Towards More Realistic Shock Models with Applications in Optimal Maintenance

A Thesis submitted by

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In partial fulfilment of the requirements for the award of the degree of

Doctor of Philosophy



Indian Institute of Technology Jodhpur

Department of Mathematics

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Dedicated To

My Brother "Rakesh Goyal"

Declaration

I hereby declare that the work presented in this thesis entitled "**Towards more realistic shock models with applications in optimal maintenance**" submitted to the Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of **Dr. Nil Kamal Hazra**. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to, any other Institute or University in India or abroad for the award of any degree or diploma.



Dheeraj Goyal
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Certificate

This is to certify that the thesis entitled **"Towards more realistic shock models with applications in optimal maintenance"** being submitted by **Dheeraj Goyal** to the Indian Institute of Technology Jodhpur, India, for the award of degree of Doctor of Philosophy in Science is a record of bonafide research work carried out by him under my supervision during the last four years. The results embodied in this thesis, to the best of my knowledge, have not been submitted for the award of any other degree/diploma anywhere. In my opinion, the thesis fulfils the requirement for the award of the degree.

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PhD Student

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List of Notation

X	underlying nonnegative random variable
$f_X(\cdot)$	probability density function of X
$F_X(\cdot)$	cumulative distribution function of X
$\bar{F}_X(\cdot)$	survival (reliability) function of X
$E(X)$	expectation (mean) of X
$r_X(\cdot)$	failure (hazard) rate function of X
$m_X(\cdot)$	mean residual (remaining) life function of X
$L_X(\cdot)$	Laplace transform of X
$\lfloor \cdot \rfloor$	floor function
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
i	$\sqrt{-1}$
$Re(z)$	real part of a complex number z
$Im(z)$	imaginary part of a complex number z
$\Gamma(z)$	Gamma function with argument z
$\|x\|$	absolute value of x
$\ln(x)$	logarithm of x with base e
I	identity matrix
$\|S\|$	number of elements in set S
$A = [A_{ij}]$	a matrix where A_{ij} represents the ij-th element of A
$A^{-1} = [A_{ij}^{-1}]$	inverse of the matrix A where A_{ij}^{-1} represents the ij-th element of A^{-1}
$\otimes$	Kronecker product
$A \otimes B = [A_{ij}B]$	Kronecker product of matrices A and B
$diag(d_1, d_2, \ldots, d_m)$	diagonal matrix of size m with i-th diagonal entry equal to d_i, $i = 1, 2, \ldots, m$
$a \stackrel{\text{def.}}{=} b$	b is defined as a
$\{M(t) : t \geq 0\} \sim NHPP(\lambda(t))$	$\{M(t) : t \geq 0\}$ follows the NHPP with rate $\lambda(t)$

List of Abbreviations

Abbreviation	Full form
st	usual stochastic
hr	hazard rate
lr	likelihood ratio
mrl	mean remaining lifetime
lt	Laplace transform
lt-r	Laplace transform ratio
r-lt-r	reversed Laplace transform ratio
d-lt-r	differentiated Laplace transform ratio
comp	component
pdf	probability density function
pmf	probability mass function
pgf	probability generating function
cdf	cumulative distribution function
mgf	moment generating function
iid	independent and identically distributed
ILR	increasing likelihood ratio
DLR	decreasing likelihood ratio
IFR	increasing failure rate
DFR	decreasing failure rate
DMRL	decreasing mean residual life
IMRL	increasing mean residual life
PH	phase-type
DPH	discrete phase-type
ML	Milttag-Leffler
MML	matrix Mittag-Leffler
HPP	homogeneous Poisson process
NHPP	non-homogeneous Poisson process
GPP	generalized Pólya process
PLP	Poisson-Lindley process
PGGP	Poisson generalized gamma process
HPGGP	homogeneous Poisson generalized gamma process
PPHP	Poisson phase-type process
FHPP	fractional homogeneous Poisson process
RPMML	renewal process of matrix Mittag-Leffler type