B_{24}}{a_{12}(1 + \lambda_2)} \right) \bar{\beta}^3 + \left(\frac{(1 + a_{11})B_{15} + a_{12}B_{25}}{a_{12}(1 + \lambda_2)} \right) x^2\bar{\beta} \\
& + \left(\frac{(1 + a_{11})B_{16} + a_{12}B_{26}}{a_{12}(1 + \lambda_2)} \right) x^2y + \left(\frac{(1 + a_{11})B_{17} + a_{12}B_{27}}{a_{12}(1 + \lambda_2)} \right) y^2\bar{\beta} \\
& + \left(\frac{(1 + a_{11})B_{18} + a_{12}B_{28}}{a_{12}(1 + \lambda_2)} \right) y^2x + \left(\frac{(1 + a_{11})B_{19} + a_{12}B_{29}}{a_{12}(1 + \lambda_2)} \right) x\bar{\beta}^2 \\
& + \left(\frac{(1 + a_{11})B_{101} + a_{12}B_{201}}{a_{12}(1 + \lambda_2)} \right) y\bar{\beta}^2 + \left(\frac{(1 + a_{11})B_{102} + a_{12}B_{202}}{a_{12}(1 + \lambda_2)} \right) xy\bar{\beta}.
\end{aligned}$$

Let $W_c(0, 0, 0)$ be the center manifold of the map (4.48) in a small neighbourhood of $\bar{\beta} = 0$. According to the center manifold theorem, there exists a center manifold

$$W_c(0, 0, 0) = \{(u, v, \bar{\beta}) \in \mathbb{R}^3 : v = h(u, \bar{\beta}), h(0, 0) = 0, Dh(0, 0) = 0\}$$

for a sufficiently small u and $\bar{\beta}$. We assume that the center manifold of the form

$$h(u, \bar{\beta}) = c_0 u^2 + c_1 u \bar{\beta} + c_2 \bar{\beta}^2 + O((|u| + |\bar{\beta}|)^3). \quad (4.49)$$

The center manifold must satisfy

$$h(-u + g_3(u, h(u, \bar{\beta}), \bar{\beta}), \bar{\beta}) - \lambda_2 h(u, \bar{\beta}) - g_4(u, h(u, \bar{\beta}), \bar{\beta}) = 0 \quad (4.50)$$

Substituting Eq.(4.49) in Eq.(4.50) and equating the coefficients of like powers to zero. After the simplification, we get

$$\begin{aligned}
c_0 &= a_{12} \left(\frac{(1+a_{11})A_{12}+a_{12}A_{22}}{(1-\lambda_2^2)} \right) + (1+a_{11})^2 \left(\frac{(1+a_{11})A_{13}+a_{12}A_{23}}{a_{12}(1-\lambda_2^2)} \right) \\
&\quad - (1+a_{11}) \left(\frac{(1+a_{11})A_{14}+a_{12}A_{24}}{(1-\lambda_2^2)} \right), \\
c_1 &= \frac{-(1+a_{11})^2 A_{16} - a_{12}(1+a_{11})(A_{26}-A_{17}) + a_{12}^2 A_{27}}{a_{12}(1+\lambda_2)^2}, \\
c_2 &= \left(\frac{(1+a_{11})A_{15}+a_{12}A_{25}}{(1-\lambda_2^2)} \right).
\end{aligned}$$

Therefore, we consider the map (4.48) restricted to the center manifold $W_c(0,0,0)$ which is given as

$$f : u \rightarrow -u + m_1 u^2 + m_2 u \bar{\beta} + m_3 u^2 \bar{\beta} + m_4 u \bar{\beta}^2 + m_5 u^3 + O((|u| + |\bar{\beta}|)^4), \quad (4.51)$$

where,

$$\begin{aligned}
m_1 &= a_{12} \left(\frac{(\lambda_2 - a_{11})A_{12} - a_{12}A_{22}}{(1+\lambda_2)} \right) + (1+a_{11})^2 \left(\frac{(\lambda_2 - a_{11})A_{13} - a_{12}A_{23}}{a_{12}(1+\lambda_2)} \right) \\
&\quad - (1+a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{14} - a_{12}A_{24}}{(1+\lambda_2)} \right), \\
m_2 &= a_{12} \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{a_{12}(1+\lambda_2)} \right) - (1+a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{a_{12}(1+\lambda_2)} \right), \\
m_3 &= 2a_{12}^2 c_1 \left(\frac{(\lambda_2 - a_{11})A_{12} - a_{12}A_{22}}{(1+\lambda_2)} \right) - 2c_1(\lambda_2 - a_{11})(1+a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{13} - a_{12}A_{23}}{a_{12}(1+\lambda_2)} \right) \\
&\quad + c_1(\lambda_2 - 1 - 2a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{14} - a_{12}A_{24}}{(1+\lambda_2)} \right) + c_0 \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{(1+\lambda_2)} \right) \\
&\quad + c_0(\lambda_2 - a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{17} - a_{12}A_{27}}{a_{12}(1+\lambda_2)} \right) + a_{12} \left(\frac{(\lambda_2 - a_{11})B_{15} - a_{12}B_{25}}{(1+\lambda_2)} \right) \\
&\quad + (1+a_{11})^2 \left(\frac{(\lambda_2 - a_{11})B_{17} - a_{12}B_{27}}{(1+\lambda_2)} \right) - (1+a_{11}) \left(\frac{(\lambda_2 - a_{11})B_{102} - a_{12}B_{202}}{(1+\lambda_2)} \right), \\
m_4 &= 2c_2 a_{12} \left(\frac{(\lambda_2 - a_{11})A_{12} - a_{12}A_{22}}{(1+\lambda_2)} \right) + c_2 a_{12}(a_{12} - (1+a_{11})) \left(\frac{(\lambda_2 - a_{11})A_{14} - a_{12}A_{24}}{a_{12}(1+\lambda_2)} \right) \\
&\quad - 2a_{12}c_2 \left(\frac{(\lambda_2 - a_{11})A_{13} - a_{12}A_{23}}{a_{12}(1+\lambda_2)} \right) + c_1 a_{12} \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{a_{12}(1+\lambda_2)} \right) \\
&\quad + c_1 a_{12} \left(\frac{(\lambda_2 - a_{11})A_{16} - a_{12}A_{26}}{a_{12}(1+\lambda_2)} \right) + a_{12} \left(\frac{(\lambda_2 - a_{11})B_{19} - a_{12}B_{29}}{a_{12}(1+\lambda_2)} \right) \\
&\quad - (1+a_{11}) \left(\frac{(\lambda_2 - a_{11})B_{101} - a_{12}B_{201}}{a_{12}(1+\lambda_2)} \right) \\
m_5 &= 2c_0 a_{12} \left(\frac{(\lambda_2 - a_{11})A_{12} - a_{12}A_{22}}{(1+\lambda_2)} \right) - 2c_0(\lambda_2 - a_{11})(1+a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{13} - a_{12}A_{23}}{a_{12}(1+\lambda_2)} \right) \\
&\quad + c_0(\lambda_2 - 1 - 2a_{11}) \left(\frac{(\lambda_2 - a_{11})A_{14} - a_{12}A_{24}}{(1+\lambda_2)} \right) + a_{12}^2 \left(\frac{(\lambda_2 - a_{11})B_{12} - a_{12}B_{22}}{(1+\lambda_2)} \right) \\
&\quad - (1+a_{11})^3 \left(\frac{(\lambda_2 - a_{11})B_{12} - a_{12}B_{22}}{a_{12}(1+\lambda_2)} \right).
\end{aligned}$$

For a system (4.51) to encounter a flip bifurcation, we require two discriminatory non-zero

quantities α_1 and α_2 , which are from [Robinson, 1998] given as follows:

$$\alpha_1 = \left(2 \frac{\partial^2 f}{\partial \beta \partial u} + \frac{\partial f}{\partial \beta} \frac{\partial f}{\partial u} \right)_{(0,0)} = 2m_2$$

$$\alpha_2 = \left(\frac{1}{2} \left(\frac{\partial^2 f}{\partial u^2} \right)^2 + \frac{1}{3} \left(\frac{\partial^3 f}{\partial u^3} \right) \right)_{(0,0)} = 2(m_1^2 + m_5).$$

We obtain the following theorem by summarizing the above analysis, together with Theorem 3.1 in [Robinson, 1998].

Theorem 4.5.1. *If $\alpha_2 \neq 0$, then the 2D Homographic map $F(x, y)$ undergoes a period-doubling bifurcation at the unique coexisting fixed point (x^*, y^*) when the parameter β varies in the small neighborhood of β_1 . Furthermore, if $\alpha_2 > 0$ ($\alpha_2 < 0$) then the period-two orbit that bifurcates from (x^*, y^*) is stable (unstable).*

4.5.1 Fold and flip bifurcation curves

In this subsection, we discuss the fold and flip bifurcation curves associated with the fixed points E_0 , E_1 and E_2 of the 2D Homographic map $F(x, y)$. Let $(\hat{x}, \hat{y})$ be one of the periodic points of a period n orbit, where $n \in \mathbb{N}$. The 2D Homographic map $F(x, y)$ and its n^{th} -iterated map $F^n(x, y)$ are given as

$$F(x, y) = (g_1(x, y), g_2(x, y)) \text{ and } F^n(x, y) = (g_1^n(x, y), g_2^n(x, y)).$$

Let

$$tr = \frac{\partial g_1^n}{\partial x} + \frac{\partial g_2^n}{\partial y} \text{ and } \tilde{D} = \frac{\partial g_1^n}{\partial x} \frac{\partial g_2^n}{\partial y} - \frac{\partial g_1^n}{\partial y} \frac{\partial g_2^n}{\partial x} + 1,$$

where tr and $\tilde{D} - 1$ respectively denote the trace and determinant of the Jacobian matrix of $F^n(x, y)$. A real number σ such that $\tilde{D}\sigma - tr = 0$ is called the reduced multiplier of the considered n periodic point. In $\mathbb{R}_+^2$ space, a contour line σ of n periodic points is a curve that satisfies the following equations:

$$\begin{aligned} F^n(\hat{x}, \hat{y}) &= (\hat{x}, \hat{y}), \\ \tilde{D}(\hat{x}, \hat{y})\sigma &= tr(\hat{x}, \hat{y}). \end{aligned} \tag{4.52}$$

Note that the equation of the contour line σ is obtained by eliminating the $(\hat{x}, \hat{y})$ in Eq.(4.52). The other properties of the reduced multiplier and contour lines can be found in [Carcasses, 1993]. A contour line $\sigma = 1$ represents the fold bifurcation curve, and contour line $\sigma = -1$ represents the flip bifurcation curve. The fold and flip bifurcation curves are generically denoted by $\Lambda_{(n)_0}^j$ and Λ_n^j respectively, where $n \in \mathbb{N}$ is the order of the cycle, and j is the differentiation between bifurcation curves of the same cycle n . In this chapter, the fold and flip bifurcation curves corresponding to the fixed point E_i are denoted by $\Lambda_{(n)_0}^{i,j}$ and $\Lambda_n^{i,j}$ respectively.

The trace and $\tilde{D}$ value of 2D Homographic map $F(x, y)$ corresponding to $n = 1$ are given as

$$\begin{aligned} tr &= \left(\frac{(\beta - x(x + \beta))}{(x + \beta)^2} \right) e^{r-x-ay} + \left(\frac{(\beta - y(y + \beta))}{(y + \beta)^2} \right) e^{s-y-bx}, \\ \tilde{D} &= \frac{(\beta^2 - \beta x(x + \beta) - \beta y(y + \beta) + (1 - ab)xy(x + \beta)(y + \beta))e^{(r+s-(1+b)x-(1+a)y)}}{(x + \beta)^2(y + \beta)^2} + 1. \end{aligned}$$

At the fixed point E_1 , the tr and $\tilde{D}$ values are given as

$$tr = \frac{\beta - W(X)(W(X) - \beta)}{W(X)} + \frac{e^{s-b(W(X)-\beta)}}{\beta} \text{ and } \tilde{D} = \left(\frac{\beta - W(X)(W(X) - \beta)}{W(X)} \right) \left(\frac{e^{s-b(W(X)-\beta)}}{\beta} \right) + 1. \quad (4.53)$$

The contour line corresponding to $\sigma = 1$ satisfies the equation $\tilde{D} = tr$. Hence, we obtain the following:

$$\left(\frac{\beta - W(X)(W(X) - \beta)}{W(X)} \right) \left(\frac{e^{s-b(W(X)-\beta)}}{\beta} \right) + 1 = \frac{\beta - W(X)(W(X) - \beta)}{W(X)} + \frac{e^{s-b(W(X)-\beta)}}{\beta}.$$

After simplifying the above equation, we obtain

$$\frac{\beta - W(X)(W(X) - \beta)}{W(X)} = 1, \quad (4.54)$$

and

$$\frac{e^{s-b(W(X)-\beta)}}{\beta} = 1. \quad (4.55)$$

Solving Eq.(4.54), we get $W(X) = -1, \beta$. Corresponding to $W(X) = \beta$, the fixed point E_1 reduces to the fixed point E_0 , and corresponding to $W(X) = -1$, the fixed point E_1 does not lie in the domain $\mathbb{R}_+^2$. Hence, we discard the $W(X) = -1, \beta$ values to obtain the fold bifurcation curves. Now, simplifying Eq.(4.55), we obtain

$$W(X) = \beta + \frac{(s - \ln(\beta))}{b}.$$

Using the definition of Lambert W function and substituting $X = e^{r+\beta}$, the fold bifurcation curve equation of the $F(x, y)$ map is given as

$$r = \ln \left(\beta + \frac{(s - \ln(\beta))}{b} \right) + \frac{(s - \ln(\beta))}{b}. \quad (4.56)$$

The contour line corresponding to $\sigma = -1$ satisfies the equation $\tilde{D} = -tr$. After substituting the values of tr and $\tilde{D}$ at the fixed point E_1 we obtain the following:

$$\left(\frac{\beta - W(X)(W(X) - \beta)}{W(X)} \right) \left(\frac{e^{s-b(W(X)-\beta)}}{\beta} \right) + 1 = - \left(\frac{\beta - W(X)(W(X) - \beta)}{W(X)} + \frac{e^{s-b(W(X)-\beta)}}{\beta} \right).$$

Simplifying the above equation, we get the following equations:

$$\frac{\beta - W(X)(W(X) - \beta)}{W(X)} = -1, \quad (4.57)$$

and,

$$\frac{e^{s-b(W(X)-\beta)}}{\beta} = -1. \quad (4.58)$$

Solving Eq.(4.57), we obtain

$$W(X) = \frac{(1 + \beta) \pm \sqrt{(1 + \beta)^2 + 4\beta}}{2}.$$

From the definition of the Lambert W function and $X = e^{r+\beta}$, the flip bifurcation curves of the $F(x,y)$ map are given as follows:

$$e^r = \left(\frac{(1+\beta) \pm \sqrt{(1+\beta)^2 + 4\beta}}{2} \right) e^{\left(\frac{(1+\beta) \pm \sqrt{(1+\beta)^2 + 4\beta}}{2} \right)}. \quad (4.59)$$

The Eq.(4.58) does not have any feasible solution; therefore, it will not contribute to determining the flip bifurcation curve corresponding to E_1 . The fold and flip bifurcation curves of the 2D Homographic map $F(x,y)$ are given by Eq.(4.56) and Eq.(4.59) respectively.

Similarly, the fold and flip bifurcation curve corresponding to the fixed point E_2 are obtained as

$$s = \ln \left(\beta + \frac{(r - \ln(\beta))}{a} \right) + \frac{(r - \ln(\beta))}{a}, \quad (4.60)$$

and

$$e^s = \left(\frac{(1+\beta) \pm \sqrt{(1+\beta)^2 + 4\beta}}{2} \right) e^{\left(\frac{(1+\beta) \pm \sqrt{(1+\beta)^2 + 4\beta}}{2} \right)}. \quad (4.61)$$

The fold bifurcation curves corresponding to the trivial fixed point E_0 are given as

$$r = \ln(\beta). \quad (4.62)$$

$$s = \ln(\beta) \quad (4.63)$$

The map $F(x,y)$ does not have any flip bifurcation curves associated with E_0 , because the first quadrant is invariant and if the map $F(x,y)$ undergoes the flip bifurcation then the period 2 point generated will be outside of the first quadrant which is not possible. We discuss the fold and flip bifurcation curves associated with different fixed points of the map $F(x,y)$ in the $\beta - r$ and $\beta - s$ parameter planes. Further, the fold and flip bifurcation curve enables us to find the stability region associated with the fixed points.

$\beta - r$ Parameter plane: Assume that s, a, b are constants and taking r, β as positive variable parameters. The fold bifurcation curves corresponding to E_0, E_1, E_2 and flip bifurcation curve corresponding to E_1 in the $\beta - r$ parameter plane are shown in Figure 4.5. From Eq.(4.56), Eq.(4.60) and Eq.(4.62), the fold bifurcation curves $\Lambda_{(1)_0}^{0,1}, \Lambda_{(1)_0}^{1,1}$ and $\Lambda_{(1)_0}^{2,1}$ associated with the fixed points E_0, E_1 , and E_2 respectively intersects at the point (e^s, s) . Since we assume that all of the parameters are positive and the flip bifurcation curve associated with the negative sign in Eq. (4.59) does not lie in the positive quadrant of the $\beta - r$ plane. As a result, the curve in Eq. (4.59) that corresponds to the negative sign is ignored. The stability region of the fixed point E_1 is denoted by $R_{stable}(E_1)$, which is enclosed by $\Lambda_{(1)_0}^{0,1}, \Lambda_{(1)_0}^{1,1}$ and $\Lambda_1^{1,1}$ curves. In the $\beta - r$ parameter plane, the (e^s, s) point is characterized as a singular point of codimension greater than two.

$\beta - s$ Parameter plane: Assume that r, a, b are constants and taking s, β as positive variable parameters. The fold bifurcation curves corresponding to E_0, E_1, E_2 and flip bifurcation curve corresponding to E_1 in the $\beta - s$ parameter plane are shown in Figure 4.6. From Eq.(4.56), Eq.(4.60) and Eq.(4.62), the fold bifurcation curves $\Lambda_{(1)_0}^{0,1}, \Lambda_{(1)_0}^{1,1}$ and $\Lambda_{(1)_0}^{2,1}$ associated with the fixed points E_0, E_1 , and E_2 respectively intersects at the point (e^r, r) . Since we assume that all of the parameters are positive and the flip bifurcation curve associated with the negative sign in Eq.(4.61) does not lie in the positive quadrant of the $\beta - s$ plane. As a result, the curve in Eq.(4.61) that corresponds to the negative sign is ignored. The stability region of the fixed point E_2 is denoted by $R_{stable}(E_2)$, which is enclosed by $\Lambda_{(1)_0}^{0,1}, \Lambda_{(1)_0}^{2,1}$ and $\Lambda_1^{2,1}$ curves. In the $\beta - s$ parameter plane, the (e^r, r) point is characterized as a singular point of codimension greater than two.

The above analysis of the fold and flip bifurcation curves leads to the following proposition.

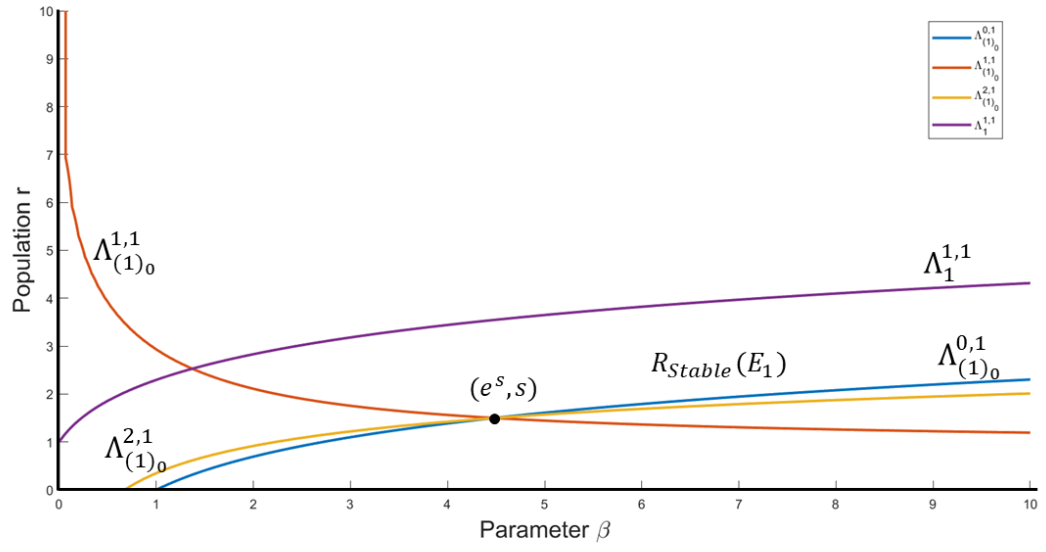


Figure 4.5 : Fold and flip bifurcation curves of $F(x,y)$ map in the $\beta - r$ parameter plane at $s = 1.5$, $a = 0.4$, $b = 0.8$.

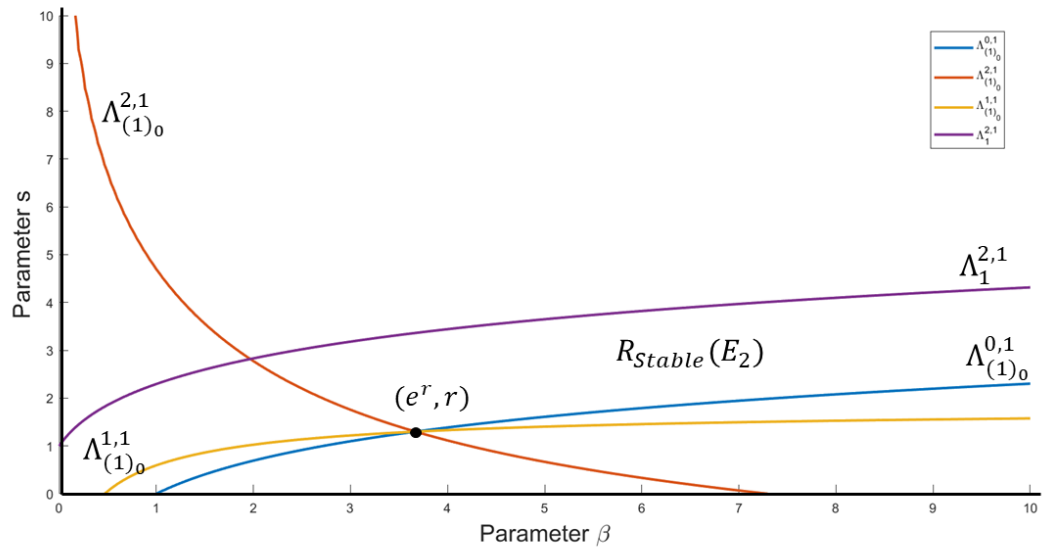


Figure 4.6 : Fold and flip bifurcation curves of $F(x,y)$ map in the $\beta - s$ parameter plane at $r = 1.3$, $a = 0.4$, $b = 0.8$.

Proposition 4.5.2. *Let $F(x,y)$ be the 2D Homographic map given by Eq.(4.3).*

(i) *In the $\beta - r$ parameter plane at the constant values of s, a, b , then*

$$\Lambda_{(1)_0}^{0,1} \cap \Lambda_{(1)_0}^{1,1} \cap \Lambda_{(1)_0}^{2,1} = \{(e^s, s)\}.$$

(ii) *In the $\beta - s$ parameter plane at the constant values of r, a, b , then*

$$\Lambda_{(1)_0}^{0,1} \cap \Lambda_{(1)_0}^{1,1} \cap \Lambda_{(1)_0}^{2,1} = \{(e^r, r)\}.$$

4.6 NUMERICAL SIMULATIONS

In this section, we give some numerical simulation results to illustrate the theoretical results related to the stability of the positive fixed point and the bifurcation analysis with respect to the parameter β . With the help of bifurcation analysis, we show some interesting dynamics of the 2D Homographic map. To perform the numerical simulations, we assume the carrying capacities r and s of the map $F(x,y)$ in the interval $(0,4]$ and discuss their dynamics. With the help of bifurcation diagrams, one can observe reverse period doubling in the system (4.3).

Example 4.6.1. *Consider $\beta = 2.4$, $r = 1.5$, $s = 1.7$, $a = 0.7$, and $b = 0.65$, then the coexisting fixed point of the 2D Homographic map $F(x,y)$ is obtained as $(x^*, y^*) = (0.192, 0.508)$. The characteristic polynomial of the Jacobian at (x^*, y^*) is obtained as*

$$\lambda^2 - 1.05124\lambda + 0.188503 = 0,$$

with eigenvalues $\lambda_1 = 0.822035735$ and $\lambda_2 = 0.229354266$. Both the eigenvalues lie within the unit disc. Hence the fixed point (x^*, y^*) is local asymptotically stable. Figure 4.7(a)-(b) shows the time series graph of the population x_n, y_n respectively, and Figure 4.8 shows that the orbits approaching the locally asymptotic stable fixed point.

Example 4.6.2. *Consider the 2D Homographic map $F(x,y)$ at the specific parameters: $r = 2.84$, $s = 1.7$, $a = 0.6$, $b = 0.4$, $\beta \in [1, 6]$. The bifurcation diagrams of the species x and y w.r.t the parameter β are shown in Figure 4.9 (a) and (b), respectively. As the parameter β passes through the critical value $\beta_1 = 1.7633$, the stable period 2 points lose their stability via reverse period doubling, and the unique coexisting equilibrium point gains stability. At $\beta_1 = 1.7633$, the unique coexisting equilibrium point is $E_3 = (1.4533, 0.3639)$. The characteristic equation of the Jacobian matrix of the map $F(x,y)$ at E_3 is given by*

$$\lambda^2 + 0.440082\lambda - 0.54783 = 0. \tag{4.64}$$

The roots of Eq.(4.64) are $\lambda_1 = -1$ and $\lambda_2 = 0.559918$. Thus the parameters $(2.84, 1.7, 0.6, 0.4, 1.7633) \in \Psi_{FB}$.

Hence, the 2D Homographic map undergoes a flip bifurcation, as shown in the theoretical analysis. The maximum Lyapunov exponent diagram of the system (4.3) at the specific parameters mentioned in Example 4.6.2 is shown in Figure 4.10.

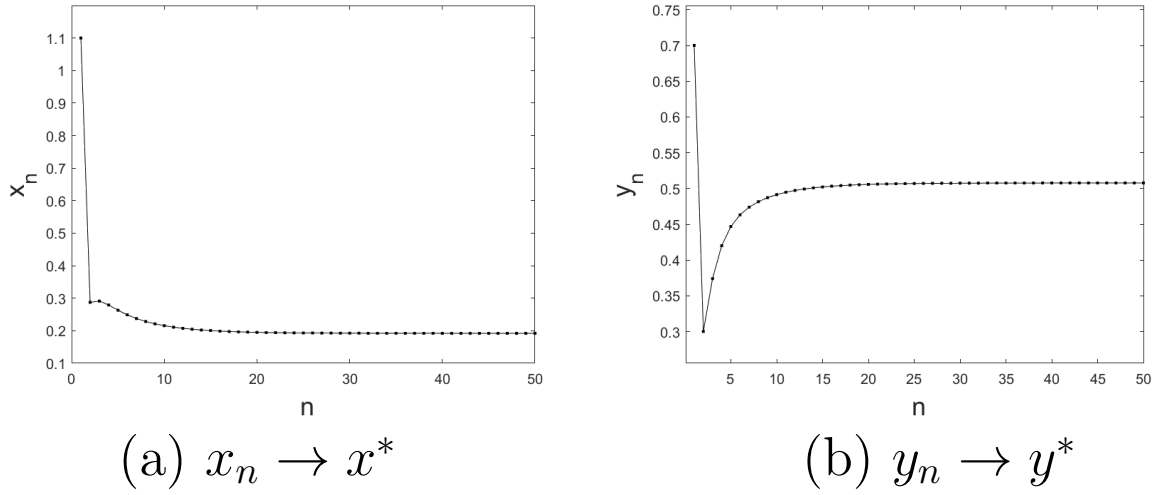


Figure 4.7 : Time series diagram of populations $\{x_n\} \rightarrow x^*$ and $\{y_n\} \rightarrow y^*$ of the map $F(x,y)$ at parameters $r = 1.5$, $s = 1.7$, $a = 0.7$, $b = 0.65$, and $\beta = 2.4$.

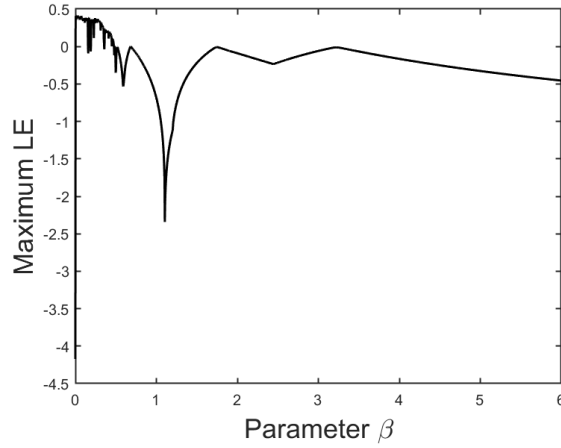


Figure 4.10 : Maximum Lyapunov exponent of map $F(x,y)$ at parameters $r = 2.84$, $s = 1.7$, $a = 0.6$, $b = 0.4$ with respect to the parameter β .

Further, we discuss the general bifurcation analysis of the system (4.3). This section focuses on the bifurcation analysis of the map $F(x,y)$ with respect to parameter β . Note that in the absence of y-species, the 2D Homographic map behaves as a one-dimensional (1D) Homographic map; therefore, their behavior is the same. The 1D Homographic Ricker map is given as

$$x_{n+1} = \frac{x_n}{x_n + \beta} e^{r-x_n},$$

where r is the carrying capacity of x species and β is the half-saturation constant of the Holling type II functional. The bifurcation diagram of the 1D Homographic map with respect to the parameter β is shown in Figure 4.11(a) and its zoomed part in Figure 4.11(b). The diagram shows that the

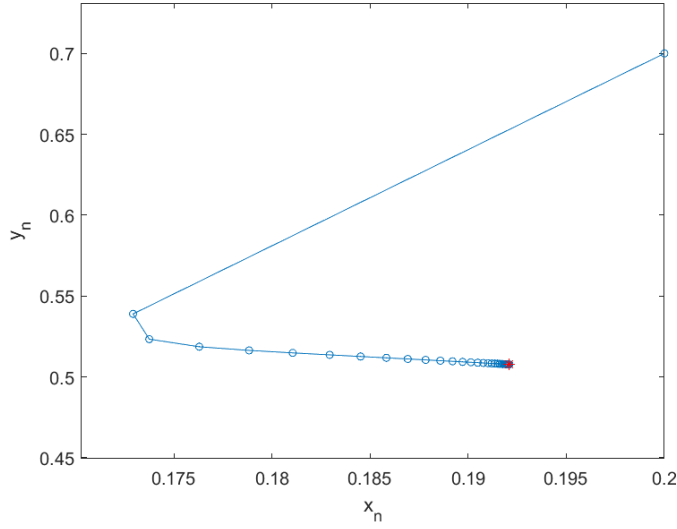


Figure 4.8 : Example of a trajectory of a locally stable coexisting positive fixed point of the map $F(x,y)$ at parameters $r = 1.5$, $s = 1.7$, $a = 0.7$, $b = 0.65$, and $\beta = 2.4$.

map exhibits chaotic behavior at the small β values and takes a reverse period doubling route to reach an equilibrium state.

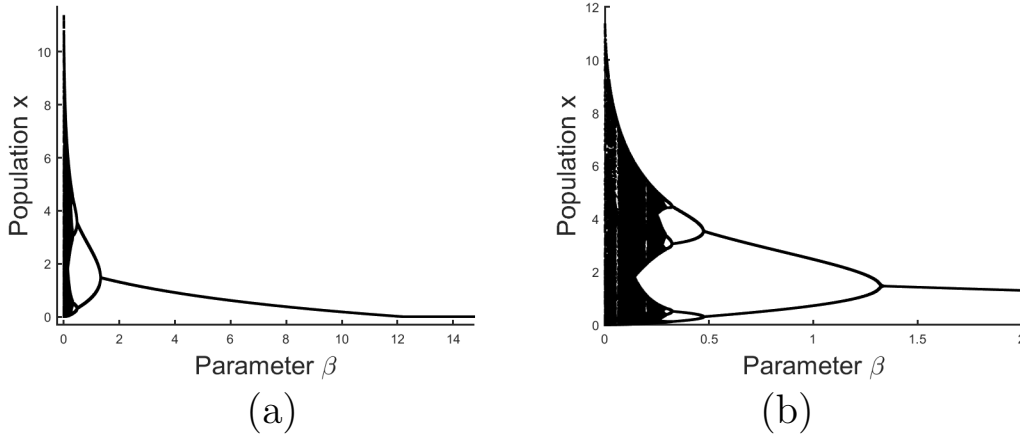


Figure 4.11 : Bifurcation diagram of 1D Homographic Ricker map with respect to parameter β (a) $r = 2.5$, (b) Zoomed part.

After being at the equilibrium position, there exists some $\beta = \beta^*$ such that for $\beta > \beta^*$, the x population goes towards extinction. As the carrying capacity r increases, i.e., $r_1 < r_2 < r_3 < r_4 \dots < r_n$ then the corresponding β_i^* after which the population goes to extinction is also increases, i.e., $\beta_1^* < \beta_2^* < \beta_3^* < \beta_4^* < \dots < \beta_n^*$. The biological meaning of the aforementioned statement is that the increase in carrying capacity r allows the x population to persist for a large range of β values. Now, we discuss the changes in the 2D Homographic map in the presence of y species. The bifurcation of the map $F(x,y)$ will be discussed in two possible cases: (i) when the competition between x and y

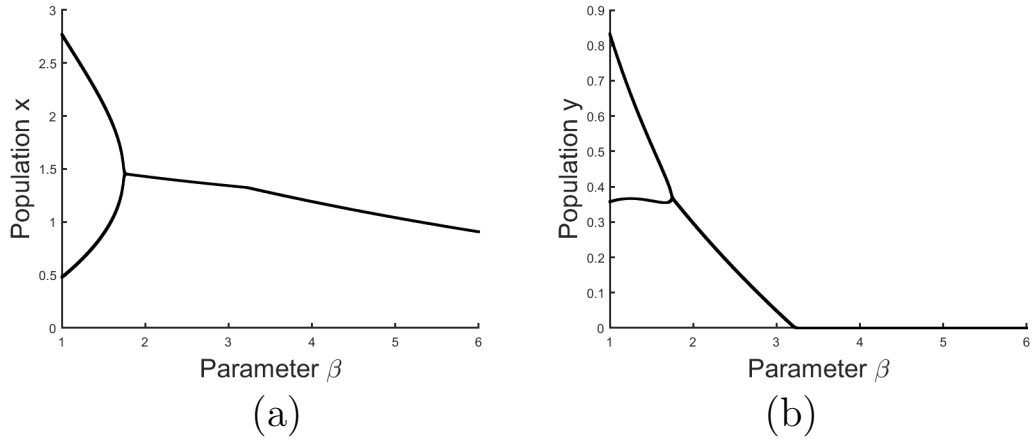


Figure 4.9 : Flip bifurcation diagrams of the x and y species of the map $F(x,y)$ at the parameters $r = 2.84$, $s = 1.7$, $a = 0.6$, $b = 0.4$ with respect to parameter β .

species are equal, i.e., $a = b$. (ii) when the competition between x and y species are not equal, i.e., $a \neq b$.

Case (i): Consider the map $F(x,y)$ at the equal competition parameters ($a = b = 0.5$). Bifurcation diagrams of x and y species with respect to the parameter β are shown in Figure 4.15 (Appendix), and from the diagrams, we have the following observations:

- (i) At small value of r say r_1 , the x population is at the equilibrium state and after a certain $\beta = \beta_1^*$ it goes to extinction. At the same time, with a constant carrying capacity of y , say s_1 , the y population is also at an equilibrium state, and at certain $\beta = \hat{\beta}_1$, the y population goes to extinction.
- (ii) For $r_2 > r_1$, x species exhibit chaotic behavior and undergoes a reverse period doubling route to reach the equilibrium state and undergo extinction for some $\beta_2^* > \beta_1^*$. With the increase in the r value, the x population shows chaotic behavior and survives for a larger range of β as shown in Figure 4.15(c). Correspondingly, at the same carrying capacity s_1 , the increase in r value changes the dynamics of y species drastically. The y population shows chaotic behavior and goes to extinction at smaller $\hat{\beta}_2 < \hat{\beta}_1$, shown in Figure 4.15(d).
- (iii) With the increase in r values the chaotic region of the x population is also increases. While at the same time, the chaotic region of y species decreases, as shown in Figure 4.15(e).
- (iv) For a sufficiently high value of r , Figure 4.15(f) illustrates that the y species shows chaotic behavior and then faces the interior crisis, which leads to direct extinction.
- (v) The maximum Lyapunov exponents corresponding to the following parameter values: $s = 0.5$, $a = b = 0.5$, $r = 2.5$ and $r = 3.5$ are calculated and plotted in Figure 4.12(a) and (b) respectively.

Case (ii)-(a): Consider the map $F(x,y)$ with different competition circumstances ($a = 0.2, b = 2$). In this scenario, the x species shows the usual reverse period doubling behavior as in the case of $a = b$, but the y species exhibits unusual behavior.

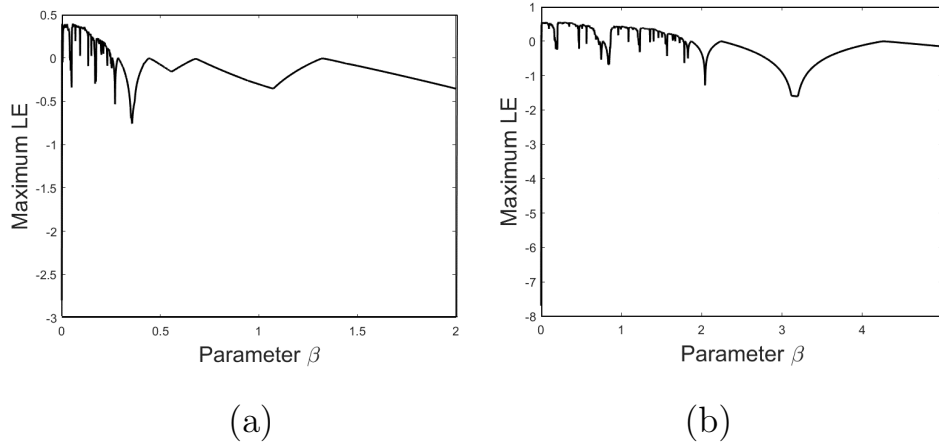


Figure 4.12 : Maximum Lyapunov exponent (LE) of the map $F(x,y)$ at parameters $a = 0.5$, $b = 0.5$, $s = 0.5$ and different r values with respect to parameter β .

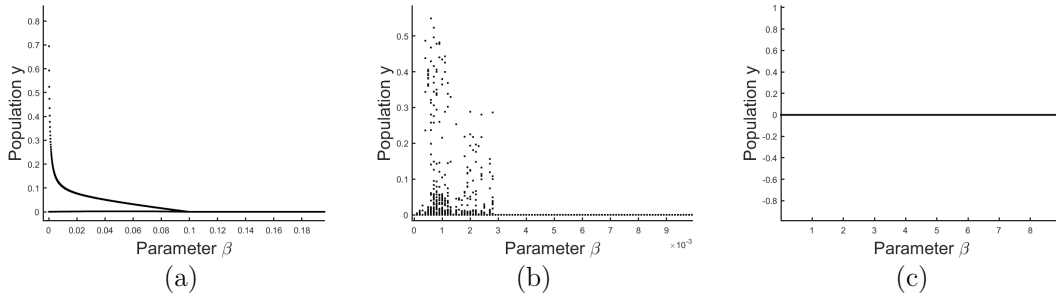


Figure 4.13 : Bifurcation diagrams showing the changes in the y species of $F(x,y)$ at parameters $s = 0.5$, $a = 0.2$, $b = 2$ values with increasing r values.

At the small value of r and constant s value, the y population is at an equilibrium state with respect to the bifurcation parameter β , but as the r value increases, the y population survives for less and less generation. For a sufficiently large value of r , the y population becomes extinct, as shown in Figure 4.13. With this type of competition in which ($a < b$), the x population survives for a large range of β value while the y population survives for a very small range of β . The y population can survive if we increase the carrying capacity of y species, too.

Case (ii)-(b): Consider the map $F(x,y)$ with the following competition parameters $a = 2$, $b = 0.2$. In this situation, the x population follows the regular reverse period-doubling bifurcation before being extinct. In this route, the x species undergoes a bubbling process. Bubbling is a process in which the period doubling route to reach the chaos is broken in between, and the system immediately takes the reverse period doubling route. In this process, a bubble-like structure is formed. Clearly, the bubble is formed between two β values, say β_1 and β_2 , such that the derivative of the function takes -1 value at $\beta = \beta_i$ for $i = 1, 2$. Table 4.1 shows the parameter values in which the chaotic, secondary, and primary bubble is formed. Figure 4.14(a), (b), and (c) shows the presence of a chaotic bubble, secondary bubble, and primary bubble, respectively, in the bifurcation diagram of

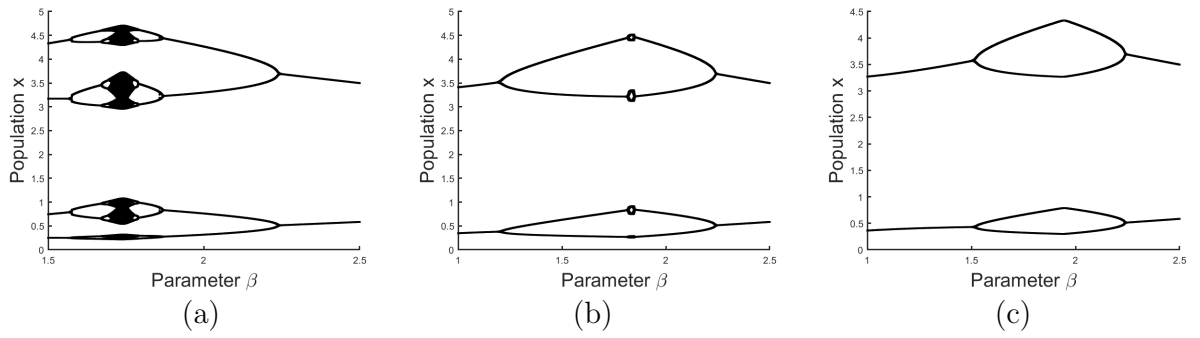


Figure 4.14 : Bubbling in the x species at the parameter $r = 3.5$, $a = 2$, $b = 0.2$ and different s values.

Nature of bubble	β_1	β_2
Chaotic Bubble	1.567	1.87
Secondary Bubble	1.197	2.241
Primary Bubble	1.507	2.249

Table 4.1 : The nature of bubble formed between β_1 and β_2 .

x species. More about the bubbling process in the population model can be found in [Bier and Bountis, 1984; Liz, 2010; Liz and Hilker, 2012; Zipkin et al., 2009]. Here, the y species also survive but does not follow the reverse period doubling route to reach the equilibrium state.

4.7 CONCLUSION

In this work, we have discussed the nonlinear dynamics of the proposed discrete-time population model, namely the 2D Homographic Ricker map. The map is obtained by introducing a Holling type II functional response as a growth function in the classical 2D Ricker-type competition model. We showed the boundedness of the solutions and the uniqueness of the coexisting fixed point of the proposed map. With the help of critical curves and singular points, we discussed the geometry of the map and proved that all the points in the domain of the 2D Homographic Ricker map are either regular, fold, or cusp in nature. Furthermore, we used the centre manifold theory to explain the local stability of the fixed points of the map $F(x,y)$. Using bifurcation theory, we obtained some conditions under which the map exhibits the flip bifurcation. Also, we discussed the fold and flip bifurcation curves associated with different fixed points of the 2D Homographic map. Finally, with the help of numerical simulations, we illustrated the stability of the coexisting positive fixed point and the bifurcation analysis of the competing x and y species.

4.8 APPENDIX

The bifurcation diagrams of the map $F(x,y)$ at different parameters are given as follows:

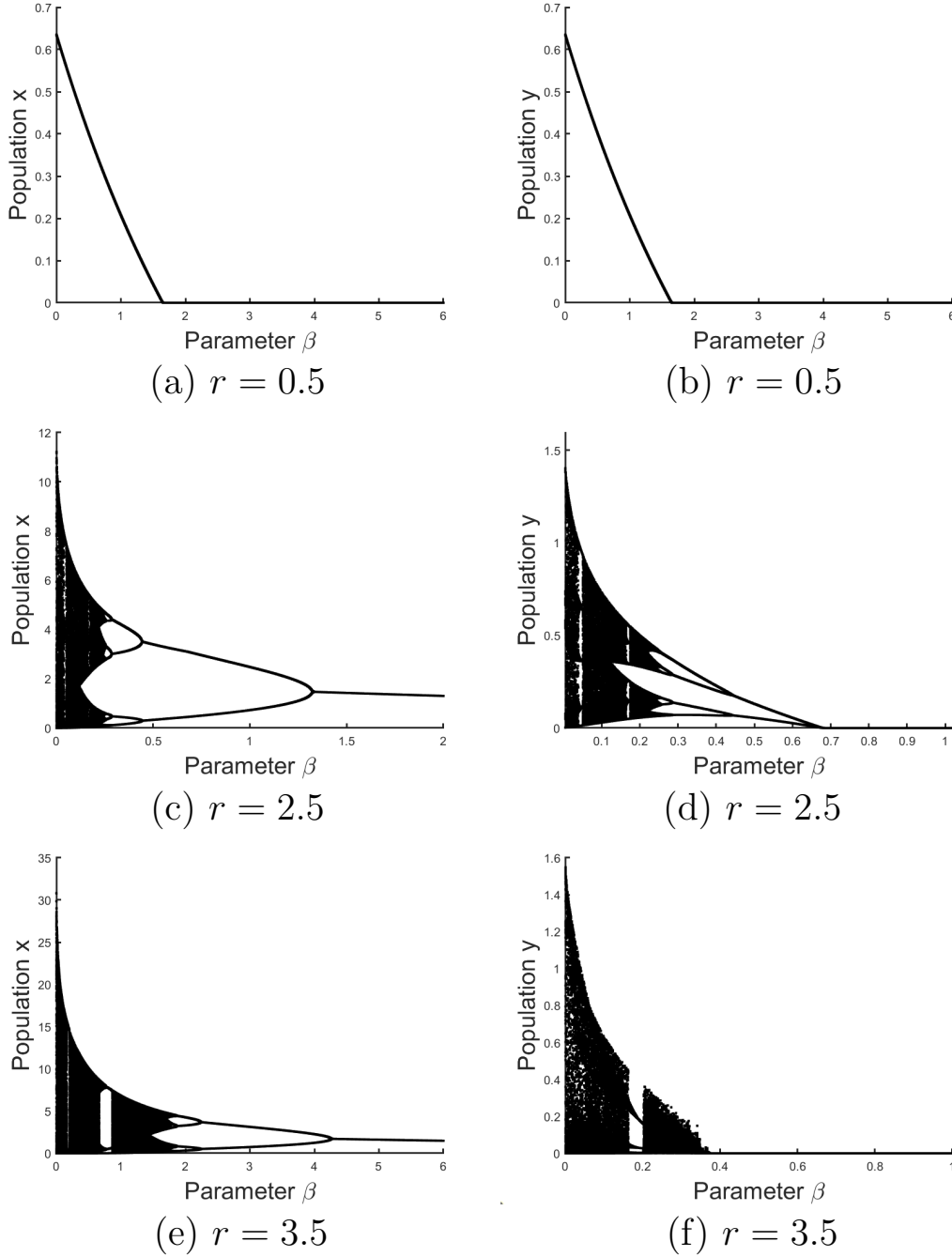


Figure 4.15 : Bifurcation diagram of x and y species of the map $F(x,y)$ with respect to the parameter β at $s = 0.5$, $a = 0.5$, $b = 0.5$ and different r values.