

Declaration

I hereby declare that the work presented in this thesis titled “*Renormalizations of Unimodal and Bimodal Maps with Low Smoothness*” submitted to the Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of *Dr. V. V. M. Sarma Chandramouli*. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to any other Institute or University in India or abroad for the award of any degree or diploma.

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Certificate

This is to certify that the thesis titled “*Renormalizations of Unimodal and Bimodal Maps with Low Smoothness*”, submitted by *Rohit Kumar (P15MA002)* to the Indian Institute of Technology Jodhpur for the award of the degree of *Doctor of Philosophy*, is a bonafide record of the research work done by him under my supervision. To the best of my knowledge, the contents of this report, in full or in parts, have not been submitted to any other Institute or University in India or abroad for the award of any degree or diploma.

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List of Symbols

Symbols	Description
$I = [a, b]$	a closed interval I with end-points a and b
I°	the interior of I
$ I $	Lebesgue measure of I
∂I	the boundary of I
$A \setminus B$	set difference of A and B
$d(x, y)$	the Euclidean distance between x and y
$Dom(f)$	domain of a map f
f^n	n fold composition of f with itself
$D^k f(c)$	k^{th} derivative of f at a point c
$Lip(f)$	Lipschitz constant of f
$h_{top}(f)$	topological entropy of f
σ	the shift map
$\begin{pmatrix} u \\ v \end{pmatrix}$	two-dimensional point

