CHAPTER 5
Discussion and Conclusions

The mathematical analysis of the optical bent waveguides model was of interest to the thesis work. The analysis was based on the operator theoretic setting defined on the square integrable norm space.

The straight waveguide model is mathematically modeled as a self-adjoint eigenvalue problem with a discontinuous differential coefficient defined on the infinite domain. The spectral theory of the self-adjoint operator is already well developed; we can use this theory to analyze the straight waveguide model. A semi-analytic study of the bent waveguide model indicates that the eigenvalue problem for this model was not self-adjoint. It is a non-self-adjoint eigenvalue problem with discontinuous differential coefficients defined on the semi-infinite domain. The bent waveguide model is not analyzed mathematically because the associated operator does not exhibit the self-adjoint property. This is mainly because there is no established general mathematical theory to analyze models whose eigenvalue problem is non-self-adjoint.

For the bent waveguide EVP analysis, it was first proved that the operator was non-self-adjoint. Since we have access to the adjoint operator, this information was used to establish the non-real eigenvalues of this operator by showing the imaginary part of the eigenvalue is non-zero. This adjoint operator was also used to express the operator in terms of the sum of the Hermitian and skew Hermitian parts.

For the bent waveguides non-self-adjoint problem, we studied the dependency between the nature of the eigenvalues and the bent radius parameter. Based on this parameter, the nature of the eigenvalues changes. Not only does the nature of the eigenvalues change, but this non-self-adjoint problem also transforms into a self-adjoint problem when the bent radius is large. In the underlying physics, when this parameter is large, the bent waveguide model transforms into the equivalent straight waveguide model.

Next, the natural question, as in the case of the self-adjoint operators, is the orthogonality of the eigenfunctions corresponding to the distinct eigenvalues. The semi-analytic study shows that eigenfunction will be orthogonal. It is found that in the case of some non-self-adjoint problems, the eigenfunctions corresponding to the distinct eigenvalues are orthogonal or sometimes not [Kumar *et al.*, 2024]. This leads to a separate analysis for each problem. For the bent waveguide problem, we found that eigenfunctions are orthogonal even though the eigenvalues were non-real. We proved this using the adjoint operator.

The boundedness of the operator involved in the bent waveguide problem and the boundedness of eigenvalues are also questions that lead to the nature of the model, such as stability or instability etc. We prove the boundedness of the non-real eigenvalues by showing the boundedness of the real and imaginary parts, i.e., the boundedness of the propagation constant. The asymptotic behavior of the eigenfunctions is also proved using the approximation theory. That leads to the asymptotic behavior of the mode of the bent waveguide problem in the cladding region, i.e., radiating energy in the cladding region.

The bent waveguide problem is sensitive based on the behavior of the bent radius as the

nature of the eigenvalues changes depending on the bent radius. In the future, we will analyze the pseudospectra of this problem to analyze the model stability. We will also explore the compactness of the bent waveguide operator and its consequences. Another important question is how to rigorously characterize the domain of the governing elliptic operator in the bent waveguide eigenvalue problem for waveguide modes, to better reflect the mathematical structure of the bent waveguide model. Specifically, this involves identifying an appropriate functional setting, or a subspace thereof, that accounts for the regularity of the solution, complexities due to curvature, and the enforcement of physically meaningful boundary conditions. A careful specification of the operator domain is critical for the spectral analysis and well-posedness of the model and the related study of adjointness.

But still try, for who knows what is possible? - Michael Faraday