

Selection of Driver Nodes in Continuous-Time Complex Networks Based on Region of Attraction

5.1 INTRODUCTION

In previous chapter, the correlation between the network structure and sensitivity for a given pair of input-output of the complex networks was presented. A controllable complex network can be driven from an initial state to the desired final state within finite time using a set of driver nodes. The set of driver nodes used to control a networked system may not be unique. There exist multiple sets of driver nodes which may be utilized to control the networked system. It is important to characterize these sets of driver nodes and identify an optimal set of driver nodes. A driver node cannot have infinite (unlimited) actuating capability and thus it is imperative to consider this limitation a priori. The limitations of actuators (driver nodes) creates more complications when the Adjacency matrix (system matrix) has unstable eigenvalues; even stability cannot be guaranteed in this situation. For unstable eigenvalues of the Adjacency matrix of the network, the region of guaranteed stability in state space is finite when limitations of actuators are considered. This region is known as the Region of Attraction (ROA) and described, generally, by ellipsoids in the n -dimensional state space. The ROA may expand or shrink depending on the control law and choice of the set of driver nodes. This raises an important question of identification of an optimal set of driver nodes from a large number of the set of nodes qualified for the set of driver nodes. This chapter proposes mathematical formulation and two algorithms based on the ROA to identify an optimal set of driver nodes. The optimal set of driver nodes is selected to maximize the stability region. The proposed mathematical formulation and algorithms are verified with the help of numerical examples using the MATLAB simulation. The proposed algorithms have also been validated by applying on a robotic network.

The remaining sections of the chapter are organized as follows. Section 5.2 describes the modeling of the complex networks with saturated actuators. Section 5.3 presents a mathematical formulation and algorithms to identify an optimal set of driver nodes of the anti-stable networked systems and transformed anti-stable subsystems with the largest ROA. Simulation results of four networks are included in Section 5.4 followed by a robotic network application in Section 5.5. Finally, concluding remarks of the chapter are included in the Section 5.6.

5.2 MODELING OF COMPLEX NETWORKS WITH SATURATED ACTUATORS

A complex network with n nodes can be represented by a graph $G := (V, E)$, where $V := \{1, 2, \dots, n\}$ and $E \subseteq V \times V$ are the set of nodes and the set of edges respectively. The weighted Adjacency matrix of the network G is described as A , where the elements $a_{ij} \in \mathbb{R}_{>0} \cup \{0\}$ are non zero if node i and j are adjacent, or, are 0 otherwise. The network G is controllable by q sets of driver nodes from the superset $N_D = \{N_1, N_2, \dots, N_q\}$, where each set i.e. N_i , $i = 1, 2, \dots, q$ contains $m \leq n$ nodes and it can control the whole network. The cardinality (number of elements of the set) of a set of driver nodes is fixed for the network G . The superset of control inputs $\bar{U}$ corresponding to the superset of driver nodes N_D is defined as $\bar{U} := \{U_1, U_2, \dots, U_q\}$, where $U_i := \{u_{i1}, u_{i2}, \dots, u_{im}\}$, $i = 1, 2, \dots, q$. The cardinality of U_i is m and U_i is the set of control inputs injected into the network

corresponding to a particular set of driver nodes N_i .

To make the above notation clearer, the aforementioned concept has been illustrated through an undirected cycle network of four nodes (green color) as shown in Figure 5.1.

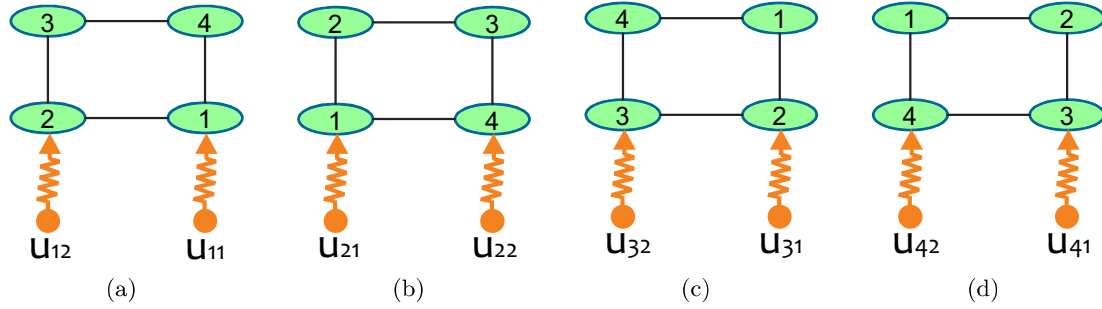


Figure 5.1: Cycle network of four nodes controllable with four sets of driver nodes with set (a) $N_1 = \{1, 2\}$, (b) $N_2 = \{1, 4\}$, (c) $N_3 = \{2, 3\}$, and (d) $N_4 = \{3, 4\}$

In this network, the total number of nodes are $n = 4$, the exact controllability theory [Yuan et al., 2013] predicts that it needs minimum two driver nodes ($m = 2$) to control the network. There are four sets of driver nodes i.e. $N_D = \{N_1, N_2, N_3, N_4\}$, where $N_1 = \{1, 2\}$, $N_2 = \{1, 4\}$, $N_3 = \{2, 3\}$, and $N_4 = \{3, 4\}$, by which the above network is controllable. The set of control inputs corresponding to four independently controllable sets of driver nodes is given by $\tilde{U} = \{U_1, U_2, U_3, U_4\}$, where $U_i \in \mathbb{R}^{2 \times 1}$, $i = 1, 2, 3, 4$. The input sets are defined as $U_1 = \{u_{11}, u_{12}\}$, $U_2 = \{u_{21}, u_{22}\}$, $U_3 = \{u_{31}, u_{32}\}$, and $U_4 = \{u_{41}, u_{42}\}$. The possible choices of sets of control inputs (red color) corresponding to the sets of driver nodes are shown in Figure 5.1. With slight abuse of notation, U_i denotes an m -dimensional vector depending on the context in subsequent presentation, i.e. $U_i := [u_{i1}, u_{i2}, \dots, u_{im}]^T$, where $i = 1, 2, \dots, q$.

Consider a networked system with control input vector U_i , and corresponding input matrix as B_i , with this, continuous-time dynamics of networked systems with saturated actuators can be written as

$$\dot{x}(t) = Ax(t) + B_i \text{sat}_i(U_i(t)) \quad (5.1)$$

where $A \in \mathbb{R}^{n \times n}$ is the Adjacency matrix, $x \in \mathbb{R}^{n \times 1}$ state vector and $U_i \in \mathbb{R}^{m \times 1}$ control input vector of the network G . The network input matrix corresponding to the control input U_i is defined as $B_i \in \mathbb{R}^{n \times m}$ and each column of B_i is i^{th} non-unity vector of n -dimension corresponding to the set of driver nodes N_i , where $i = 1, 2, \dots, q$. The sat_i is a vector valued standard saturation function of appropriate dimensions i.e. $\text{sat}_i: \mathbb{R}^m \rightarrow \mathbb{R}^m$ is defined as $\text{sat}_i(U_i) = [\text{sat}_{i1}(u_{i1}), \text{sat}_{i2}(u_{i2}), \dots, \text{sat}_{im}(u_{im})]$ i.e., $\text{sat}_{ij}(u_{ij}) = \text{sign}(u_{ij}) \min\{u_{ij, \max}, |u_{ij}|\}$, where $u_{ij, \max} > 0$, $\forall i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$ is the saturation limit of the j^{th} component of the control input $U_i \in \mathbb{R}^m$. Without loss of generality, it is assumed that $u_{ij, \max} = 1$, $\forall i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

Remark 5.1. For a single input ($m = 1$) controllable networked system, cardinality of the set of driver nodes is 1, however, driver node may not be unique i.e. multiple choices of driver node may exist.

The control law for the continuous-time system defined in Eq.(5.1) is as follows

$$U_i(t) := -K_i x(t) = -R^{-1} B_i^T P_i x(t) \quad (5.2)$$

where $R > 0$ is a chosen matrix and matrix $P_i > 0$ is obtained by solving the following Continuous-Time Algebraic Riccati Equation (CARE) for some $Q > 0$

$$A^T P_i + P_i A - P_i B_i R^{-1} B_i^T P_i + Q = 0 \quad (5.3)$$

The control law Eq.(5.2) is divided into two parts, low gain and high gain, i.e. $U := U_L + U_H := -\frac{1}{\Phi} R^{-1} B_i^T P_i x - \frac{1}{\Psi} R^{-1} B_i^T P_i x$, where, $\Phi^{-1} + \Psi^{-1} = 1$ and $\Phi, \Psi > 0$. For a special case, the value of $\Phi = 2$ and $\Psi = 2$ are considered in this chapter. Hence, for simplicity, it is being assumed that $U_L = U_H = -\frac{1}{2} R^{-1} B_i^T P_i x$. The low gain part is not allowed to saturate, however, the overall control is allowed [Turner and Postlethwaite, 2001].

Remark 5.2. The chapter discusses low and high gain approach to obtain the estimate of the ROA and performance for the complex networked systems under the Linear Quadratic (LQ) control law [Turner and Postlethwaite, 2001], however, the presented theory is valid for any control law of the form $U = -Kx$.

Under the control law Eq.(5.2), the system Eq.(5.1) is asymptotically stable for all states such that these states lie in the invariant ellipsoid $\mathcal{E}(P_i, \delta_i)$ [Gutman and Hagander, 1985], and is given as

$$\mathcal{E}(P_i, \delta_i) = \{x : x^T P_i x \leq \delta_i\} \quad (5.4)$$

where $\delta_i (> 0)$ is the radius of the invariant ellipsoid $\mathcal{E}(P_i, \delta_i)$ for the system defined in Eq.(5.1).

This radius is obtained by using the low gain part of the control law Eq.(5.2). For continuous-time system Eq.(5.1), the radius of the invariant ellipsoid is given as

$$\delta_i := \min_j \left(\frac{4r_j^2 u_{j,max}^2}{b_j^T P_i b_j} \right) \quad (5.5)$$

where $i = 1, 2, \dots, q; j = 1, 2, \dots, m$, r_j is the j^{th} diagonal entry of chosen matrix $R > 0$, $u_{j,max}$ is the maximum limit of the j^{th} element of the control vector U and b_j is the j^{th} column vector of the input matrix $B_i \in \mathbb{R}^{n \times m}$.

Without loss of generality, assume $u_{j,max} = 1$ and $r_j = 1$ for $j = 1, 2, \dots, m$. Proof of Eq.(5.5) is given in [Gutman and Hagander, 1985; Henrion et al., 1999; Turner and Postlethwaite, 2001; Li and Lin, 2015]. It is obvious from Eqs.(5.3), (5.5), that with different sets of driver nodes, the corresponding input matrix B_i and matrix $P_i > 0$ will change and this eventually changes δ_i and the corresponding ROA.

The ROA of the invariant ellipsoids can be obtained using LHG technique presented in [Turner and Postlethwaite, 2001; Lin, 2009]. The state-feedback LQ control law is used in this technique to obtain an estimation of the ROA. Authors also obtained the semi-global stability using this technique for the linear system with actuator saturation. In [Hu and Lin, 2000], the authors obtained the large basin of attraction and low convergence rate using the LMI approach. This approach is developed for constructing feedback laws and it can be used to increase guaranteed stability and accomplish disturbance rejection [Hu et al., 2002; Hu and Lin, 2002, 2003]. In [Zhou, 2013], the authors obtained the Domain of Attraction (DOA) using LMI method for testing the local as well as global stability of the nonlinear system. In [Li and Lin, 2014], authors extended the work of [Zhou, 2013], and proposed special properties of the inputs which lies in the different regions of the input space. Authors also obtained the better estimation of the DOA and less conservative stability conditions using an existing method such as piecewise quadratic Lyapunov functions presented in [Saber et al., 1996]. In [Li and Lin, 2015], authors described the complete characterization of the

set of maximal invariant ellipsoids based on the algebraic computation. Recently, in [Singh, 2015], the author proposed a method to fix the convergence rate of states using parametric form of matrix P so as to limit the ROA of the networked system. However, this chapter focuses on ROA instead of convergence rate of states of the networked system.

5.3 SELECTION OF OPTIMAL DRIVER NODES

This section is divided into four subsections. The first explains the problem setup, second computes the volume of ROA of anti-stable networked system defined in Eq.(5.1) (all eigenvalues of A are unstable) with saturated actuators and third computes the volume of ROA of unstable networked subsystems of Eq.(5.1) with saturated actuators. The computational aspects for large complex networks are discussed in subsection 5.3.4.

5.3.1 Problem Setup

A complex network of homogeneous nodes with q sets of driver nodes from the superset $N_D = \{N_1, N_2, \dots, N_q\}$, where each set i.e. N_i , $i = 1, 2, \dots, q$ contains $m \leq n$ nodes, having capability to control the network acting alone, is considered. For a complex network with a large number of nodes, all sets of driver nodes i.e. superset N_D can be obtained by applying the structural controllability theory [Lin, 1974], maximum matching algorithm [Liu et al., 2011] and exact controllability theory [Yuan et al., 2013]. However, all of the above methods are computationally difficult to use [Jia and Barabási, 2013] and require a brute force search to obtain the set of driver nodes. Consider different superset of control inputs corresponding to the superset of driver nodes N_D as $\bar{U} := \{U_1, U_2, \dots, U_q\}$, where $U_i := \{u_{i1}, u_{i2}, \dots, u_{im}\}$, $i = 1, 2, \dots, q$ and cardinality of U_i is m , a region where stability, with saturated inputs, is guaranteed may expand or shrink. This chapter presents a mathematical formulation and algorithms to find an optimal set of inputs such that the largest ROA for a given state-feedback control law of the form given in Eq.(5.2) can be achieved. The objective of this chapter is to characterize the set of driver nodes N_D such that ROA becomes the largest in comparison to region obtained for all other sets of driver nodes from the superset of driver nodes N_D for given control law Eq.(5.2).

5.3.2 Region of Attraction and Selection of Optimal Set of Driver Nodes of Anti-Stable Networked Systems With Saturated Actuators

This section computes a volume of ROA for the networked system defined in Eq.(5.1) with all unstable eigenvalues of the matrix A . It is to be noted that in absence of self-loops, all elements on diagonal of matrix A are zero. This necessitates the existence of positive eigenvalues of matrix A . Consider an anti-stable networked system Eq.(5.1) with saturated actuators. The networked system is controllable with q sets of driver nodes and cardinality of each set of driver nodes is m . The invariant ellipsoid (or ROA) corresponding to input U_i of the networked system Eq.(5.1) depends on the positive definite matrix P_i which can be obtained by solving the CARE defined in Eq.(5.3).

Theorem 5.1. For the networked system defined in Eq.(5.1) with control input U_i , control law Eq.(5.2) and radius of the invariant ellipsoid (or ROA) defined in Eq.(5.5):

(i) The volume Vol_i of the ROA is given by

$$Vol_i = \frac{Vol(S(0, \sqrt{\delta_i}))}{\sqrt{\det(P_i)}} \quad (5.6)$$

where $Vol(S(0, \sqrt{\delta_i}))$ is the volume of n -dimensional Ball ($n \geq 3$) of radius $\sqrt{\delta_i}$ and center at origin, and $\det(P_i)$ is the determinant of the positive definite matrix P_i .

The value of $Vol(S(0, \sqrt{\delta_i}))$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009] is given as

$$Vol(S(0, \sqrt{\delta_i})) = \frac{(\pi \delta_i)^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} \quad (5.7)$$

where n is the number of nodes in the networked system Eq.(5.1) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\Re(\zeta) > 0$).

(ii) Largest Region of Attraction (ROA) for the state-feedback control law of the form given in Eq.(5.2) corresponds to the i^{th} set of driver nodes for which the volume becomes the largest i.e. *Largest Volume* = $\max_i \{Vol_i\}$, $i = 1, 2, \dots, q$.

Proof: To prove the first part, the outer boundaries of the contractively invariant ellipsoids $\mathcal{E}(P_i, \delta_i) = \{x : x^T P_i x \leq \delta_i\}$ of the linear anti-stable networked system Eq.(5.1) are obtained from the following equation

$$x^T P_i x = \delta_i \quad (5.8)$$

and the volume of the ROA is given by

$$Vol_i = \underbrace{\int_{x_{jmin}}^{x_{jmax}} \dots \int_{x_{jmin}}^{x_{jmax}}}_{x_{jmin}, j=1, \dots, n} dx_1 dx_2 \dots dx_n \quad (5.9)$$

where x_{jmin} and x_{jmax} are the minimum and maximum limits of state vector x in j^{th} direction.

Now assume that $x = T_i y$, where $T_i > 0$ is chosen a positive definite matrix and $y \in \mathbb{R}^n$ state vector, then the Eq.(5.8) can be rewritten as

$$y^T T_i^T P_i T_i y = \delta_i \quad (5.10)$$

The positive definite matrix T_i is chosen in such a way that $T_i^T P_i T_i = I_{n \times n}$, where $I_{n \times n}$ is the identity matrix of dimension $n \times n$. This implies that $T_i = ((M_i \sqrt{D_i})^{-1})^T$, where M_i is the eigenvector matrix of positive definite matrix P_i and matrix D_i is the diagonal matrix (elements are the eigenvalues of the matrix P_i).

Using a change of variables $x = T_i y$, it follows $\underbrace{dx_1 dx_2 \dots dx_n}_{x_{jmin}, j=1, \dots, n} = (\frac{1}{\sqrt{\det(P_i)}}) \underbrace{dy_1 dy_2 \dots dy_n}_{y_{jmin}, j=1, \dots, n}$. With this substitution, Eq.(5.9) becomes

$$Vol_i = (\frac{1}{\sqrt{\det(P_i)}}) \underbrace{\int_{y_{jmin}}^{y_{jmax}} \dots \int_{y_{jmin}}^{y_{jmax}}}_{y_{jmin}, j=1, \dots, n} dy_1 dy_2 \dots dy_n \quad (5.11)$$

The Eq.(5.10) becomes $y^T y = \delta_i$ i.e. $y_1^2 + y_2^2 + \dots + y_n^2 = \delta_i$ by using the change of variables.

Now, the volume of n -dimensional Ball ($n \geq 3$) is defined as

$$Vol(S(0, \sqrt{\delta_i})) := \underbrace{\int_{y_{jmin}}^{y_{jmax}} \dots \int_{y_{jmin}}^{y_{jmax}}}_{y_{jmin}, j=1, \dots, n} dy_1 dy_2 \dots dy_n \quad (5.12)$$

with center at origin and the radius $\sqrt{\delta_i}$.

As it is well known fact that $\int_0^\pi \sin \theta d\theta \int_0^\pi \sin^2 \theta d\theta \dots \int_0^\pi \sin^n \theta d\theta = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$, where, $\Gamma(\cdot)$ is the standard Gamma Function. Now, assume $y_1 = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{n-1}$; $y_2 = r \cos \theta_1 \sin \theta_2 \dots \sin \theta_{n-1}$; $y_3 = r \cos \theta_2 \sin \theta_3 \dots \sin \theta_{n-1}$; $\dots$; $y_{n-1} = r \cos \theta_{n-2} \sin \theta_{n-1}$; $y_n = r \cos \theta_{n-1}$, where, $0 \leq t \leq \sqrt{\delta}$, $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i = 2, 3, \dots, n$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009]. Then

$$dy_1 dy_2 \dots dy_n = ((-1)^{n-1} r^{n-1} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{n-2} \theta_{n-1}) (dr d\theta_1 d\theta_2 \dots d\theta_{n-1}) \quad (5.13)$$

By substituting Eq.(5.13) in Eq.(5.12) with the definition of standard Gamma Function, and using Eq.(5.11) and Eq.(5.12), the Eq.(5.6) is obtained. Using Eq.(5.6) and Eq.(5.7), the volume becomes

$$Vol_i = \frac{(\pi \delta_i)^{\frac{n}{2}}}{\sqrt{\det(P_i)} \Gamma(\frac{n}{2} + 1)} \quad (5.14)$$

After obtaining the volume of ROA corresponding to each set of driver nodes, the largest ROA for the state-feedback control law Eq.(5.2) is obtained corresponds to the i^{th} set of driver nodes for which the volume of ROA becomes largest. This completes the proof of Theorem 5.1.

Corollary 5.1. For $n = 2$, the region is area. The area $Area_i$ of the ellipse (region) is given by

$$Area_i = \frac{\pi \delta_i}{\sqrt{\det(P_i)}} \quad (5.15)$$

$Area_i > Area_j \iff \delta_j^2 \det(P_i) < \delta_i^2 \det(P_j)$, $i \neq j = 1, 2$. The i^{th} node is chosen as a driver node of the networked system Eq.(5.1), corresponding to the largest area.

The computation of $Area_{max} / Vol_{max}$ of the ROA for n nodes anti-stable networked systems with q sets of driver nodes is summarized in the Algorithm 5.1.

Algorithm 5.1. Find maximum Area / Volume of the ROA and corresponding to the set of driver nodes of an anti-stable networked system Eq.(5.1)

Input : Anti-stable networked system $G := (V, E)$, number of nodes n , sets of driver nodes q , cardinality of driver nodes sets m , $Q > 0$ and $R = I_{m \times m}$

Output : Maximum Area / Volume of the ROA and corresponding to the i^{th} set of driver nodes of the networked system Eq.(5.1)

- 1: for $i = 1 : q$ do
- 2: Identify input matrix B_i corresponding to control input U_i
- 3: Obtain the positive definite matrix $P_i > 0$ by solving the CARE defined in Eq.(5.3)
- 4: Calculate the radius δ_i of ROA using Eq.(5.5)
- 5: Calculate the $Area_i / Vol_i$ of ROA using Eq.(5.15) or Eq.(5.14)
- 6: end for
- 7: Calculate $Area_{max} / Vol_{max}$ as
 $Area_{max} / Vol_{max} = \max_i \{Area_i / Vol_i \mid i = 1, 2, \dots, q\}$
- 8: index=1
- 9: for $i = 1 : q$ do
- 10: if $Area_{max} = Area_i$ or $Vol_{max} = Vol_i$ then
- 11: $max_{index}(index, 1) = i$
- 12: index=index+1

13: end if
14: end for
15: return $Area_{max} / Vol_{max}$ and the corresponding indices of sets of driver nodes

Remark 5.3. Whenever ROA of two or more sets of driver nodes equals to the largest ROA of the networked system, then any one of them can be selected as the optimal set of driver nodes.

5.3.3 Region of Attraction and Selection of Optimal Set of Driver Nodes of Unstable Networked Systems With Saturated Actuators

This section considers the unstable networked systems which have stable and unstable eigenvalues with saturated actuators. The networked system is controllable with q sets of driver nodes and cardinality of each set of driver nodes is m . The original networked system Eq.(5.1) is transformed into two subsystems. The first subsystem has all unstable eigenvalues and the second subsystem has all stable eigenvalues. It is to be noted that for stable eigenvalues of matrix A , Region of Attraction (ROA) becomes infinite. Furthermore, for a system with stable and unstable eigenvalues, a subsystem with all unstable eigenvalues need to be considered. In what follows, the subsystem with unstable eigenvalues is considered and obtain the ROA. In this case, the original networked system is transformed using a non singular transformation matrix V i.e $x = Vz$, where x is the state vector of original network and z is state vector of transformed system.

The transformed continuous-time networked system has an anti-stable subsystem as well as a stable subsystem of the original networked system Eq.(5.1) defined as

$$\dot{z}(t) = \tilde{A}z(t) + \tilde{B}_i \text{sat}_i(U_i(t)) \quad (5.16)$$

where $U_i \in \mathbb{R}^{m \times 1}$; $|u_{ij,max}| = 1$; $i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

The Adjacency matrix $\tilde{A}$ can be partitioned as [Hu and Lin, 2001; Horn and Johnson, 2012]

$$\tilde{A} = V^{-1}AV = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \quad (5.17)$$

where $\tilde{A}_1 \in \mathbb{R}^{l \times l}$ anti-stable and $\tilde{A}_2 \in \mathbb{R}^{(n-l) \times (n-l)}$ stable subsystem matrices.

The control input matrix $\tilde{B}_i$ is partitioned as

$$\tilde{B}_i = V^{-1}B_i = \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} \quad (5.18)$$

where $\tilde{B}_{1i} \in \mathbb{R}^{l \times m}$, $\tilde{B}_{2i} \in \mathbb{R}^{(n-l) \times m}$, and n is the number of nodes (order) in the networked system Eq.(5.1) and l is the order of the anti-stable subsystem respectively. Here, the input matrix $\tilde{B}_{1i}$ is denoted as $\tilde{B}_{1i} = [\tilde{b}_{i1}^1, \tilde{b}_{i2}^1, \dots, \tilde{b}_{im}^1]$, where the column $\tilde{b}_{ij}^1$ represents the j^{th} column of the input matrix $\tilde{B}_{1i}$ of the transformed anti-stable subsystem and similarly the input matrix $\tilde{B}_{2i}$ is denoted as $\tilde{B}_{2i} = [\tilde{b}_{i1}^2, \tilde{b}_{i2}^2, \dots, \tilde{b}_{im}^2]$, where the column $\tilde{b}_{ij}^2$ represents that the j^{th} column of the input matrix $\tilde{B}_{2i}$ of the transformed stable subsystem.

The transformed continuous-time networked system Eq.(5.16) can be rewritten as

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} \tilde{B}_{1i} \\ \tilde{B}_{2i} \end{bmatrix} \text{sat}_i(U_i(t)) \quad (5.19)$$

Now, the continuous-time anti-stable subsystem of Eq.(5.19) is described as follows

$$\dot{z}_1(t) = \tilde{A}_1 z_1(t) + \tilde{B}_{1i} \text{sat}_i(U_i(t)) \quad (5.20)$$

Remark 5.4. To obtain the ROA of the unstable networked systems, only anti-stable subsystem Eq.(5.20) required to be considered. For the stable subsystem, global stabilization is possible with saturated actuators [Hu and Lin, 2001].

The control law for the continuous-time anti-stable subsystem defined in Eq.(5.20) is as follows

$$U_i(t) := -Hz_1 \quad (5.21)$$

where $H = R_1^{-1} \tilde{B}_{1i}^T \tilde{P}_{1i}$ is a feedback gain matrix.

The positive definite matrix $\tilde{P}_{1i} > 0$ is the solution of the following CARE for the continuous-time anti-stable subsystem Eq.(5.20) for some $\tilde{Q}_1 > 0$

$$\tilde{A}_1^T \tilde{P}_{1i} + \tilde{P}_{1i} \tilde{A}_1 - \tilde{P}_{1i} \tilde{B}_{1i} R_1^{-1} \tilde{B}_{1i}^T \tilde{P}_{1i} + \tilde{Q}_1 = 0 \quad (5.22)$$

where $R_1 > 0$ are chosen diagonal matrix.

The radius ($\tilde{\delta}_{1i}$) of the invariant ellipsoid ($\varepsilon(\tilde{P}_{1i}, \tilde{\delta}_{1i}) = \{z_1 : z_1^T \tilde{P}_{1i} z_1 \leq \tilde{\delta}_{1i}\}$) of the continuous-time anti-stable subsystem Eq.(5.20) is defined as

$$\tilde{\delta}_{1i} = \min_j \left(\frac{4r_{1i}^2 u_{ij,max}^2}{\tilde{b}_{ij}^{1,T} \tilde{P}_{1i} \tilde{b}_{ij}^1} \right) \quad (5.23)$$

where $\tilde{b}_{ij}^1$ is the j^{th} column of the input matrix $\tilde{B}_{1i}$ of the transformed anti-stable subsystems. Without loss of generality, it is assumed that $u_{ij,max} = 1$ and $r_{1i} = 1$ for $i = 1, 2, \dots, q$ and $j = 1, 2, \dots, m$.

Theorem 5.2. For the anti-stable subsystem defined in Eq.(5.20) with control law Eq.(5.21), the radius of invariant ellipsoid (or ROA) is defined in Eq.(5.23)

(i) The volume of the ROA of the anti-stable subsystem Eq.(5.20) is given by

$$Vol_{1i} = \frac{(\pi \tilde{\delta}_{1i})^{\frac{l}{2}}}{\Gamma(\frac{l}{2} + 1) \sqrt{\det(\tilde{P}_{1i})}} \quad (5.24)$$

where $\det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$, l is the number of nodes in the anti-stable subsystem Eq.(5.20) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\Re(\zeta) > 0$).

(ii) Largest ROA for the state-feedback control law of the form given in Eq.(5.21) corresponds to the i^{th} set of driver nodes for which volume becomes the largest i.e.

Largest Volume = $\max_i \{Vol_{1i}\}$, $i = 1, 2, \dots, q$.

Proof: To prove the first part, the outer boundaries of the contractively invariant ellipsoids $\varepsilon(\tilde{P}_{1i}, \tilde{\delta}_{1i}) = \{z_1 : z_1^T \tilde{P}_{1i} z_1 \leq \tilde{\delta}_{1i}\}$ of the linear anti-stable subsystem Eq.(5.20) are obtained from the following equation

$$z_1^T \tilde{P}_{1i} z_1 = \tilde{\delta}_{1i} \quad (5.25)$$

and the volume of the ROA is given by

$$Vol_{1i} = \underbrace{\int_{z_{1,min}}^{z_{1,max}} \dots \int_{z_{1,min}}^{z_{1,max}}}_{j=1, \dots, l} dz_{1_1} dz_{1_2} \dots dz_{1_l} \quad (5.26)$$

where $z_{1_{j_{\min}}}$ and $z_{1_{j_{\max}}}$ are the minimum and maximum limits of state vector z_1 in j^{th} direction.

Now assume that $z_1 = \tilde{T}_{1i} y_1$, where $\tilde{T}_{1i} > 0$ is a chosen positive definite matrix and $y_1 \in \mathbb{R}^l$ state vector, then the Eq.(5.25) can be rewritten as

$$y_1^T \tilde{T}_{1i}^T \tilde{P}_{1i} \tilde{T}_{1i} y_1 = \tilde{\delta}_{1i} \quad (5.27)$$

The positive definite matrix $\tilde{T}_{1i}$ is chosen in such a way that $\tilde{T}_{1i}^T \tilde{P}_{1i} \tilde{T}_{1i} = I_{l \times l}$, where $I_{l \times l}$ is the identity matrix of dimension $l \times l$. This implies that $\tilde{T}_{1i} = ((\tilde{M}_{1i} \sqrt{\tilde{D}_{1i}})^{-1})^T$, where $\tilde{M}_{1i}$ is the eigenvector matrix of positive definite matrix $\tilde{P}_{1i}$ and matrix $\tilde{D}_{1i}$ is the diagonal matrix (elements are the eigenvalues of the matrix $\tilde{P}_{1i}$).

Using a change of variables $z_1 = \tilde{T}_{1i} y_1$, it follows $\underbrace{dz_{1_1} dz_{1_2} \dots dz_{1_l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}} = \left(\frac{1}{\sqrt{\det(\tilde{P}_{1i})}} \right) \underbrace{dy_{1_1} dy_{1_2} \dots dy_{1_l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}}$. With this substitution, Eq.(5.26) becomes

$$Vol_{1i} = \left(\frac{1}{\sqrt{\det(\tilde{P}_{1i})}} \right) \underbrace{\int_{y_{1_{j_{\min}}}}^{y_{1_{j_{\max}}}} \dots \int_{j=1, \dots, l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}} \underbrace{dy_{1_1} dy_{1_2} \dots dy_{1_l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}} \quad (5.28)$$

The Eq.(5.27) becomes $y_1^T y_1 = \tilde{\delta}_{1i}$ i.e. $y_{1_1}^2 + y_{1_2}^2 + \dots + y_{1_l}^2 = \tilde{\delta}_{1i}$ by using the change of variables.

Now, the volume of l -dimensional Ball ($l \geq 3$) is defined as

$$Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}})) := \underbrace{\int_{y_{1_{j_{\min}}}}^{y_{1_{j_{\max}}}} \dots \int_{j=1, \dots, l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}} \underbrace{dy_{1_1} dy_{1_2} \dots dy_{1_l}}_{\substack{y_{1_{j_{\min}}}, j=1, \dots, l}} \quad (5.29)$$

with center at origin and the radius $\sqrt{\tilde{\delta}_{1i}}$.

As it is well known fact that $\int_0^\pi \sin \theta d\theta \int_0^\pi \sin^2 \theta d\theta \dots \int_0^\pi \sin^l \theta d\theta = \frac{\pi^{\frac{l}{2}}}{\Gamma(\frac{l}{2}+1)}$, where, $\Gamma(\cdot)$ is the standard Gamma Function. Now, assume $y_{1_1} = r \sin \theta_1 \sin \theta_2 \dots \sin \theta_{l-1}$; $y_{1_2} = r \cos \theta_1 \sin \theta_2 \dots \sin \theta_{l-1}$; $y_{1_3} = r \cos \theta_2 \sin \theta_3 \dots \sin \theta_{l-1}$; $\dots$; $y_{1_{l-1}} = r \cos \theta_{l-2} \sin \theta_{l-1}$; $y_{1_l} = r \cos \theta_{l-1}$, where, $0 \leq t \leq \sqrt{\tilde{\delta}_{1i}}$, $0 \leq \theta_1 \leq 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i = 2, 3, \dots, l$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009]. Then

$$dy_{1_1} dy_{1_2} \dots dy_{1_l} = ((-1)^{l-1} r^{l-1} \sin \theta_2 \sin^2 \theta_3 \dots \sin^{l-2} \theta_{l-1}) (dr d\theta_1 d\theta_2 \dots d\theta_{l-1}) \quad (5.30)$$

By substituting Eq.(5.30) in Eq.(5.29) with the definition of standard Gamma Function, and using Eq.(5.28) and Eq.(5.29), the volume of the ROA is obtained as

$$Vol_{1i} = \frac{Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}}))}{\sqrt{\det(\tilde{P}_{1i})}} \quad (5.31)$$

where $Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}}))$ is the volume of l -dimensional Ball ($l \geq 3$) of radius $\sqrt{\tilde{\delta}_{1i}}$ and center at origin, and $\det(\tilde{P}_{1i})$ is the determinant of the positive definite matrix $\tilde{P}_{1i}$.

The value of $Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}}))$ [Smith and Vamanamurthy, 1989; Boyd and Vandenberghe, 2009] is given as

$$Vol_1(S(0, \sqrt{\tilde{\delta}_{1i}})) = \frac{(\pi \tilde{\delta}_{1i})^{\frac{l}{2}}}{\Gamma(\frac{l}{2}+1)} \quad (5.32)$$

where l is the number of nodes in the subsystems Eq.(5.20) and $\Gamma(\cdot)$ is the standard Gamma Function represented as $\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t} dt$, where, the real part of the complex number ζ is positive ($\mathbb{R}(\zeta) > 0$).

By using Eq.(5.31) and Eq.(5.32), the volume becomes

$$Vol_{li} = \frac{(\pi \tilde{\delta}_{li})^{\frac{l}{2}}}{\sqrt{\det(\tilde{P}_{li})} \Gamma(\frac{l}{2} + 1)} \quad (5.33)$$

After obtaining the volume of ROA corresponding to each set of driver nodes, the largest ROA for the state-feedback control law Eq.(5.21) is obtained corresponds to the i^{th} set of driver nodes for which the volume of ROA becomes the largest. This completes the proof of Theorem 5.2.

Corollary 5.2. For $l = 2$, it requires to compute area of the ROA of the anti-stable subsystem Eq.(5.20). The area of ROA is same as given in Eq.(5.15) with appropriate replacement of associated variables.

The computation of $Area_{max} / Vol_{max}$ of the ROA for an unstable networked system of order n with dimension of anti-stable subsystem l , $l < n$ is summarized in the Algorithm 5.2.

Algorithm 5.2. Find maximum Area/Volume of the ROA and the corresponding to the set of driver nodes of an unstable networked system Eq.(5.1)

Input : Unstable networked systems $\mathbf{G} := (\mathbf{V}, \mathbf{E})$, number of nodes n , number of anti-stable nodes l , sets of driver nodes q , cardinality of driver nodes sets m , $\tilde{Q}_1 > 0$ and $\mathbf{R}_1 = \mathbf{I}_{m \times m}$, where $\mathbf{I}_{m \times m}$ is the identity matrix of dimension $m \times m$.

Output : Maximum Area/Volume of the ROA and corresponding to the i^{th} set of driver nodes of the networked system Eq.(5.1).

- 1: Partition matrix $\tilde{\mathbf{A}}$ using Eq.(5.17)
- 2: for $i = 1 : q$ do
- 3: Identify $\mathbf{B}_i \in \mathbb{R}^{n \times m}$ of the original networked system Eq.(5.1)
- 4: Partition $\tilde{\mathbf{B}}_i$ using Eq.(5.18)
- 5: Calculate the positive definite matrix $\tilde{\mathbf{P}}_{li}$ using Eq.(5.22)
- 6: Calculate the radius $\tilde{\delta}_{li}$ of ROA using Eq.(5.23)
- 7: Calculate the $Area_i / Vol_i$ of ROA using Eq.(5.15) or Eq.(5.24)
- 8: end for
- 9: Calculate $Area_{max} / Vol_{max}$ as
 $Area_{max} / Vol_{max} = \max_i \{Area_i / Vol_i \mid i = 1, 2, \dots, q\}$
- 10: index=1
- 11: for $i = 1 : q$ do
- 12: if $Area_{max} = Area_i$ or $Vol_{max} = Vol_i$ then
- 13: $max_{index}(index, 1) = i$
- 14: index=index+1
- 15: end if
- 16: end for
- 17: return $Area_{max} / Vol_{max}$ and the corresponding indices of sets of driver nodes

Remark 5.5. Whenever the largest ROA for two (or more) set of driver nodes is equal, then it can identify any one set of them as a set of driver nodes.

Remark 5.6. For some applications, the presented theories and algorithms can also be used to obtain a specific driver node (or a set of the specific driver nodes) based on the shape of the ROA.

5.3.4 Computational Aspects for Large Complex Networks

In this subsection, the computational aspects of the proposed method for selecting the optimal set of driver nodes based on ROA for the large complex networks are discussed. The choice of input influences size of this region. In this chapter, inputs are selected based on maximizing this region. The proposed algorithms in this chapter also use brute-force method to obtain largest region. This is the first work which investigated selection of driver node by considering region of attraction. The main computationally intensive aspect of the proposed method is to find the solution matrix P of CARE for each set of driver nodes in large complex networks. In this regard, the authors of [Benner et al., 2008; Heyouni and Jbilou, 2009; Benner and Faßbender, 2011] described the methods like Alternating Direction Implicit (ADI) iteration, Newton iteration, Cholesky factor, Schur, etc to compute the P matrix i.e. the solution of CARE for a large dimension system. These methods are already used by some authors in similar computational requirements. Authors of [Benner et al., 2008] used the combination of ADI iteration with the Newton iteration method and efficiently computed the stabilizing solution of the large CARE. Recently, the authors of [Summers et al., 2016; Tzoumas et al., 2016; Fitch and Leonard, 2016] used these methods [Benner and Faßbender, 2011] to find the inversion of controllability Gramian to obtain further computational ease by computing low rank approximations. In proposed approach, the matrix P is used to compute the ROA's of the complex networks for all possible sets of minimum driver nodes and then an optimal set of driver node is identified corresponding to maximum ROA. The proposed method has computational complexity of $O(nq^3)$, where n is the total number of nodes and q is the total number of the sets of driver nodes of the complex network. Computational aspect discussed above could be applied to large discrete-time complex networks as well. This analysis is done in next chapter.

5.4 SIMULATION RESULTS AND DISCUSSION

This section considers three numerical examples to verify the theoretical results. First example consist of only unstable eigenvalues whereas second example has both stable and unstable eigenvalues. The third example consist of multiple-input networked system with both stable and unstable eigenvalues.

Example 5.1. Consider an undirected continuous-time anti-stable network of two nodes as shown in Figure 5.2.

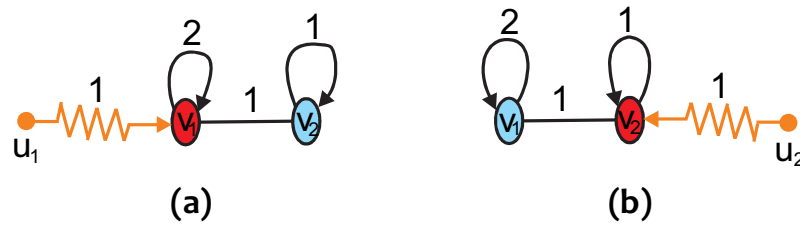


Figure 5.2 : Two nodes network with input (a) u_1 , and (b) u_2

The Adjacency and control input matrices are given as:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The matrix B_i depends on the i^{th} driver node in order to control the networked system.

The eigenvalues of the network are $\lambda(A) = \{0.3820, 2.6180\}$. Let $Q = I_2$, $r_i = 1$, $i = 1, 2$ and solve the CARE Eq.(5.3) for different input matrices B_i . The area of the ROA's corresponding to driver nodes 1 and 2 are obtained using Eq.(5.15) are 0.2473 and 0.2014 respectively. Now the maximum area of the ROA is calculated using Algorithm 5.1 is 0.2473. The ROA's of the networked system considered in Example 5.1 for different driver nodes, are plotted in the Figure 5.3. From the Figure 5.3, the area of first ROA (blue color) which is larger than the second ROA (black color) is obtained. In conclusion, choosing the first node as the driver node is advantageous because the maximum area of the ROA is obtained. According to Table 5.1, results clearly validate the effectiveness of the theoretical results.

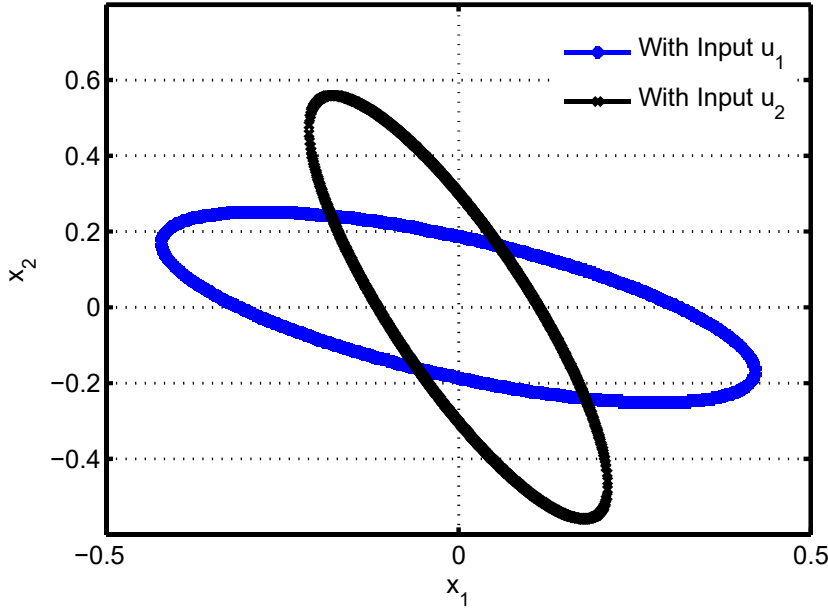


Figure 5.3 : Region of Attractions (ROA's) of the network considered in Example 5.1 for different control input matrices B_i

Table 5.1 : Area of the ROA for different driver nodes of network considered in Example 5.1

Driver Node	Area of ROA	$\frac{Area_i}{Area_{max}}$	Remark
1	0.2490	1	Maximum Area
2	0.2023	0.8124	19% Reduction

Example 5.2. Consider a directed continuous-time unstable network of four nodes as shown in Figure 5.4. The Adjacency matrix and control input matrices of the network are given as:

$$A = \begin{bmatrix} 0 & 0.0178 & 0.3410 & 0.5807 \\ 0.0659 & 0 & 0.6175 & 0.6207 \\ 0.5694 & 0.5547 & 0 & 0.5997 \\ 0.4501 & 0.0190 & 0.0143 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

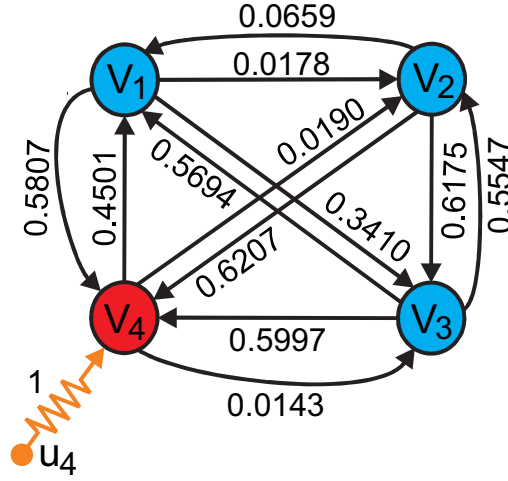


Figure 5.4 : Four nodes network with control input u_4 and it can also be controlled by other individual driver node i.e. u_1, u_2 & u_3

Depending on the choice of a particular node as a driver node, the input matrix B_i has four possibilities. The eigenvalues of the network shown in Figure 5.4, are $\lambda(A) = \{0.9613, 0.1318, -0.7706, -0.3225\}$. Here two eigenvalues are positive, so the network will be partitioned into two subsystems according to Eq.(5.19). Now, the anti-stable subsystem Eq.(5.20) is analyzed. Let $\tilde{Q}_1 = I_2$, $R_1 = 1$ and solve the CARE Eq.(5.22) for different input matrices B_i corresponding to different driver nodes. The ROA's of the network considered in Example 5.2 for different driver nodes are plotted in the Figure 5.5.

From the Figure 5.5, the largest area is obtained when fourth node is selected as the driver node (magenta color) vis-a-vis region corresponding to node 1, 2, 3 (blue, black and red colors). In conclusion, choosing the fourth node as the driver node is advantageous because the maximum area of the ROA is obtained corresponding to this node.

Area's of the ROA corresponding to different driver nodes calculated using Algorithm 5.2 are given in Table 5.2. Now from the Table 5.2, the maximum area of the ROA is 0.9449 corresponding to driver node 4. According to Table 5.2, the results clearly validate the effectiveness of the theoretical results.

Table 5.2 : Area of the ROA for different driver nodes of network considered in Example 5.2

Driver Node	Area of ROA	$\frac{Area_i}{Area_{max}}$	Remark
4	0.9449	1	Maximum Area
1	0.7917	0.8378	17% Reduction
3	0.3910	0.4138	59% Reduction
2	0.3796	0.4017	60% Reduction

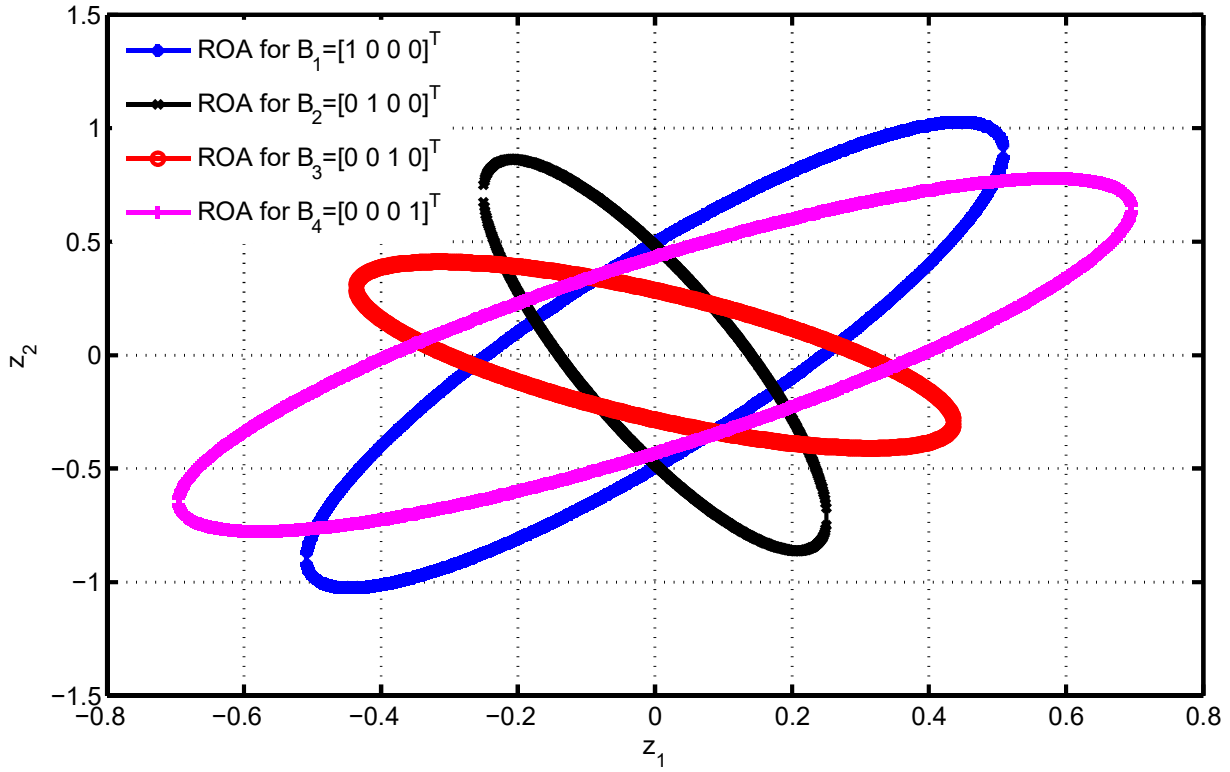


Figure 5.5 : Region of Attractions (ROA's) of the network considered in Example 5.2 for different control input matrices B_i

Example 5.3. Consider an undirected multiple-input network of eight nodes with self-loop as shown in Figure 5.6.

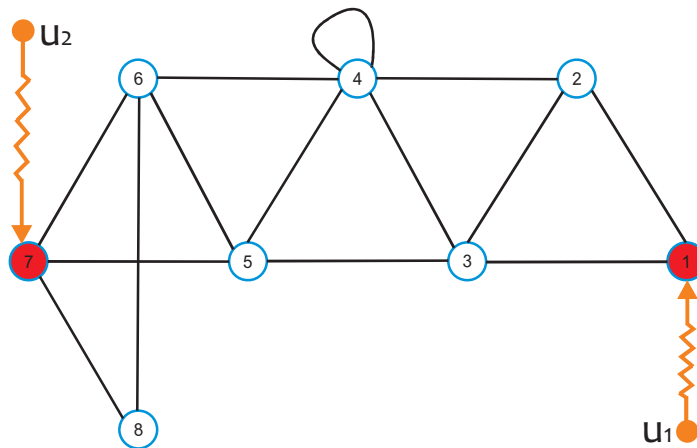


Figure 5.6 : Network of eight nodes controllable with a set of two driver nodes (red color) $\{1, 7\}$

The Adjacency matrix and control input matrices of the network shown in Figure 5.6 are given as:

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_7 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \\
B_8 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_9 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

The given network is controllable with a set of two driver nodes ($m = 2$) and there exist twelve distinct possibilities of such set of driver nodes i.e. $N_1 = \{1, 2\}$, $N_2 = \{1, 3\}$, $N_3 = \{1, 7\}$, $N_4 = \{1, 8\}$, $N_5 = \{2, 3\}$, $N_6 = \{2, 6\}$, $N_7 = \{2, 8\}$, $N_8 = \{3, 6\}$, $N_9 = \{3, 7\}$, $N_{10} = \{6, 7\}$, $N_{11} = \{6, 8\}$, $N_{12} = \{7, 8\}$. These choices of set of the driver nodes correspond to input matrices $B_1 \in \mathbb{R}^{8 \times 2}$ to $B_{12} \in \mathbb{R}^{8 \times 2}$. The eigenvalues of the given network are $\lambda(A) = \{-1.9054, -1.5694, -1, -1, -0.0622, 0.6744, 2.1368, 3.7257\}$. Here three eigenvalues are unstable and remaining are stable, so the network will be divided into two subsystem according to Eq.(5.19). Now the anti-stable subsystem Eq.(5.20) is analyzed here. Using the LQ control design method with $\tilde{Q}_1 = I_3$, $R_1 = 1$ and solve the CARE defined in Eq.(5.22) for different input matrices B_i corresponding to different set of driver nodes. Using Eq.(5.23), the radius of the ROA $\tilde{\delta}_{1i}$, $i = 1, 2, \dots, q$ are calculated as $\tilde{\delta}_{11} = 0.3645$, $\tilde{\delta}_{12} = 0.3670$, $\tilde{\delta}_{13} = 0.5861$, $\tilde{\delta}_{14} = 0.5652$, $\tilde{\delta}_{15} = 0.3853$, $\tilde{\delta}_{16} = 0.5661$, $\tilde{\delta}_{17} = 0.5539$, $\tilde{\delta}_{18} = 0.5405$, $\tilde{\delta}_{19} = 0.5801$, $\tilde{\delta}_{110} = 0.3914$, $\tilde{\delta}_{111} = 0.3688$ and $\tilde{\delta}_{112} = 0.3636$ with $\tilde{b}_{ij}$, $i = 1, 2, \dots, 12$ and $j = 1, 2$ respectively.

According to Table 5.3, the maximum volume of the ROA is 0.019344, corresponding to the set of driver nodes $N_3 = \{1, 7\}$. It is observed that the consequence of the set of driver nodes from network stability point of view depends upon the volume of the ROA. The ROA's of the network considered in Example 5.3 for different driver nodes are plotted in the Figure 5.7. From the Figure

5.7, the largest volume is obtained when third set of nodes is selected as the driver nodes set (red color) vis-a-vis region corresponding to other sets of nodes. In conclusion, choosing the third set of driver nodes as the driver nodes is advantageous because the maximum volume of the ROA is obtained corresponding to this set. Notice that the algorithms developed here give a method to select an optimal set of driver nodes among all possible sets of driver nodes. The effectiveness of the theoretical developments is clearly visible from Table 5.3.

Table 5.3 : Volume of the ROA for different sets of driver nodes of network considered in Example 5.3

Set of Driver Nodes	Corresponding Input Matrix B_i	Volume of the ROA	$\frac{Vol_i}{Vol_{max}}$	Remark
$N_3 = \{1, 7\}$	B_3	0.019344	1	Maximum Volume
$N_1 = \{1, 2\}$	B_1	0.001961	0.1013	90% Reduction
$N_2 = \{1, 3\}$	B_2	0.003093	0.1598	85% Reduction
$N_4 = \{1, 8\}$	B_4	0.015088	0.7799	33% Reduction
$N_5 = \{2, 3\}$	B_5	0.001366	0.0706	93% Reduction
$N_6 = \{2, 6\}$	B_6	0.010501	0.5428	46% Reduction
$N_7 = \{2, 8\}$	B_7	0.014966	0.7736	33% Reduction
$N_8 = \{3, 6\}$	B_8	0.005917	0.3058	70% Reduction
$N_9 = \{3, 7\}$	B_9	0.015557	0.8042	20% Reduction
$N_{10} = \{6, 7\}$	B_{10}	0.002636	0.1362	87% Reduction
$N_{11} = \{6, 8\}$	B_{11}	0.003251	0.1680	84% Reduction
$N_{12} = \{7, 8\}$	B_{12}	0.001321	0.0682	94% Reduction

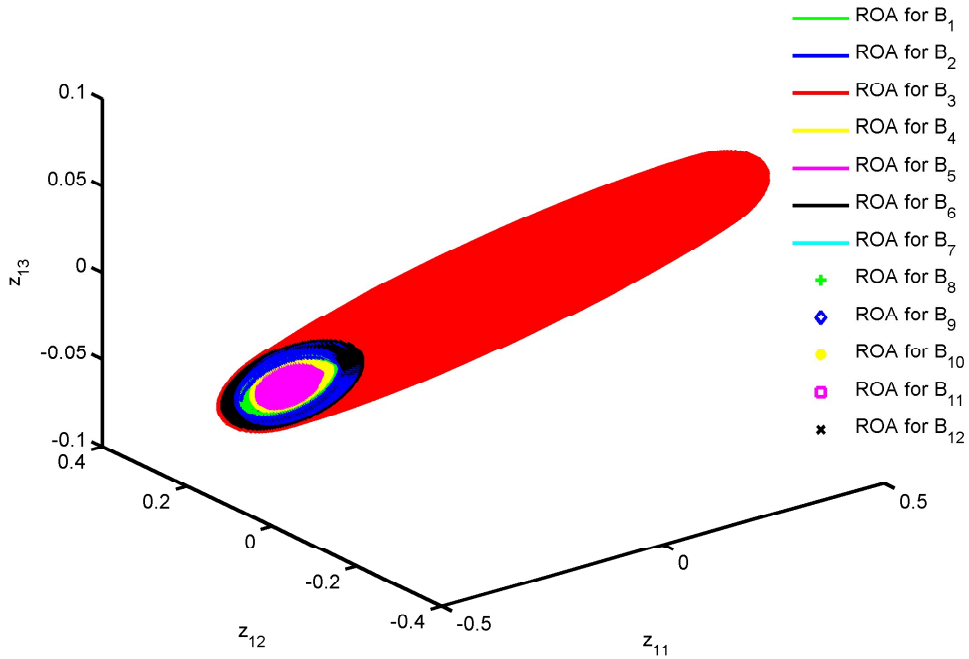


Figure 5.7 : Region of Attractions (ROA's) of the network considered in Example 5.3 for different control input matrices B_i

5.5 APPLICATION TO ROBOT FORMATION CONTROL

In this section the proposed algorithms are used in formation control of robotic network. Consider a continuous-time network of nine robots shown in Figure 5.8. In this network, the control objective is to make different formations of robots with the largest stability region by controlling a set of multiple robots (set of driver nodes).

Remark 5.7. The presented theories and algorithms can be used to obtain a specific driver node (or a set of the specific driver nodes) which ensures the largest region or largest region in some specified directions (i.e the x,y coordinates)[Balch and Arkin, 1998; Lawton et al., 2003; Tanner, 2004].

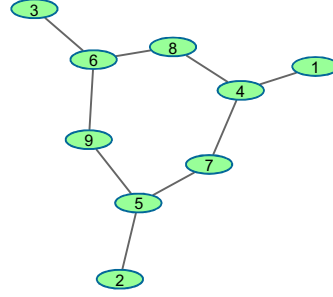


Figure 5.8 : Network of nine robot nodes independently controllable with different sets of three driver nodes

The exact control theory predicts that it needs minimum three driver nodes ($m = 3$) to control the entire network. There are total seventeen distinct choices of the set of driver nodes i.e. $N_1 = \{1, 2, 3\}$, $N_2 = \{1, 2, 7\}$, $N_3 = \{1, 2, 8\}$, $N_4 = \{1, 2, 9\}$, $N_5 = \{1, 3, 7\}$, $N_6 = \{1, 3, 8\}$, $N_7 = \{1, 3, 9\}$, $N_8 = \{1, 7, 9\}$, $N_9 = \{1, 8, 9\}$, $N_{10} = \{2, 3, 7\}$, $N_{11} = \{2, 3, 8\}$, $N_{12} = \{2, 3, 9\}$, $N_{13} = \{2, 7, 8\}$, $N_{14} = \{2, 8, 9\}$, $N_{15} = \{3, 7, 8\}$, $N_{16} = \{3, 7, 9\}$ and $N_{17} = \{7, 8, 9\}$. The input matrices corresponding to these set of driver nodes are $B_1 \in \mathbb{R}^{9 \times 3}$ to $B_{17} \in \mathbb{R}^{9 \times 3}$. The above network can be controlled by the control input vector corresponding to seventeen independently controllable sets of driver nodes given by $\bar{U} = \{U_1, U_2, \dots, U_{17}\}$, where $U_i \in \mathbb{R}^{3 \times 1}$, $i = 1, 2, \dots, 17$. The ROA of the network is governed by the choice of a particular set of nodes selected as the set of driver nodes.

The Adjacency matrix and different possibilities of control input matrices of the network shown in Figure 5.8 corresponding to $N_1, N_2, \dots, N_{17}$ are given as

$$A = \begin{bmatrix} 0 & 0 & 0 & 0.0580 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0630 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0590 & 0 & 0 & 0 \\ 0.0580 & 0 & 0 & 0 & 0 & 0 & 0.0850 & 0.0920 & 0 \\ 0 & 0.0630 & 0 & 0 & 0 & 0 & 0.1610 & 0 & 0.0720 \\ 0 & 0 & 0.0590 & 0 & 0 & 0 & 0 & 0.1700 & 0.1010 \\ 0 & 0 & 0 & 0.0850 & 0.1610 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0920 & 0 & 0.1700 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0720 & 0.1010 & 0 & 0 & 0 \end{bmatrix},$$

[illegible]

$$B_{5\{1,3,7\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_{6\{1,3,8\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_{7\{1,3,9\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_{8\{1,7,9\}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

[illegible]

$$B_{13\{2,7,8\}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_{14\{2,8,9\}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B_{15\{3,7,8\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, B_{16\{3,7,9\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_{17\{7,8,9\}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of the Adjacency matrix of the network shown in Figure 5.8 are $\lambda(A) = \{-0.2407, -0.1760, -0.0884, 0, 0, 0, 0.0884, 0.1760, 0.2407\}$, which indicates that the network is unstable. The given network has six unstable eigenvalues and then, the network is transformed into two subsystems, anti-stable subsystem and stable subsystem of sixth order and third order respectively.

The transformed Adjacency $\tilde{A}_1 \in \mathbb{R}^{6 \times 6}$ and input matrices $\tilde{B}_{1i} \in \mathbb{R}^{6 \times 3}$, where $i = 1, 2, \dots, 17$ of the anti-stable subsystem are given as

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0884 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1760 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2407 \end{bmatrix}, \tilde{B}_{11\{1,2,3\}} = \begin{bmatrix} 0.2929 & 0.8375 & -0.3092 \\ -0.4550 & 0.3647 & 0.5135 \\ -0.6010 & -0.0715 & -0.7219 \\ 0.4076 & -0.1791 & -0.1514 \\ 0.0215 & 0.1982 & -0.1458 \\ -0.0800 & -0.0945 & -0.1249 \end{bmatrix},$$

$$\tilde{B}_{12\{1,2,7\}} = \begin{bmatrix} 0.2929 & 0.8375 & -0.3229 \\ -0.4550 & 0.3647 & 0.1223 \\ -0.6010 & -0.0715 & 0.0735 \\ 0.4076 & -0.1791 & 0.1397 \\ 0.0215 & 0.1982 & 0.5381 \\ -0.0800 & -0.0945 & -0.3587 \end{bmatrix}, \tilde{B}_{13\{1,2,8\}} = \begin{bmatrix} 0.2929 & 0.8375 & 0.1137 \\ -0.4550 & 0.3647 & 0.1738 \\ -0.6010 & -0.0715 & 0.3110 \\ 0.4076 & -0.1791 & 0.2103 \\ 0.0215 & 0.1982 & -0.3859 \\ -0.0800 & -0.0945 & -0.4867 \end{bmatrix},$$

$$\tilde{B}_{14\{1,2,9\}} = \begin{bmatrix} 0.2929 & 0.8375 & -0.0107 \\ -0.4550 & 0.3647 & -0.5926 \\ -0.6010 & -0.0715 & -0.1018 \\ 0.4076 & -0.1791 & -0.4638 \\ 0.0215 & 0.1982 & -0.0230 \\ -0.0800 & -0.0945 & -0.3217 \end{bmatrix}, \tilde{B}_{15\{1,3,7\}} = \begin{bmatrix} 0.2929 & -0.3092 & -0.3229 \\ -0.4550 & 0.5135 & 0.1223 \\ -0.6010 & -0.7219 & 0.0735 \\ 0.4076 & -0.1514 & 0.1397 \\ 0.0215 & -0.1458 & 0.5381 \\ -0.0800 & -0.1249 & -0.3587 \end{bmatrix},$$

$$\begin{aligned}
\tilde{B}_{16\{1,3,8\}} &= \begin{bmatrix} 0.2929 & -0.3092 & 0.1137 \\ -0.4550 & 0.5135 & 0.1738 \\ -0.6010 & -0.7219 & 0.3110 \\ 0.4076 & -0.1514 & 0.2103 \\ 0.0215 & -0.1458 & -0.3859 \\ -0.0800 & -0.1249 & -0.4867 \end{bmatrix}, \tilde{B}_{17\{1,3,9\}} = \begin{bmatrix} 0.2929 & -0.3092 & -0.0107 \\ -0.4550 & 0.5135 & -0.5926 \\ -0.6010 & -0.7219 & -0.1018 \\ 0.4076 & -0.1514 & -0.4638 \\ 0.0215 & -0.1458 & -0.0230 \\ -0.0800 & -0.1249 & -0.3217 \end{bmatrix}, \\
\tilde{B}_{18\{1,7,9\}} &= \begin{bmatrix} 0.2929 & -0.3229 & -0.0107 \\ -0.4550 & 0.1223 & -0.5926 \\ -0.6010 & 0.0735 & -0.1018 \\ 0.4076 & 0.1397 & -0.4638 \\ 0.0215 & 0.5381 & -0.0230 \\ -0.0800 & -0.3587 & -0.3217 \end{bmatrix}, \tilde{B}_{19\{1,8,9\}} = \begin{bmatrix} 0.2929 & 0.1137 & -0.0107 \\ -0.4550 & 0.1738 & -0.5926 \\ -0.6010 & 0.3110 & -0.1018 \\ 0.4076 & 0.2103 & -0.4638 \\ 0.0215 & -0.3859 & -0.0230 \\ -0.0800 & -0.4867 & -0.3217 \end{bmatrix}, \\
\tilde{B}_{110\{2,3,7\}} &= \begin{bmatrix} 0.8375 & -0.3092 & -0.3229 \\ 0.3647 & 0.5135 & 0.1223 \\ -0.0715 & -0.7219 & 0.0735 \\ -0.1791 & -0.1514 & 0.1397 \\ 0.1982 & -0.1458 & 0.5381 \\ -0.0945 & -0.1249 & -0.3587 \end{bmatrix}, \tilde{B}_{111\{2,3,8\}} = \begin{bmatrix} 0.8375 & -0.3092 & 0.1137 \\ 0.3647 & 0.5135 & 0.1738 \\ -0.0715 & -0.7219 & 0.3110 \\ -0.1791 & -0.1514 & 0.2103 \\ 0.1982 & -0.1458 & -0.3859 \\ -0.0945 & -0.1249 & -0.4867 \end{bmatrix}, \\
\tilde{B}_{112\{2,3,9\}} &= \begin{bmatrix} 0.8375 & -0.3092 & -0.0107 \\ 0.3647 & 0.5135 & -0.5926 \\ -0.0715 & -0.7219 & -0.1018 \\ -0.1791 & -0.1514 & -0.4638 \\ 0.1982 & -0.1458 & -0.0230 \\ -0.0945 & -0.1249 & -0.3217 \end{bmatrix}, \tilde{B}_{113\{2,7,8\}} = \begin{bmatrix} 0.8375 & -0.3229 & 0.1137 \\ 0.3647 & 0.1223 & 0.1738 \\ -0.0715 & 0.0735 & 0.3110 \\ -0.1791 & 0.1397 & 0.2103 \\ 0.1982 & 0.5381 & -0.3859 \\ -0.0945 & -0.3587 & -0.4867 \end{bmatrix}, \\
\tilde{B}_{114\{2,8,9\}} &= \begin{bmatrix} 0.8375 & 0.1137 & -0.0107 \\ 0.3647 & 0.1738 & -0.5926 \\ -0.0715 & 0.3110 & -0.1018 \\ -0.1791 & 0.2103 & -0.4638 \\ 0.1982 & -0.3859 & -0.0230 \\ -0.0945 & -0.4867 & -0.3217 \end{bmatrix}, \tilde{B}_{115\{3,7,8\}} = \begin{bmatrix} -0.3092 & -0.3229 & 0.1137 \\ 0.5135 & 0.1223 & 0.1738 \\ -0.7219 & 0.0735 & 0.3110 \\ -0.1514 & 0.1397 & 0.2103 \\ -0.1458 & 0.5381 & -0.3859 \\ -0.1249 & -0.3587 & -0.4867 \end{bmatrix}, \\
\tilde{B}_{116\{3,7,9\}} &= \begin{bmatrix} -0.3092 & -0.3229 & -0.0107 \\ 0.5135 & 0.1223 & -0.5926 \\ -0.7219 & 0.0735 & -0.1018 \\ -0.1514 & 0.1397 & -0.4638 \\ -0.1458 & 0.5381 & -0.0230 \\ -0.1249 & -0.3587 & -0.3217 \end{bmatrix}, \tilde{B}_{117\{7,8,9\}} = \begin{bmatrix} -0.3229 & 0.1137 & -0.0107 \\ 0.1223 & 0.1738 & -0.5926 \\ 0.0735 & 0.3110 & -0.1018 \\ 0.1397 & 0.2103 & -0.4638 \\ 0.5381 & -0.3859 & -0.0230 \\ -0.3587 & -0.4867 & -0.3217 \end{bmatrix}
\end{aligned}$$

The volumes of the ROA corresponding to the different set of driver nodes, obtained by applying the Algorithm 5.2, are given in the Table 5.4. From the Table 5.4, the obtained volume of ROA corresponding to set of driver nodes $N_{17} = \{7, 8, 9\}$ is the largest as compared to all other sets

of driver nodes of the network. It is clear from the Table 5.4 that if the optimal set of driver nodes is not selected as per the proposed algorithm, the stability region of network may reduce upto 95%. In conclusion, the set of driver nodes $N_{17} = \{7, 8, 9\}$ should be chosen as the optimal set of driver nodes of the network since it corresponds to the largest region of stability.

Table 5.4 : Volume of the ROA for different sets of driver nodes in application of robot formation control

Set of Driver Nodes	Corresponding Input Matrix B_i	Volume of ROA	$\frac{Vol_i}{Vol_{max}}$	Remark
$N_{17} = \{7, 8, 9\}$	B_{17}	0.7168	1	Maximum Volume
$N_1 = \{1, 2, 3\}$	B_1	0.0367	0.0512	95% Reduction
$N_2 = \{1, 2, 7\}$	B_2	0.0493	0.0688	94% Reduction
$N_3 = \{1, 2, 8\}$	B_3	0.1122	0.1565	85% Reduction
$N_4 = \{1, 2, 9\}$	B_4	0.0709	0.0990	91% Reduction
$N_5 = \{1, 3, 7\}$	B_5	0.1013	0.1414	86% Reduction
$N_6 = \{1, 3, 8\}$	B_6	0.0564	0.0786	93% Reduction
$N_7 = \{1, 3, 9\}$	B_7	0.0458	0.0639	94% Reduction
$N_8 = \{1, 7, 9\}$	B_8	0.1781	0.2485	76% Reduction
$N_9 = \{1, 8, 9\}$	B_9	0.1462	0.2039	80% Reduction
$N_{10} = \{2, 3, 7\}$	B_{10}	0.0559	0.0780	93% Reduction
$N_{11} = \{2, 3, 8\}$	B_{11}	0.0727	0.1014	90% Reduction
$N_{12} = \{2, 3, 9\}$	B_{12}	0.0604	0.0843	92% Reduction
$N_{13} = \{2, 7, 8\}$	B_{13}	0.2295	0.3202	68% Reduction
$N_{14} = \{2, 8, 9\}$	B_{14}	0.1482	0.2067	80% Reduction
$N_{15} = \{3, 7, 8\}$	B_{15}	0.3135	0.4374	57% Reduction
$N_{16} = \{3, 7, 9\}$	B_{16}	0.1786	0.2491	76% Reduction

5.6 SUMMARY

The chapter has presented a mathematical formulation and algorithms to identify an optimal set of driver nodes. The proposed theory and algorithms enable to identify the most suitable set of driver nodes by guaranteeing the largest ROA for a given control law. It has been observed through numerical example that 60% reduction in the stability region occurs in worst case if driver node is not selected using the proposed algorithm with a single input, whereas for the multiple inputs, in the worst case, the reduction of stability region is upto 95%. A close congruence between theoretical and simulation results has been obtained. The proposed theory and algorithms have been applied successfully for formation control of a robotics network. This chapter has considered only continuous-time models of complex network. However, a large number of applications require intermittent exchange of information in discrete-time setting. The next chapter will discuss discrete-time models of complex networks to choose optimal driver nodes.

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