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Controllability of corona product of signed graphs

Complex networks, which include systems like neural and transportation networks, pose significant challenges in management and control. These challenges are especially pronounced in large-scale systems where traditional controllability tests are computationally intensive. Recent advances have improved our understanding of network controllability, including for signed networks, by establishing efficient criteria and guidelines. Graph products, such as Cartesian, Corona, and Kronecker products, are effective tools for constructing and analyzing large signed networks, offering insights through the combination of smaller network components.

In this chapter, we first analyze the controllability of the corona product of two signed graphs. Later we extend it to study the controllability of signed corona graphs in terms of the controllability of the seed graph.

4.1 INTRODUCTION

A signed graph is a mathematical structure used to model systems where interactions between elements (nodes) can be either positive or negative. These nodes could represent diverse entities such as individuals, proteins, or devices, and the edges between them indicate the relationships or interactions, carrying either a positive or negative sign to denote the nature of the interaction. Signed graphs extend the concept of complex networks by introducing an additional layer of signed interactions, which is crucial in various real-world applications, such as social networks (where relationships can be friendly or antagonistic) or biological networks (where interactions can be cooperative or inhibitory).

Many real-world systems, including social, information, and biological networks, have been extensively studied [Almaas *et al.*, 2004; Li and Cai, 2004; Guimera *et al.*, 2005]. Abstract networks, such as those representing scientific collaboration, social interactions, and communication, are also prevalent [Newman, 2001; Carrington *et al.*, 2005; Albert *et al.*, 1999]. A key challenge in network science and engineering is managing these complex networks. Practical engineering problems often frame network controllability issues, such as optimizing a network's layout to efficiently control power distribution with minimal substations [Arianos *et al.*, 2009]. Despite significant research efforts, controlling complex networks remains a considerable challenge. The controllability of such systems refers to the ability to guide them from any initial state to a desired state through external inputs.

In recent years, advances in understanding network controllability for signed networks

have led to the development of various efficient conditions [Liu *et al.*, 2011; Long *et al.*, 2018]. Liu et al. explored the least number of driver nodes required to control directed networks [Liu *et al.*, 2011], which was further extended to determine the minimum set of driver nodes needed for full network control [Yuan *et al.*, 2013]. Research focused on the control input matrix and linear nodal dynamics [Hou *et al.*, 2018], providing guidelines for designing the matrix. Hao et al. introduced a straightforward controllability condition for networks having diagonalizable topology matrices [Hao *et al.*, 2018], extended to non-diagonalizable matrices [Hao *et al.*, 2019]. Group controllability of multi-agent systems with delays was defined [Liu *et al.*, 2016], and necessary criteria for two time-scale multi-agent networks were established using singular perturbation methods [Long *et al.*, 2018]. However, applying these controllability criteria to specialized signed networks still poses difficulties.

The topological structures of networks like the Internet and distribution networks can be represented using the Corona product of two graphs. Previous studies have explored various aspects of the Corona product in graph applications [Gao *et al.*, 2019; Zhang *et al.*, 2019; Liu *et al.*, 2018]. The concept of metric dimension for Corona products, relevant in engineering, was introduced [Kuziak *et al.*, 2013; Aisyah *et al.*, 2019], while coloring problems within Corona product graphs were examined [Huigol and Sariram, 2016; Kaliraj *et al.*, 2018]. Sharma et al. defined Corona graphs as a model for growing networks [Sharma and Adhikari, 2015], with a deterministically growing model for generating small-world networks using the Corona product established [Sharma *et al.*, 2019]. This model was later extended to weighted networks by Qi et al. [Qi *et al.*, 2018]. However, research on the controllability of networks formed by the Corona product is limited.

For a networked system of n agents is represented by

$$\dot{X} = AX + BU \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}$ is a system matrix, $B \in \mathbb{R}^{n\times n}$ is an input matrix, $X \in \mathbb{R}^n$ is state vector representing states of the agents, and $U \in \mathbb{R}^n$ control input vector. The controllability of signed graphs is crucial because it determines whether a desired collective behavior can be achieved in the network, regardless of the initial state, considering the signed nature of the interactions.

Since large signed networks often consist of smaller components, such as communities [Girvan and Newman, 2002] and motifs [Milo *et al.*, 2002], graph products offer a useful approach for constructing signed networks from smaller graphs. This method provides valuable insights into analyzing large-scale signed networks. Graph products include the Cartesian product [Chapman *et al.*, 2014], Corona product [Frucht and Harary, 1970], and Kronecker product [Leskovec *et al.*, 2010a], among others. The concept of the Corona product was introduced by Frucht and Harary in [Frucht and Harary, 1970]. Various topological features of signed networks, such as the profile [Lai and Chang, 2004] and minimum sum [Williams, 1993] of the Corona product, have been examined. Additionally, several studies have investigated the spectra of the Corona product of two graphs, including signed graphs [Barik *et al.*, 2007; McLeman and McNicholas, 2011; Cui and Tian, 2012; Wang and Zhou, 2013; Liu and Zhou, 2014; Chen and Liao, 2017].

Signed corona product is such a method and is defined as follows. Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1, \mu_1)$ and $G^s_{\mu_1} = (V_{\mu_2}, E^s_2, \sigma_2, \mu_2)$ be signed graphs having m and n nodes, respectively. Then their corona product $G^s_{\mu_1} \circ G^s_{\mu_2}$ is a signed graph G^s obtained by taking one copy of $G^s_{\mu_1}$ and m copies of $G^s_{\mu_2}$ and then joining i^{th} node of $G^s_{\mu_1}$ to every node of the ith copy of $G^s_{\mu_2}$, for all i, with a signed edge. The sign of this new edge between i^{th} node $u \in V_{\mu_1}$ and j^{th} node $v \in V_{\mu_2}$, in the ith copy of $G^s_{\mu_2}$, is given by $\mu_1(u)\mu_2(v)$ where μ_i is a marking scheme defined by $\sigma_i, i = 1,2$. Then the number of nodes in G^s are $n(m+1)$ [Adhikari *et al.*, 2023].

A multi-agent network system, a subset of complex netwroks, each agent (represented as a node) interacts with others according to the signed edges of the graph. Consider a multi-agent networked system represented by a signed graph $G^s_\mu = (V_\mu, E^s, \sigma, \mu)$ where the sets $V_\mu = \{v_1, v_2, \ldots, v_n\}$ and $E^s \subset V_\mu \times V_\mu$ represent the agents and the interactions between them, respectively. If the networked system follows the leader-follower framework, the set V_μ can be partitioned into leaders (V^l_μ) and followers (V^f_μ) sets. The interactions among the agents are represented by the adjacency matrix A, and an entry in i^{th} represents the presence of negative/positive relation between v_i and v_j in V_μ. The Laplacian matrix of G^s_μ is given by $L = D - A$. Let the laplacian spectrum of G^s_μ is $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

Lemma 4.1.1. *[Williams and Lawrence, 2007] An LTI control system denoted by a pair of matrices (A, B) is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair (λ, v) of A such that $v^T B = 0$.*

$$\dot{X} = AX + BU \tag{4.2}$$

is uncontrollable if and only if for every left eigenvector y of A, $y^T B \neq 0$, where $A \in \mathbb{R}^{n\times n}$ is a matrix of the system, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $X \in \mathbb{R}^n$ is a state vector of the the system, and $U \in \mathbb{R}^m$ is the control input vector, respectively.

4.2 CONTROLLABILITY OF CORONA PRODUCT OF SIGNED GRAPHS

In this section, we characterize conditions under which the controllability of corona product graph is implied from the controllability and structural properties of underlying signed graphs.

A corona product of two signed graphs $G^s_{\mu_1}$ and $G^s_{\mu_1}$ combines these graphs in a way that introduces new nodes and edges, while the signs of the edges are determined by the product of the signs of the corresponding edges in the original graphs. The corona product creates a larger, more complex network, and understanding its controllability involves studying how the properties of the original signed graphs influence the dynamics of the resulting graph.

The controllability of the corona product graph is governed by the interplay between the controllability of the original signed graphs and the structural properties of the corona product itself. By analyzing the Laplacian matrices of the signed graphs and applying controllability tests, such as the Popov–Belevitch–Hautus (PBH) test, we can derive conditions under which the multi-agent system represented by the corona product graph is controllable.

According to Popov–Belevitch–Hautus (PBH) test, a system $\dot{X} = AX + BU$ is uncontrollable if and only if there exists an eigenpair (λ, v) of A such that $v^T B = 0$. Let the controllability dynamics of the signed graphs G_1^s and G_2^s of order n_1 and n_2 be described by

$$\begin{aligned} &\dot{X}_1 = L_1 X_1 + B_1 U_1, \text{ and} \\ &\dot{X}_2 = L_2 X_1 + B_2 U_2. \end{aligned} \tag{4.3}$$

where $X_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in$, and U_i are the state vector, input matrix, and control input for $i = 1, 2$. Now, the controllability of the multi-agent networked system with graph $G^s = G_1^s \circ G_2^s$ is given by

$$\dot{X} = LX + BU \tag{4.4}$$

where

$$\begin{aligned} L &= \begin{bmatrix} L_1 + mI_n & -\mu[V_2] \otimes diag(\mu[V_1]) \\ -\mu[V_2]^T \otimes diag(\mu[V_1]) & (L_2 + I_m) \otimes I_n \end{bmatrix} \text{ and} \\ B &= \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \otimes I_n \end{bmatrix}. \end{aligned} \tag{4.5}$$

Theorem 4.2.1. *[Adhikari* et al.*, 2023] Let $G_{\mu_1}^s = (V_{\mu_1}, E_1^s, \sigma_1, \mu_1)$ be a signed graph of order n and $G_{\mu_2}^s = (V_{\mu_2}, E_2^s, \sigma_2, \mu_2)$ be a signed graph on m nodes. Let $V_{\mu_2} = \{v_1, v_2, \ldots, v_m\}$. Let (λ_i, X_i) be a signed Laplacian eigenpair of $G_{\mu_1}^s$, and (η_j, Y_j) are signed Laplacian eigenpair of $G_{\mu_2}^s$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. And let $d^- = d_j^-$ is the negative degree of $v_j \in V_{\mu_2}$. Then a signed Laplacian eigenpair of $G_{\mu_1}^s \circ G_{\mu_2}^s$ is given by $(\gamma_\pm^{(i)}, Z_\pm^{(i)})$ where*

$$\begin{aligned} \gamma_\pm^{(i)} &= \frac{2d^- + 1 + m + \lambda_i \pm \sqrt{((2d^- + 1) - (m + \lambda_i))^2 + 4m}}{2} \\ Z_\pm^{(i)} &= \left[\mu[V_{\mu_2}] \otimes \frac{X_i}{\gamma_\pm^{(i)} - (2d^- + 1)} diag\left(\mu[V_{\mu_1}]\right) X_i \right] \end{aligned} \tag{4.6}$$

for $i = 1, 2, \ldots, n$. Let $\eta_m = 2d^-$. In addition, if all the nodes in $G_{\mu_2}^s$ are homogeneously marked (either positively or negatively) then the eigenpairs of $G_{\mu_1}^s \circ G_{\mu_2}^s$ are given by

$$\left(\eta_j + 1, \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix} \right), \; i = 1, 2, \ldots, n, j = 1, 2, \ldots, m - 1 \tag{4.7}$$

where e_i is the standard basis in $\mathbb{R}^n$.

The controllability of the graph obtained by the corona product of a pair of signed graphs, according to [Adhikari *et al.*, 2023], is given by the following theorem.

Theorem 4.2.2. *Let $G_{\mu_1}^s = (V_{\mu_1}, E_1^s, \sigma_1, \mu_1)$ be a signed graph of order n and $G_{\mu_2}^s = (V_{\mu_2}, E_2^s, \sigma_2, \mu_2)$ be a signed graph of order m. The corona product graph $G^s = G_{\mu_1}^s \circ G_{\mu_2}^s$ is controllable if and only if*

- *$G_{\mu_2}^s$ is controllable*
- *if there exists a common eigenvalue of $G_{\mu_1+}^s$ and $G_{\mu_2}^s$ where $\tau = \gamma^{(t)} = \eta_{j_1} + 1 = \cdots = \eta_{j_s} + 1$, then $a_t X_t^T B_1 \neq 0$ $(t < n)$ or $\left(\mu[V_{\mu_2}] \otimes \frac{a_t}{\gamma_\pm^{(i)} - (2d^- + 1)} diag(\mu[V_{\mu_1}]) + \sum_{k=1}^{s} \sum_{i=1}^{n} y_{j_k} \otimes a_{ij_k} e_i \right)^T (B_2 \otimes I_n) \neq 0$, $(j = 2, \ldots, m)$ holds, where $s \geq 1$ is the geometric multiplicity of η_{j_1} and $a_{1j_1}, \ldots, a_{1j_s}, \ldots, a_{nj_1}, \ldots, a_{nj_s}$ are arbitrary coefficients which are not all zero.*

Proof. Based on Lemma 4.1.1, the corona product $G_\mu^s = G_{\mu_1}^s \circ G_{\mu_2}^s$ is controllable if the eigenvectors of $L(G_\mu^s)$ are not in the null space of B. According to Theorem 4.2.1, $L(G_\mu^s)$ will have eigen-pairs $\left(\gamma_\pm^{(i)}, Z_\pm^{(i)}\right)$ and $\left(\eta_j+1, \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix}\right)$ corresponding to the eigenvalues $\lambda_i \in \Gamma(G_1^s)$ and $\eta_j \in \Gamma(G_2^s)$, for $i = \{1,2,\ldots,n\}, j = \{1,\ldots,m-1\}$ respectively. Each $\gamma_\pm^{(i)}$ and η_j+1 has multiplicities 1 and n, respectively. The controllability of G_μ^s can be expressed in the following cases.

1. For an eigenvalue $\lambda_i \in \Gamma(G_1^s)$, consider the corresponding eigen-pair $\left(\gamma_+^{(i)}, Z_+^{(i)}\right)$ in $L(G_\mu^s)$, for $i = 1,2,\ldots,n$. Then, $(Z_+^{(i)})^T B$ is

$$\begin{aligned}(Z_+^{(i)})^T B &= \left[\mu\left[V_{\mu_2}\right] \otimes \frac{X_i}{\gamma_+^{(i)} - (2d^- + 1)} diag\left(\mu\left[V_{\mu_1}\right]\right) X_i\right]^T \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \otimes I_m \end{bmatrix} \\ &= \left[X_i^T B_1 \quad \mu([V_{\mu_2}])^T B_2 \otimes \frac{1}{\gamma_+^{(i)} - (2d^- + 1)} \left(diag\left(\mu\left[V_{\mu_1}\right]\right) X_i\right)^T\right]\end{aligned} \tag{4.8}$$

   If the graph $G_{\mu_1}^s$ is controllable, $X_i^T B_1 \neq 0$. According to the first condition of the theorem, G_2 is controllable, $B_2 \neq 0$ and $\mu([V_{\mu_2}])^T B_2 \neq 0$ and $\left(diag\left(\mu\left[V_{\mu_1}\right]\right) X_i\right) \neq 0$ because it depends only on single eigenvector. So, $(Z_+^{(i)})^T B \neq 0$, for $i = 1,2,\ldots,n$. Hence, Equation 4.8 not equal to 0.

2. The eigenvalue $\eta_j \in L(G_{\mu_2}^s)$ has a corresponding eigenvalue $(\eta_j + 1) \in L(G_\mu^s)$ with multiplicity n, for $j = 1,2,3,\ldots,m-1$. The n eigenvectors corresponding to the eigenvalue $\eta_j + 1 \in L(G_\mu^s)$ can be represented as a linear combination and is given by $Z_j = \sum_{i=1}^n c_i Z_{ij}$, where $Z_{ij} = \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix}$ and $i = /1,2,\ldots,n/$. Thus, according to equation **??**,

$$\begin{aligned}Z_j^T B &= \left[Y_j \otimes \sum_{i=1}^n c_i e_i\right]^T \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \otimes I_n \end{bmatrix} \\ &= \begin{bmatrix} 0 & c_1 Y_j^T B_2 & c_2 Y_j^T B_2 & \ldots & c_n Y_j^T B_2 \end{bmatrix}\end{aligned} \tag{4.9}$$

   Since $G_{\mu_2}^s$ is controllable then we deduce $Z_j^T B_2 \neq 0$. That means there must exist at least one $c_i \neq 0$, such that $Z_j^T B \neq 0$.

3. If there exist $i_1, i_2, \ldots, i_p$, such that $\eta_{i_1} = \eta_{i_2} = \cdots = \eta_{i_p} = \eta \in \Gamma(G_{\mu_2}^s)$. Then let $Z_{i_1}, \ldots, Z_{i_p}$ be the linearly independent eigenvectors corresponding to the eigenvalues $(\eta_{i_1}+1, \eta_{i_2}+1, \ldots, \eta_{i_p}+1)$ in $L(G_\mu^s)$ where $Z_{i_k} = \left[Y_k \otimes \sum_{j=1}^m \mathbf{0} a_{ji_k} e_j\right]$, $k \in \{1,\ldots,p\}$ and scalers $(a_{1i_k}, \ldots, a_{i_k})$ not all are zero. Since, all the p eigenvalues are equal, we can represent $Z_p = \sum_{k=1}^p \sum_{j=1}^n a_{ji_k} Z_{i_k}$ as the eigenvector corresponding to $\eta + 1$ in $L(G_\mu^s)$. Then

$$\begin{aligned}Z_p^T B &= \left[\sum_{k=1}^p \left(Z_{i_k} \otimes \sum_{j=1}^n a_{ji_k} e_j\right)\right]^T \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \otimes I_m \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \sum_{k=1}^p a_{1i_k} Z_{i_k}^T B_2 & \sum_{k=1}^p a_{2i_k} Z_{i_k}^T B_2 & \ldots & \sum_{k=1}^p a_{ni_k} Z_{i_k}^T B_2 \end{bmatrix}\end{aligned} \tag{4.10}$$

   If $G_{\mu_2}^s$ is controllable, $Z_{i_k}^T B_2 \neq \mathbf{0}$ and also $\sum_{k=1}^p a_{ji_k} Z_{i_k}^T B_2 \neq 0$, where all a_{ji_k} are not zero for $k = 1,2,\ldots,p$, $j = 1,2,\ldots,n$. Hence, $Z_p^T B \neq 0$.

4. Let $\gamma_\pm^{(i)}$ and $\gamma_\pm^{(j)}$ are the eigenvalue pairs of $L(G)$ corresponding to λ_i and $\lambda_j \in L(G_{\mu_1}^s)$, respectively. If $\gamma_+^{(i)} = \gamma_-^{(j)}$ (or you may take any two among the four pairs of eigenvalues) then

$$\frac{2d^- + 1 + m + \lambda_i + \sqrt{((2d^- + 1) - (m + \lambda_i))^2 + 4m}}{2} = \frac{2d^- + 1 + m + \lambda_j - \sqrt{((2d^- + 1) - (m + \lambda_i))^2 + 4m}}{2}$$

It implies that λ_i, $\lambda_j < 0$ or do not belong to $L(G^s_{\mu_1})$. This means $\gamma_+^{(i)} = \gamma_+^{(j)}$ suggests $\lambda_i = \lambda_j$.

Therefore, lets consider the case if $\gamma_+^{(i_1)} = \gamma_+^{(i_2)} = \ldots, \gamma_+^{(i_p)} \in L(G)$ then the corresponding eigenvectors are $Z_+^{(i_1)}, Z_+^{(i_2)}, \ldots, Z_+^{(i_p)}$ and since these are linearly independent we can write $Z_+^{(i)} = \sum_{k=1}^{p} a_{i_k} Z_+^{(i_k)}$ where all coefficients a_{i_k} are not zero. Then,

$$\begin{aligned}
(Z_+^{(i)})^T B &= \sum_{k=1}^{p} a_{i_k} (Z_+^{(i_k)})^T B \\
&= \begin{bmatrix} \mu\,[V_2] \otimes \sum_{k=1}^{p} a_{i_k} \dfrac{\sum_{k=1}^{p} a_{i_k} X_i}{\frac{1}{\gamma_+^{(i)} - (2d^- + 1)} diag\,(\mu\,[V_1])\, X_i} \end{bmatrix}^T \begin{bmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & B_2 \otimes I_m \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=1}^{p} a_{i_k} X_i^T B_1 & \mu([V_{\mu_2}])^T B_2 \otimes \sum_{k=1}^{p} a_{i_k} \frac{1}{\gamma_+^{(i)} - (2d^- + 1)} \left(diag\left(\mu\left[V_{\mu_1}\right]\right) X_i\right)^T \end{bmatrix}
\end{aligned} \tag{4.11}$$

If G_{μ_1} is controllable, $\sum_{k=1}^{p} a_{i_k} X_i^T B_1 \neq 0$. If $G^s_{\mu_1}$ is uncontrollable then equation (4.11) is equal to zero if $B_2 \neq 0$ and $(diag\,(\mu\,[V_1])\, X_i) \neq 0$.
The case for $\gamma_-^{(i_1)} = \gamma_-^{(i_2)} = \ldots, \gamma_-^{(i_p)} \in L(G)$ is similar.

5. Let $\lambda_i \in \Gamma(G^s_{\mu_1})$ and $\eta_j \in \Gamma(G^s_{\mu_2})$. Then $\gamma_\pm^{(i)}$ and $\eta_j + 1$ are the eigenvalues of $L(G^s_{\mu})$ corresponding to λ_i and η_j, respectively. If $\gamma_+^{(i)} = (\eta_j + 1)$, then $\eta_j \leq m$ [Hong *et al.*, 1997]. There are different bounds available for largest Laplacian eigenvalue of signed graphs [Belardo *et al.*, 2010]. $G^s_{\mu_1}$ is a connected graph so $\lambda_i = 0$. So, for the common eigenvalues $\tau = \lambda_i = \eta_{j_1} + 1 = \cdots = \eta_{j_s} + 1$ the corresponding eigenvectors are $v_{sr} = v_t a_t + \sum_{k=1}^{s} \sum_{i=1}^{n} v_{ij_k} a_{ij_k}$ where $a_t, a_{1j_1}, \cdots, a_{1j_s}, \cdots, a_{nj_1}, \cdots, a_{nj_s}$ are scalers which all are not zero are free scalars which are not all zero. The vectors $v_{j_1}, \cdots, v_{j_s}$ represent the eigenvectors corresponding to $\eta_{j_1} + 1, \cdots, \eta_{j_s} + 1$, which are linearly independent. So, $v_{sr}^T B$ is given by

$$\begin{aligned}
v_{sr}^T B &= \begin{bmatrix} \mu[V_{\mu_2}] \otimes \frac{a_t}{\gamma_\pm^{(i)} - (2d^- + 1)} diag(\overset{a_t X_t}{\mu[V_{\mu_1}]}) + \sum_{k=1}^{s} \sum_{i=1}^{n} y_{j_k} \otimes a_{ij_k} e_i \end{bmatrix}^T \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \otimes I_m \end{bmatrix} \\
&= \begin{bmatrix} a_t X_t B_1 & \left(a_t \mu\left[V_{\mu_2}\right] \otimes \frac{1}{\gamma_+^{(i)} - (2d^- + 1)} diag\left(\mu\left[V_{\mu_1}\right]\right) X_i + \sum_{k=1}^{s} \left(Z_{i_k} \otimes \sum_{j=1}^{n} a_{ij_k} e_i\right) \right)^T (B_2 \otimes I_n) \end{bmatrix}
\end{aligned}$$

If there does not exist $a_t \neq 0$ such that $a_t X_t B_1 \neq 0$ and hence $v_{sr}^T B \neq 0$. Because the first condition of Theorem 4.2.2 holds, $B_2 \neq 0$. Therefore, the corona product is controllable.

□

4.3 CONCLUSION

Studying the controllability of signed graphs, particularly in the context of their corona product, is essential for understanding the dynamics of complex networks with signed interactions. These insights are vital for applications ranging from social network analysis to biological systems, where the nature of the interactions significantly impacts the overall behavior and controllability of the network.