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Stochastically stable chaos for q-deformed unimodal maps

3.1 INTRODUCTION

In this chapter, we describe the deformation schemes inspired by Heine and Tsallis in reference to q-deformed physical systems related to the quantum group structures and the statistical mechanics. We discuss the dynamics of deformed one-dimensional maps, in particular q-logistic map, q-Gaussian map and q-Ricker map. These maps have attractors coexisting with the stable fixed point which lies on the boundary of invariant interval. We focus on typical asymptotic behavior, which can be observed for large sets of maps and for almost all initial conditions. We show that the deformed maps have strong chaotic properties with positive frequency in parameter space. This leads to the fact that the deformed maps have a chaotic behaviour for a large space of deformed parameter q than the canonical maps.

This chapter is organized as follows. In Section 3.2, we consider the deformed logistic map of type-3 denoted by $F_{a,\varepsilon}(x)$. For this deformed map, we discuss the basic dynamics like existence of fixed point, periodic attractor and their stability. Further we describe the abundance of chaotic maps. We compute the parameters (a_n^*, ε_n) such that $F_{a_n^*,\varepsilon_n}$ is a Misiurewicz map. Then we show that in the neighborhood of a_n^*, the map $F_{a,\varepsilon}$ has an absolutely continuous invariant probability measure (acip). In Section 3.3, we discuss the dynamics of type-1 deformed Gaussian map denoted by $G_{c,q}(x)$. We show that, for a given sequence of parameters $\{q_n\} \to q^*$, there exist two sequences of Misiurewicz parameters $\{c_{1n}\}$ and $\{c_{2n}\}$ which are converging to c^* in such a way that the corresponding sequences of Misiurewicz maps G_{c_{1n},q_n} and G_{c_{2n},q_n} converges to the Misiurewicz map G_{c^*,q^*}. Further, we prove that these maps admits absolutely continuous invariant probability measure in the neighborhood of Misiurewicz parameters. In Section 3.4, we apply type-3 deformation on Ricker map and discuss the stochastically stable chaos for deformed Ricker map.

3.2 Q-LOGISTIC MAP

One-dimensional logistic map is given by

$$f_a(x) = ax(1-x) \tag{3.1}$$

where the parameter $a \in [0,4]$ and $x \in [0,1]$.

Type-3 deformation of a real number x is given by

$$[x]_\varepsilon = \frac{x}{1+\varepsilon(1-x)}. \tag{3.2}$$

where $\varepsilon \in (-1,\infty)$. The deformation $[x]_\varepsilon \to x$, as the deformation parameter $\varepsilon \to 0$.

Consider this deformation on x of $f_a(x)$ then we obtain the following q-logistic map.

$$\begin{aligned} f_a([x]_\varepsilon) &= \frac{ax(1+\varepsilon)(1-x)}{(1+\varepsilon-\varepsilon x)^2} \\ &\equiv F_{a,\varepsilon}(x) \end{aligned} \tag{3.3}$$

The q-logistic map $F_{a,\varepsilon}(x)$ becomes canonical logistic map as the limit $\varepsilon \to 0$, which is equivalent to $q \to 1$. We consider $\varepsilon \in (-1,2]$ in our discussion, because of all non-transient dynamics takes place in this bounded interval.

3.2.1 Dynamics of q-logistic map

The fixed points of the q-logistic map obtained by solving $F_{a,\varepsilon}(x) = x$ are given by

$$\begin{aligned} x^* &= 0 \\ x^*_+ &= 1 - \frac{a+a\varepsilon-2\varepsilon+\sqrt{a^2(1+\varepsilon)^2-4a\varepsilon(1+\varepsilon)}}{2\varepsilon^2} \\ x^*_- &= 1 - \frac{a+a\varepsilon-2\varepsilon-\sqrt{a^2(1+\varepsilon)^2-4a\varepsilon(1+\varepsilon)}}{2\varepsilon^2}. \end{aligned}$$

The fixed points x^*_+ and x^*_- will be real only if $a \geq \frac{4\varepsilon}{1+\varepsilon}$. The maximum value of $\frac{4\varepsilon}{1+\varepsilon}$ within the restricted value of ε is obtained at $\varepsilon = 2$; therefore, x^*_+ and x^*_- will be real for $a \geq 8/3$. Now we discuss the existence of the fixed points in the parameter plane (a,ε) which is shown in Fig. 3.1. The fixed point x^*_+ exists in the region R1 and it is always unstable. The other fixed point x^*_- exists for the region $R1 \cup R2$, whereas $x^* = 0$ exists in the region $R1 \cup R3$.

The stability of fixed points is shown in Fig. 3.2. In the region $S1$ the fixed point x^*_- is unique metric attractor. At the boundary between $S1 \cup S3$ and $S2 \cup S4$, the period doubling bifurcation occurs. In the region $S2$, the x^*_- takes the route of Feigenbaum period doubling to achieve chaos. In the region $S3$, the fixed point $x^* = 0$ coexists with the stable fixed point x^*_- and $x^* = 0$ is the unique attractor in $S5$. In the region $S4$, the $x^* = 0$ coexists with other attractors like periodic window, cantor attractor and chaotic attractor.

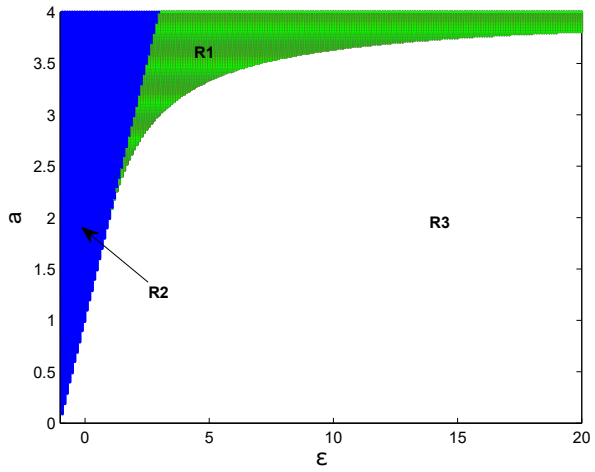


Figure 3.1. : Region of existence of fixed points.

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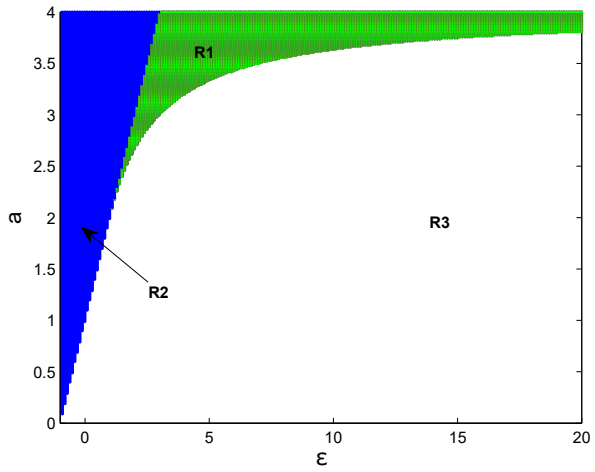


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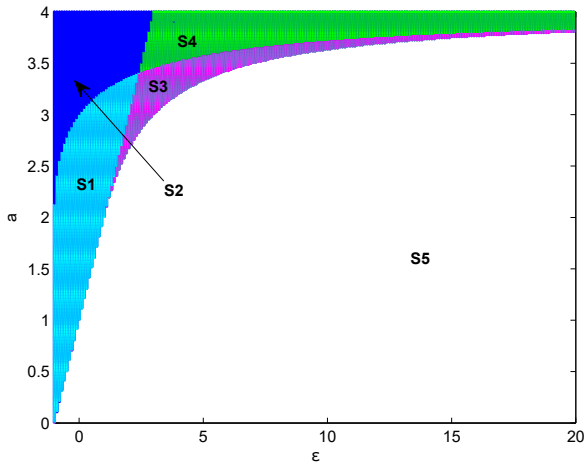


Figure 3.2. : Region of stability of fixed points.

3.2.2 Periodic attractor

Lemma 3.2.1. *The q-logistic map $F_{a,\varepsilon}(x)$ is S-unimodal map and it has non-flat critical point at c with $c = \frac{1+\varepsilon}{2+\varepsilon}$, which is unique for given ε.*

Proof. The Schwarzian Derivative of $F_{a,\varepsilon}(x)$ is

$$SF_{a,\varepsilon}(x) = \frac{-6(1+2\varepsilon+\varepsilon^2)}{(1+\varepsilon-\varepsilon x)^2(1-\varepsilon-\varepsilon x)^2}$$

Clearly $SF_{a,\varepsilon}(x)$ is negative for all $a \in [0,4]$ and $\varepsilon \in (-1,2]$. The second derivative $F''_{a,\varepsilon}(x) \neq 0$ at $x = \frac{1+\varepsilon}{2+\varepsilon}$. □

We know that an S-unimodal map with non-flat critical point has unique metric attractor. From [Graczyk *et al.*, 2004], every metric attractor of the q-logistic map $F_{a,\varepsilon}(x)$ is of the following types:

M1 a periodic attractor, or

M2 a Cantor attractor of measure zero, or

M3 a finite union of intervals.

There is at most one metric attractor of type M2 or M3. In the first two cases, metric attractor must attract the orbits of the critical point. The q-deformed one-dimensional maps can have at most two attracting periodic orbit. If the fixed point $x^* = 0$ is the only periodic attractor (region *S5* in Fig. 3.2) then it attract the orbits of the critical point. If the fixed point $x^* = 0$ coexists with other periodic attractor then the periodic attractor lie on the interval $(0,1)$ must attract the orbits of the critical point.

Lemma 3.2.2. *For $\varepsilon \in (-1,2)$, the q-logistic map $F_{a,\varepsilon}(x)$ has coexisting periodic attractor and has a unique metric attractor of type M2 or M3 which is attracting almost all initial conditions.*

Proof. The fixed points of the q-logistic map $F_{a,\varepsilon}(x)$ are $x^*=0$, x^*_+ and x^*_-. The fixed point $x^*=0$ is stable for $a\in(0,1+\varepsilon)$. The derivative $F'_{a,\varepsilon}(x^*_-)=1$ for $a=1+\varepsilon$ and $a=\frac{4\varepsilon}{1+\varepsilon}$. For $\varepsilon\in(-1,1)$, the fixed point x^*_- becomes stable if $a>1+\varepsilon$. For $\varepsilon\in(1,2)$, the fixed point x^*_- becomes stable if $a>\frac{4\varepsilon}{1+\varepsilon}$, and the fixed point $x^*=0$ coexists with the fixed point x^*_- for $a\in(\frac{4\varepsilon}{1+\varepsilon},1+\varepsilon)$. The 2-cycle of $F_{a,\varepsilon}(x)$ is given by

$$P_2=\frac{a+2\varepsilon+3a\varepsilon+a^2+2\varepsilon^2\pm\sqrt{\gamma}}{2\varepsilon^2},$$

where $\gamma=-a(a+\varepsilon+1)(3a+4\varepsilon-a\varepsilon-a^2)$. The 2-cycle become stable if $\gamma>0$, which gives $a>\frac{3-\varepsilon\sqrt{(\varepsilon+1)(\varepsilon+9)}}{2}$. When the 2-cycle born, the fixed point x^*_- loses its stability for $a>\frac{3-\varepsilon\sqrt{(\varepsilon+1)(\varepsilon+9)}}{2}$. Continue in this way, we obtain period doubling bifurcation route to chaos. □

From [Kozlovski, 1997], we recall the following result:

Remark 3.2.3. *If $f_{a_1,a_2,\ldots a_l}$ is an l-parameter family of real analytic S–unimodal maps of an interval, depending in a real analytic fashion on the parameter(s), such that there are at least two non conjugated maps in the family, in which one map with a stable periodic attractor, then maps with a stable periodic attractor are dense in the set of parameters $\{(a_1,a_2\ldots a_l)\}$.*

Lemma 3.2.4. *For $\varepsilon\in(-1,2)$, q-logistic map $F_{a,\varepsilon}(x)$ has a dense set of stable periodic attractor.*

Proof. There exist parameters a_i,a_j such that the maps $F_{a_i,\varepsilon}(x)$ and $F_{a_j,\varepsilon}(x)$ are non-conjugated to each other, since these maps have different periodic attractors. Therefore, by Remark 3.2.3, the stable periodic orbits are dense in parameter space (a-space). □

3.2.3 Abundance of strongly chaotic map

Consider the family $f_\lambda(x)=\lambda f(x)(mod\ 1)$ where f is C^3 with $f(0)=f(1)=0$ and assume that f has a unique critical point which is non-degenerate. Jakobson [Jakobson, 1981] proved that for a given class of map f_λ, there is a set of parameter values λ so that $f_\lambda(x)$ has an invariant measure absolutely continuous with respect to Lebesgue measure. This invariant measure exists for the parameters which are chosen in such a way that the critical point of the map is contained in the preimage of an unstable periodic orbit. Then we obtain the set $\Lambda_1=\{\lambda : f_\lambda$ has an acip with respect to Lebesgue measure$\}$ has a positive measure. The map corresponding to each parameter of the set Λ_1 has sensitive dependence on initial conditions. This leads to the stochastically stable chaotic behaviour of the map.

Definition 3.2.5. *Misiurewicz point is a parameter value at which the critical point is not periodic itself but lands onto the repelling periodic orbit in a finite number of iterations. The map at Misiurewicz point is known as the Misiurewicz map.*

The setup shown below is based on [De Melo and Van Strien, 2012]. We define a special class of $\mathscr{MU}$-map. The C^2-unimodal map $F_a : I\to I$ of parameter a is called $\mathscr{MU}$–map if

1. Each F_a has non-degenerate critical point which says second derivative at critical point is nonzero i.e., $D^2f(c)\neq 0$. We may expect that the critical point c does not depend on the parameter a.

2. Each F_a has fixed point on the boundary of I which is repelling.

3. The map $(x,a) \longrightarrow (F_a(x), D_x F_a(x), D_x^2 F_a(x))$ is C^1.

4. Suppose F_{a^*} is Misiurewicz map at the parameter value $a = a^*$. The forward orbit of $F_{a^*}(c)$ does not lie in the neighborhood U of the critical point and $F_{a^*}(c)$ is mapped onto unstable periodic cycle P^* in finite number of steps which implies F_{a^*} has no stable periodic attractors. We can find $\xi(a) \in I$ which is the small perturbation in unstable periodic orbit $P(a)$ of the map such that

   - $a \mapsto \xi(a)$ is differentiable;
   - $\xi(a^*) = F_{a^*}(c)$;
   - $P(a)$ moves continuously with a and $P(a^*) = P^*$;
   - $\xi(a)$ is mapped onto $P(a)$ in the same number of steps as $\xi(a^*)$ onto the corresponding point of $P(a)$.

5. $\frac{d}{da}(\xi(a) - F_a(c))|_{a=a^*} \neq 0$.

The deformed map $F_{a,\varepsilon}(x)$ acts as one-parameter family $F_{a,\varepsilon_n}(x)$ for fixed $\varepsilon = \varepsilon_n$. Clearly, the map $F_{a,\varepsilon_n}(x)$ satisfies conditions 1-3 of $\mathscr{MU}$ family. Now we prove the rest of two conditions from the following Proposition.

Proposition 3.2.6. *For each $\varepsilon_n \in (-0.95, 2) \setminus \{0\}$ and $n \in \mathbb{N}$, there exist a sequence $\{a_n^*\}$ such that $F_{a_n^*,\varepsilon_n}(x)$ is Misiurewicz map and satisfy condition 4 and condition 5 for the class of $\mathscr{MU}$-map.*

Proof. The critical point $c = \frac{1+\varepsilon}{2+\varepsilon}$ of the map $F_{a,\varepsilon}(x)$ is mapped onto unstable fixed point x_-^* in exactly three iterates, that is,

$$c \longrightarrow c_1 \longrightarrow c_2 \longrightarrow c_3 = x_-^*,$$

where $c_n = F_{a,\varepsilon}^n(c)$ for $n \geq 1$.

By MATLAB computations, we observe that for $a \in (3.3722, 4)$, the fixed point x_-^* is unstable for each $\varepsilon \in (-1, 2)$. For each ε_n, we compute a_n^* by checking the condition $F_{a,\varepsilon_n}^3(x) - \{x_-^*\}_{\varepsilon=\varepsilon_n} = 0$ for the parameter a at $x = c$. By numerical computation, we obtained a sequence $\{a_n^*\}$ of Misiurewicz parameters using 295 points of $\varepsilon \in (-0.95, 2) \setminus \{0\}$ and taking $a \in (3.3722, 4)$. The sequence $\{a_n^*\}$ shown in Fig. 3.3 for each ε_n. When $\varepsilon < 0$, the Misiurewicz parameter a is greater than the value of Misiurewicz parameter of canonical logistic map which is $a^* = 3.6785\ldots$ and when $\varepsilon > 0$, the value of $\{a_n^*\}$ is smaller than a^*.

By the Singer's theorem [Singer, 1978], the family of S-unimodal map on interval I can have at most one attracting periodic orbit in $I_1 = (0,1)$ and the critical point or boundary point is attracted by this periodic orbit. Furthermore, each neutral point is attracting. By the construction of a_n^*, the critical point itself mapped onto unstable cycle x_-^*. Hence $F_{a_n^*,\varepsilon_n}$ is a Misiurewicz map since it has neither stable nor neutral cycle.

Finally, we verify condition 5 of class $\mathscr{MU}$-map numerically for fixed $\varepsilon = \varepsilon_n$ and unique a_n^* corresponding to ε_n. We define

$$G(x;a) := F_{a,\varepsilon_n}(x)$$

$$x_-^*\big|_{\varepsilon=\varepsilon_n} := z(\varepsilon_n, a)$$

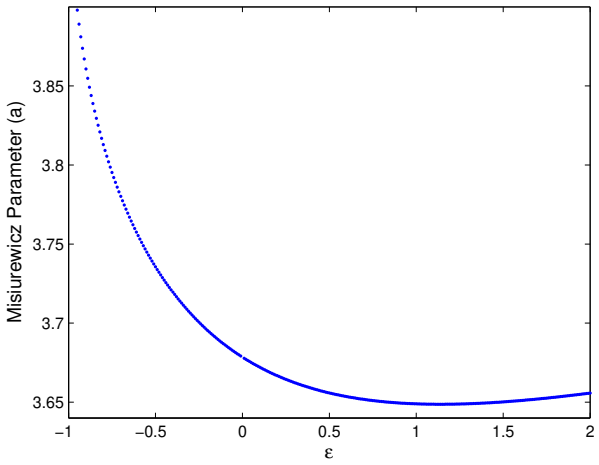


Figure 3.3. : Misiurewicz parameters a_n^* for 295 points of $\varepsilon \in (-0.95, 2)\backslash\{0\}$.

For $a \to a_n^*$ we define $\xi(a)$ by

$$\xi(a_n^*) = F_{a_n^*,\varepsilon_n}(c)$$

$$G^2(\xi(a);a) = z(\varepsilon_n, a)$$

where $\xi(a)$ is a point in the small neighborhood of $F_{a,\varepsilon_n}(c)$, with the same number of iterations as $F_{a_n^*,\varepsilon_n}(c)$. Let

$$\xi_1(a) := G(\xi(a);a)$$

and write condition 5 as

$$\frac{d}{da}(\xi(a) - G(c;a))\Big|_{a=a_n^*} \neq 0$$

Then from the definition of ξ we get

$$\begin{aligned} z'(a) &= \frac{d}{da}G(G(\xi;a);a) \\ &= \frac{\partial G}{\partial x}(\xi_1;a)\frac{d}{da}G(\xi;a) + \frac{\partial G}{\partial a}(\xi_1;a) \\ &= \frac{\partial G}{\partial x}(\xi_1;a)\left\{\frac{\partial G}{\partial x}(\xi;a)\frac{d\xi}{da} + \frac{\partial G}{\partial a}(\xi;a)\right\} + \frac{\partial G}{\partial a}(\xi_1;a) \\ \frac{d\xi}{da} &= \left(\frac{\partial G}{\partial x}(\xi;a)\right)^{-1}\left\{\left(\frac{\partial G}{\partial x}(\xi_1;a)\right)^{-1}\left(\frac{dz}{da} - \frac{\partial G}{\partial a}(\xi_1;a)\right) - \frac{\partial G}{\partial a}(\xi;a)\right\} \end{aligned} \tag{3.4}$$

By substituting $a = a_n^*$ and using definition of $\xi(a)$ and $\xi_1(a)$ in Eq. (3.4), condition 5 can be verified numerically.

Since we have $F_{a_n^*,\varepsilon_n}(c) = \xi(a_n^*)$ and $\frac{\partial F}{\partial a}(a_n^*, \varepsilon_n) \neq 0$ then by implicit function theorem for any ε close to ε_n there is a unique $a_n^*(\varepsilon)$ which satisfy condition 5. □

As a consequence of $\mathscr{MU}$-map, we immediately obtained following Lemma, which extends the result of [De Melo and Van Strien, 2012] to our family.

Lemma 3.2.7. *As $F_{a,\varepsilon}$ is a class of $\mathscr{MU}$-family then for each $\varepsilon = \varepsilon_n$, there exist two positive constants γ, C and subsets $\Lambda_n \subset (0,4]$ of parameters containing a_n^* as Lebesgue density point such that*

$$| D_x F_{a,\varepsilon_n}^{n_1}(F_{a,\varepsilon_n}(c)) | \geq Ce^{\gamma n_1} \quad \text{for all } a \in \Lambda_n \text{ and all } n_1 \geq 1.$$

Remark 3.2.8. *The critical point c should not accumulate too fast on the critical point for the construction of the good parameter sets Λ_n which is depending upon ε_n. For sufficiently small $\alpha > 0$, we require the following estimate :*

$$| F_{a,\varepsilon_n}^{n_1}(c) - c | \geq e^{-\alpha n_1} \quad \text{for all } a \in \Lambda_n \quad \text{and all } n_1 \geq 1.$$

For the class of S-unimodal map f with non-flat critical point, if f has a periodic attractor or a cantor attractor then f admits unique natural measure supported on the attractor [De Melo and Van Strien, 2012]. On account of the interval attractor, f does not guarantee the presence of natural measure. In the converse case if f admits an acip μ then f has interval attractor with unique natural measure supported by the attractor. Keller [Keller, 1990] proved that f exists acip if and only if f is an S-unimodal interval map with non-flat critical point of order l and also has positive Lyapunov exponent for almost all points. Indeed for practically $\limsup_{n_1 \to \infty} \frac{1}{n_1} \log | Df^{n_1}(x) |$ exists and have same value for almost all x. Nowicki and Van Strien [Nowicki and van Strien, 1991] showed that acip measure of f can be also ensure by much less condition. Namely, the summability condition is finite. i.e.,

$$\sum_{n_1=0}^{\infty} | Df^{n_1}(f(c)) |^{-1/l} < \infty,$$

then there exists an acip measure of f with a density L^p for all $p < \frac{l}{l-1}$.

The above discussion together with Proposition 3.2.6 and Lemma 3.2.7, we conclude the following Lemma:

Lemma 3.2.9. *As F_{a,ε_n} is an S-unimodal map and is class of $\mathscr{MU}$ family, then for all $a \in \Lambda_n$ we have*

1. *F_{a,ε_n} exists an acip measure μ_{a,ε_n} with a L^p density for any $p < 2$.*
2. *μ_{a,ε_n} is a natural measure that describes the asymptotics of almost all orbits.*
3. *The Lyapunov exponent of F_{a,ε_n} is positive for almost all points.*

Stability under random perturbation can be expressed as follows. Let f be an interval map with unique acip measure μ. To ensure stability of f our purpose is to show that the asymptotics are slightly affected when we replace f^n by $(f+t_1) \circ \ldots \circ (f+t_n)$, where t_i chosen at random in a small interval $[-\hat{\varepsilon}, \hat{\varepsilon}]$ following some probability distribution $\theta_{\hat{\varepsilon}}$. For each $\hat{\varepsilon} > 0$ we consider Markov chain $\chi^{\hat{\varepsilon}}$ on the Borel set E, the transition probabilities of which are given by $P^{\hat{\varepsilon}}(x,E) = \int_E \theta_{\hat{\varepsilon}}(y - fx)dy$. There is a unique probability measure $m_{\hat{\varepsilon}}$ that is stationary under $\chi^{\hat{\varepsilon}}$.

$$m_{\hat{\varepsilon}} = \int P^{\hat{\varepsilon}}(x,E)\, dm_{\hat{\varepsilon}}(x)$$

for every Borel set E. Then f is strongly stochastic stable under $\chi^{\hat{\varepsilon}}$ if

$$\lim_{\hat{\varepsilon} \to 0} \left\| \frac{dm_{\hat{\varepsilon}}}{dx} - \frac{d\mu}{dx} \right\|_{L^1(dx)} = 0.$$

Lemma 3.2.7 and Remark 3.2.8 together with the condition that $F_{a,\varepsilon}$ is topologically mixing proves that unique acip measure μ_a is stochastically stable [Baladi and Viana, 1996]. So the only condition which we have to prove is that $F_{a,\varepsilon}$ is topologically mixing. Since there is a full measure set of initial points whose orbits are distributed according to μ_a, and these orbits will be dense on the unique interval attractor. Hence every open set contains a point whose orbits are dense is topologically mixing and so $F_{a,\varepsilon}$ is topologically mixing.

From the above discussion and by Lemmas 3.2.7 and 3.2.9, we obtain the following theorem, which ensures the existence of absolutely continuous invariant probability measure and its properties for the map $F_{a,\varepsilon}$.

Theorem 3.2.10. *In the parameter plane, $F_{a,\varepsilon}$ are strongly chaotic with a positive frequency: For each $\varepsilon = \varepsilon_n \in (-0.95, 2) \setminus \{0\}$, there exists a Misiurewicz parameter a_n^* and positive measure subsets $\Lambda_n(\varepsilon) \subset (0,4]$ containing a_n^* such that if $a \in \Lambda_n(\varepsilon)$ then we have*

1. *There is no periodic attractor for F_{a,ε_n} and transitive interval attractor is the only metric attractor.*

2. *F_{a,ε_n} exists an acip μ_{a,ε_n} which is strongly stochastic stable and has density in L^p, $1 \leq p < 2$. The asymptotic distribution of almost all orbits under F_{a,ε_n} is defined by μ_{a,ε_n}.*

3. *The Lyapunov exponent of F_{a,ε_n} is positive almost everywhere, especially at the critical points.*

3.3 Q-GAUSSIAN MAP

In this section we introduce the q-Gaussian map with the deformation of type-1. Our goal is to prove the existence of absolutely continuous invariant measure for q-Gaussian map as similar to q-logistic map which is discussed in Section 3.2.

Let $g_c(x)$ be the Gaussian map, which is given by

$$g_c(x) = e^{-bx^2} + c, \tag{3.5}$$

where $b \neq 0$ and c is real parameter. This one-dimensional interval map with two-parameters qualitatively describes the dynamics of electrical circuit with a nonlinear diode which shows chaotic behaviour.

Type-1 deformation of real number x recalling from the Chapter 1 is given by

$$[x]_q = \frac{1-q^x}{1-q}, \tag{3.6}$$

such that $[x]_q \to n$ when $q \to 1$.

Applying type-1 deformation on x of $g_c(x)$, we obtain the following q-Gaussian map.

$$g_c([x]_q) = e^{-b\left(\frac{1-q^x}{1-q}\right)^2} + c \tag{3.7}$$

$$\equiv G_{c,q}(x) \tag{3.8}$$

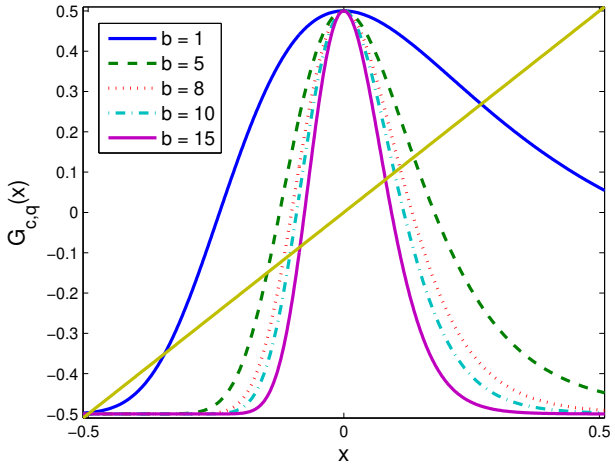


Figure 3.4. : Graph of q-Gaussian map at $c = -0.5$ and $q = 0.1$.

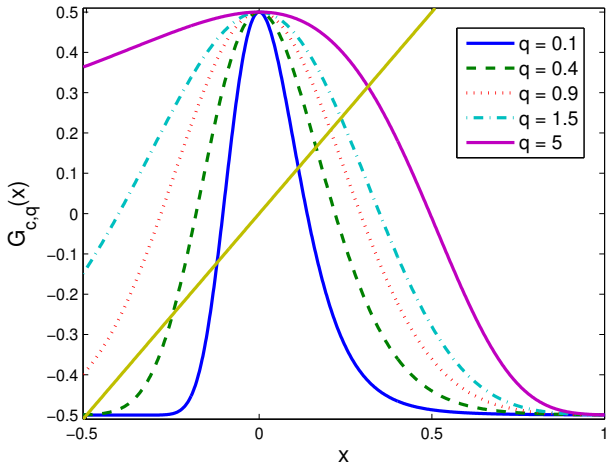


Figure 3.5. : Graph of q-Gaussian map at $c = -0.5$ and $b = 7.5$..

The q-Gaussian map is one-dimensional unimodal map with three parameters b, c and deformed parameter q, where $q > 0$, $b > 0$ and $c \in (-1,1)$. In this discussion, we consider $b = 7.5$, $q \in \{(0,2) \setminus \{1\}\}$ and $c \in (-1,1)$. As $q \to 1$, $[x]_q \to x$, the original Gaussian map can be obtained. As the parameter b changes from 1 to 15, the peak of the map is getting sharper and sharper, and it becomes symmetrical about the line $x = C$ (where C is critical point), and it is shown in Fig. 3.4. As we decrease the parameter q from 5 to 0.1, the graph of the map becomes symmetrical about the line $x = C$, which is shown in Fig. 3.5. Note that, the parameter c decides the peak of the map. As we increase c from -1 to 0, the graph is moving upward, which is shown in Fig. 3.6. Similarly as c varies from 0 to 1, the graph is moving downward.

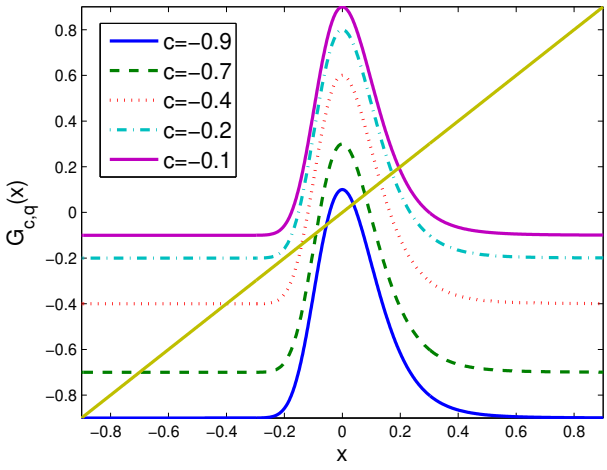


Figure 3.6. : Graph of q-Gaussian map at $q = 0.1$ and $b = 7.5$.

3.3.1 Fixed points

The q-Gaussian map $G_{c,q}(x)$ is transcendental equation so we need to employ the Newton-Raphson method to obtain the fixed points of the map. As the parameter c varies from -1 to 1, initially we get three fixed points in an order $x^* < x^*_- < x^*_+$. Out of these three fixed points, two of them x^* and x^*_- are negative and other one x^*_+ is positive. Later, we have only one fixed point x^*_+. For the different values of q, fixed points of the map are shown in Fig. 3.7. The thick line (black colour) indicates the stable fixed points, whereas the thin line (grey colour) indicates unstable fixed points. The fixed point x^* exists as coexisting attractor with other periodic attractors. The fixed point x^*_- is always unstable and x^*_+ undergoes period doubling bifurcation route to chaos. In the region where x^*_+ is unstable, we have other metric attractors (periodic orbit, a cantor set or a transitive interval attractor).

Remark 3.3.1. *The map* $G_{c,q}(x) = e^{-b\left(\frac{1-q^x}{1-q}\right)^2} + c$*—satisfies the following properties:*

a) $G_{c,q}(0) = 1 + c$*. The maximum value of the map is given by* $1 + c$ *which is attained at* $x = 0$*.*

b) The map has unique critical point $C = 0$ *which is non-degenerate i.e.* $G''_{c,q}(0) \neq 0$*.*

c) This is C^∞ *unimodal map from* $[-1,1]$ *to* $[-1,1]$ *with negative Schwarzian derivative.*

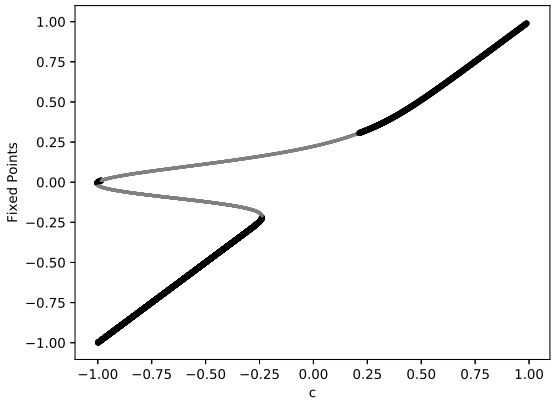
The $g_c(x)$ *and* $[.]_q$ *are given by the Eqs.* (3.5) *and* (3.6) *respectively. Then* $S(g \circ [\cdot]_q)(x)$*, can be written as*

$$S(g \circ [\cdot]_q)(x) = (S(g) \circ [\cdot]_q)(x) \cdot ([x]'_q)^2 + S([\cdot]_q)(x).$$

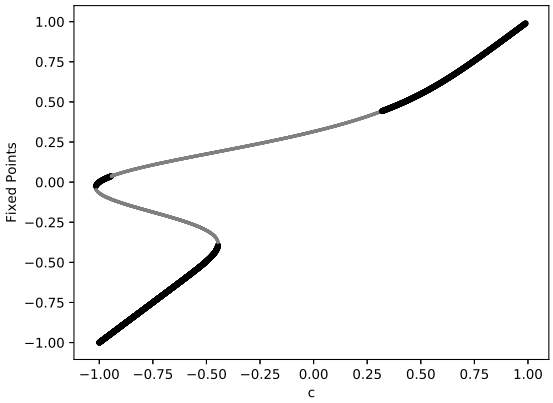
where $S(g)(x) = \frac{-4b^2x^4+3}{2x^2}$ *and* $S([\cdot]_q)(x) = \frac{-(\log q)^2}{2}$.

$$\begin{aligned} S(G_{c,q})(x) &= S(g \circ [\cdot]_q)(x) \\ &= -\frac{1}{2}(\log q)^2 \left(\frac{4b^2(q^x)^2\{(1-q^x)^4 + 3(1-q)^4\}}{2(1-q^x)^2(1-q)^4} + 1 \right). \end{aligned}$$

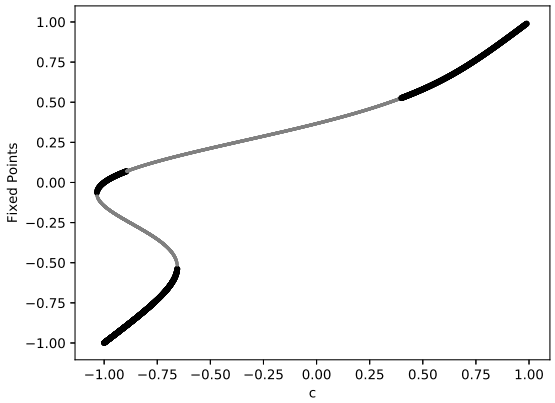
clearly which is negative for all c *and* q*.*



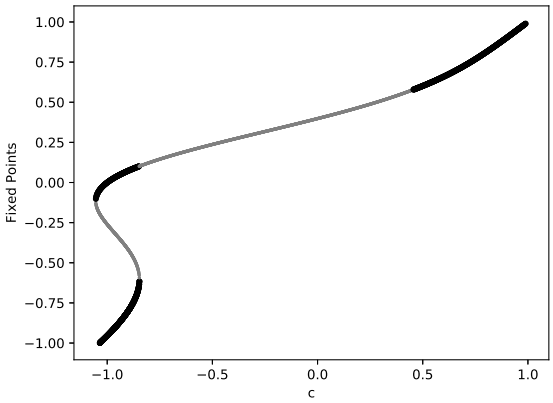
(i) q = 0.1



(ii) q = 0.5



(iii) q = 1.01



(iv) q = 1.5

Figure 3.7. : Fixed points of q-Gaussian map at different values of deformed parameter q and $c \in (-1,1)$.

3.3.2 Abundance of chaotic maps

As similar to Section 3.2.3 first we need to verify the conditions of $\mathscr{MU}$–map for $G_{c,q}$. From Remark 3.3.1, it is easy to see that conditions 1 and 3 are satisfied for $\mathscr{MU}$–map. Since $G_{c,q}(x)$ does not have any fixed point on the boundary of invariant interval, condition 2 can be relaxed. Rest of two conditions 4 and 5 for $G_{c,q}(x)$ can be described as:

As the q-Gaussian map undergoes direct as well as reverse period doubling bifurcation route to chaos when the parameter changes monotonically. Therefore, there exists parameters c_i $(i = 1,2)$ such that $G_{c_i,q}$ are Misiurewicz maps.

The forward orbit of $G_{c_i,q}$ does not lie in the neighborhood U of C (where C denotes the critical point of $G_{c_i,q}$). The critical point is mapped onto unstable periodic cycles P_1^* and P_2^* in three iterations for the parameter c_1 and c_2 respectively. This implies that $G_{c_i,q}$ has no stable periodic attractors. For each c near c_i $(i = \{1,2\})$, there exists points $\xi(c), \tilde{\xi}(c) \in I$ corresponding to c_1 and c_2 such that $G_{c_1,q} = \xi(c_1)$, $G_{c_2,q} = \tilde{\xi}(c_2)$ and the maps $c \to \xi(c)$, $c \to \tilde{\xi}(c)$ is differentiable. The

kneading sequence of $\xi(c), \tilde{\xi}(c)$ under $G_{c_i,q}$ is the same for each c sufficiently close c_i. Condition 5 gives $\frac{d}{dc}(\xi(c) - G_{c,q}(0))|_{c=c_1} \neq 0$ and $\frac{d}{dc}\left(\tilde{\xi}(c) - G_{c,q}(0)\right)|_{c=c_2} \neq 0$.

Conditions 4 and 5 will lead to the following Propositions.

Proposition 3.3.2. *For $n \in \mathbb{N}$, each $q_n \in (0, q^*)$, there exist two sequences $\{c_{1n}\}$ and $\{c_{2n}\}$ such that $G_{c_{1n},q_n}(x)$ and $G_{c_{2n},q_n}(x)$ are Misiurewicz maps. Further as $q_n \to q^*$ when $n \to \infty$, the sequences $\{c_{1n}\}$ and $\{c_{2n}\}$ converge to c^* in such a way that $G_{c^*,q^*}(x)$ is Misiurewicz map.*

Proof. We propose an algorithm to calculate the sequences of parameters $\{c_{1n}\}$ and $\{c_{2n}\}$ corresponding to $q_n \in (0, q^*)$. For a given q_n, first we calculate the fixed points for $c \in (-1, 1)$, and then we compute the interval region I_c of parameter c for which positive fixed point is unstable in this region. Then for each $q_n \in (0, 2)$, we compute the Misiurewicz parameter in such a way that third iterate of the map $G_{c,q_n}(x)$ at critical point lands onto the unstable fixed point x_+^*. We calculate the Misiurewicz parameters by using the following algorithm:

(1) For each $q_n \in (0, 2)$ and for a given c, we calculate the fixed points of $G_{c,q_n}(x)$ which are the roots of the equation $G_{c,q_n}(x) - x = 0$ by using Newton—Raphson method.

(2) Repeat the step-1 for each $c \in (-1, 1)$. For every c, we can get at most three fixed points in which two of them are negative and one is positive (say x_+^*).

(3) We determine the region I_c in which x_+^* is unstable by verifying the condition $|\, G'_{c,q_n}(x_+^*)\,| > 1$. For some values of q, the region I_c is shown in Table 3.1.

q value	I_c
0.1	(-0.982, 0.211)
0.6	(-0.933, 0.335)
0.9	(-0.904, 0.382)
0.99	(-0.895, 0.397)
1.05	(-0.889, 0.403)
1.2	(-0.875, 0.421)
1.4	(-0.857, 0.444)
2	(-0.805, 0.499)

Table 3.1. : Region I_c for different values of q.

(4) By computations using MATLAB, we obtain two Misiurewicz parameters $c_1, c_2 \in I_c$ which satisfy Equation $G_{c,q_n}^3(0) - x_+^* = 0$.

(5) Repeat the steps 1–4 for each $q_n \in (0, 2)$ and we get the sequences $\{c_{1n}\}$ and $\{c_{2n}\}$ of Misiurewicz parameters.

We employ the above algorithm for $q \in (0, 2)$ and $c \in (-1, 1)$, and then we obtained the two sequences $\{c_{1n}\}$ and $\{c_{2n}\}$ of Misiurewicz parameters which are shown in Fig. 3.8. Further we observe that the sequences $\{c_{1n}\}$ and $\{c_{2n}\}$ converge to c^* as $q_n \to q^*$. By computation, we get $c^* = -0.482$ and $q^* = 1.586$. This completes the proof. □

Remark 3.3.3. *For the fixed q value and $c \in I_c$, the q-Gaussian map has unique metric attractor which is a periodic orbit, a Cantor set or a transitive interval attractor.*

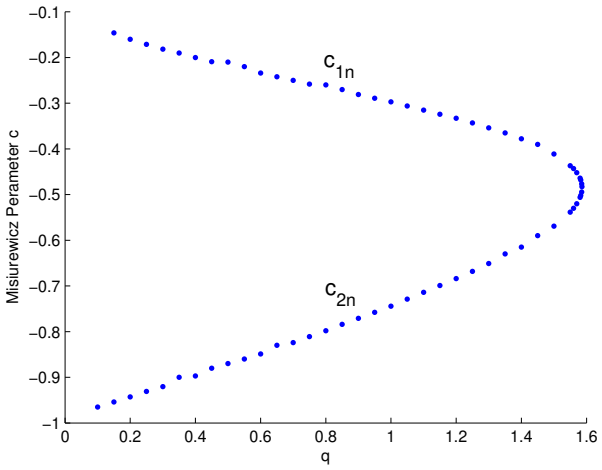


Figure 3.8. : Misiurewicz parameters c_{1n} and c_{2n} of q-Gaussian map for different q_n.

Proposition 3.3.4. *For each $q \in (0, q^*) \setminus \{1\}$, there exist c_i ($i = 1$ and 2) such that $G_{c,q}(x)$ is a one-parameter family with Misiurewicz map $G_{c_i,q}$ fulfilling condition 5.*

Proof. The critical point $C = 0$ is mapped onto positive fixed point x_+^* which is unstable for $c \in (\phi_1, \phi_2)$ in exactly three iterates for c_i.

The procedure to find c_i is given in Proposition 3.3.2. Now we verify condition 5 numerically. Define

$$F(x;c) = G_{c,q}(x)$$
$$F^n(x;c) = F(F^{n-1}(x;c);c) = G_{c,q}^n(x) \text{ for } n > 1.$$

For c close to c_1 we may define $\xi(c)$ by

$$\xi(c_1) = G_{c_1,q}(0),$$
$$F^2(\xi(c);c) = x_+^*,$$

where x_+^* is fixed point corresponding to c and $\xi(c)$ will be point close to $G_{c,q}(0)$, with same type of forward orbit as $G_{c^*,q}(0)$.

$$\xi_1(c) = F(\xi(c);c).$$

Let us write condition 5 as

$$\frac{d}{dc}(\xi(c) - F(C;c))\Big|_{c=c_1} \neq 0$$

Similar to Eq. (3.4) we can get value of $\frac{d\xi}{dc}$ which is given by

$$\frac{d\xi}{dc} = \left(\frac{\partial F}{\partial x}(\xi;c)\right)^{-1} \left\{ \left(\frac{\partial F}{\partial x}(\xi_1;c)\right)^{-1} \left(\frac{dx_+^*}{dc} - \frac{\partial F}{\partial c}(\xi_1;c)\right) - \frac{\partial F}{\partial c}(\xi;c) \right\} \tag{3.9}$$

As x_+^* is the solution of $G_{c,q}(x) - x = 0$, so by differentiating it with respect to c we get

$$\frac{dx_+^*}{dc} = \frac{(1-q)^2}{2b\log q(x_+^* - c)q^{x_+^*}(1-q^{x_+^*}) - (1-q)^2}$$

By substituting $c = c_1$ and using definition of $\xi(c)$ and $\xi_1(c)$ in Eq. (3.9), condition 5 can be verified numerically. Condition 5 can be checked similarly for $c = c_2$ and the corresponding $\bar{\xi}(c)$. □

The discussion provided for Lemmas 3.2.7 and 3.2.9 in Section 3.2 will also follow for $G_{c,q}(x)$ and consequently we obtain the following theorem for existence of absolutely continuous invariant probability measure for the given q-Gaussian map.

Theorem 3.3.5. *In the parameter plane, $G_{c,q}$ is strongly chaotic with a positive frequency: For each $q = q_n \in (0, q^*)$, there exists two sequences of positive measure subsets Λ_{jn} containing c_{jn} for $j = \{1,2\}$ such that for each $c \in \Lambda_{jn}$, we have*

1. *There is no periodic attractor for G_{c,q_n} and transitive interval attractor is the only metric attractor.*

2. *G_{c,q_n} exists an acip μ_{c,q_n} which is strongly stochastic stable and has density in L^p, $1 \leq p < 2$. The asymptotic distribution of almost all orbits under G_{c,q_n} is defined by μ_{c,q_n}.*

3. *The Lyapunov exponent of G_{c,q_n} is positive almost everywhere, especially at the critical points.*

Remark 3.3.6. *The Gaussian map $g_c(x) = e^{-bx^2} + c$ is S-unimodal map with unique, non-flat critical point $C = 0$. For a given $b = 7.5$, using the algorithm presented in Proposition 3.3.2, we obtain $c_1 = -0.7437$ and $c_2 = -0.2980$ such that $g_{c_1}(C)$ and $g_{c_2}(C)$ falls onto an unstable periodic orbit, the corresponding induced map also satisfies Lemmas 3.2.7 and 3.2.9 for c close to c_1 and c_2. This lead to the fact that for the Gaussian map $g_c(x)$, there exists a positive measure set of parameter c such that g_c admits an acip and c_1, c_2 are Lebesgue density points.*

3.4 Q-RICKER MAP

The Ricker map is given by

$$r_{\lambda,\beta}(x) = \lambda x e^{-\beta x},$$

where $\lambda > 1$ and $\beta > 0$. The state space of the Ricker map is $[0,\infty)$ but non transient dynamics occurs on a bounded interval.

Consider type-3 deformation $[x]_\varepsilon$ given by Eq. (3.2) on x of $r_{\lambda,\beta}(x)$ then we obtain the following q-Ricker map:

$$R_{\lambda,\beta,\varepsilon}(x) = r_{\lambda,\beta}([x]_\varepsilon) = \frac{\lambda x}{1+\varepsilon-\varepsilon x} e^{\frac{-\beta x}{1+\varepsilon-\varepsilon x}}, \tag{3.10}$$

where $\varepsilon \in (-1,\infty)$. For convenience of notation, we use $R(x)$ in place of $R_{\lambda,\beta,\varepsilon}(x)$.

3.4.1 Fixed points and stability

To calculate the fixed points of $R(x)$, we solve

$$R(x) - x = 0,$$

which gives

$$x\left[\frac{\lambda x}{1+\varepsilon-\varepsilon x} e^{\frac{-\beta x}{1+\varepsilon-\varepsilon x}} - 1\right] = 0.$$

The $x^* = 0$ is the trivial fixed point and the non-trivial fixed point x_1^* is given by

$$x_1^* = \left(\frac{1}{\varepsilon} + \frac{1}{\varepsilon^2}\right) \frac{\beta}{W\left(0, \frac{-\beta e^{-\beta/\varepsilon(1+\varepsilon)}}{\lambda\varepsilon}\right)},$$

where W is Lambert W Function. For real values of Lambert W Function, we have $\frac{-\beta e^{-\beta/\varepsilon}(1+\varepsilon)}{\lambda\varepsilon} > -\frac{1}{e}$.

3.4.2 Abundance of chaotic maps

In this subsection, we show that the q-Ricker map $R_{\lambda,\beta,\varepsilon}$ has stochastically stable chaos. The q-Ricker map $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}$ acts as one-parameter family R_λ for fixed β and ε. The q-Ricker map R_λ is a C^2-unimodal map with negative Schwarzian derivative and satisfies the conditions (1-3) of $\mathscr{MU}$ family. Now we prove the rest of two conditions by the following proposition.

Proposition 3.4.1. *For a fixed $\beta = \tilde{\beta}_i$ and for each $\varepsilon = \varepsilon_n \in (0,1)$ where $n \in \mathbb{N}$, there exist a sequence $\{\lambda_n^*\}$ such that $R_{\lambda_n^*,\tilde{\beta}_i,\varepsilon_n}(x)$ is Misiurewicz map and the map really moves with the parameter λ close to λ_n^*.*

Proof. We propose an algorithm to calculate the sequence of Misiurewicz parameters $\{\lambda_n^*\}$ corresponding to $\varepsilon_n \in (0,1)$ and for each $\tilde{\beta}_i$. First we calculate the fixed points $x \in (0,1)$ (say x_1^*) and then we find the interval I_λ of parameter $\lambda \in (0,17)$ for which the fixed point x_1^* is unstable. Then for each $\varepsilon \in (0,1)$, we compute the Misiurewicz parameter in such a way that third iterate of the map $R_\lambda(x)$ at critical point falls onto the unstable fixed point x_1^*. We calculate the Misiurewicz parameters by using the steps of the following algorithm:

(1) Fix the parameter $\beta = \tilde{\beta}_i \in \{2,3,4,5\}$ where $1 \le i \le 4$.

(2) For each $\varepsilon_n \in (0,1)$ and for a given λ, we calculate the fixed point x_1^* of $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}(x)$ by using Newton Raphson method. The x_1^* is the non-trivial fixed point and it is the solution of the equation $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}(x) - x = 0$ for $x \in (0,1)$.

(3) Repeat the step-1 for each $\lambda \in (0,17)$. For every λ we get one fixed point in the interval (0,1).

(4) We find the range of λ (say I_λ) for which x_1^* is unstable by verifying the condition $|R'_{\lambda,\tilde{\beta}_i,\varepsilon_n}(x_1^*)| > 1$.

(5) We solve the equation

$$R^3_{\lambda,\tilde{\beta}_i,\varepsilon_n}(c) - x_1^* = 0$$

where c is the critical point of $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}$ and $\lambda \in I_\lambda$. The Misiurewicz parameters λ_n^* corresponding to ε_n is the solution of the above equation, which is obtained by simulations (using MATLAB softwar).

(6) Repeat the steps 2-5 for each $\varepsilon_n \in (0,1)$ and we get a sequence $\{\lambda_n^*\}$ of Misiurewicz parameters.

(7) For each $\tilde{\beta}_i \in \{2,3,4,5\}$, we get the sequences $\{\lambda_n^*\}_{\tilde{\beta}_i}$.

By the above Proposition and previous discussion, for fixed $\varepsilon_n \in (0,1)$ there exists nonempty, open sets E_n containing ε_n such that for all $\varepsilon \in E_n$, there exists positive measure set Γ_λ such that if $\lambda \in \Gamma_\lambda$ then the map $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}$ admits an absolutely continuous invariant probability measure which is strongly stochastic stable in the sense of Baladi and Viana [Baladi and Viana, 1996]. For all $\lambda \in \Gamma_\lambda$, the map $R_{\lambda,\tilde{\beta}_i,\varepsilon_n}$ also has positive Lyapunov exponent for almost all initial conditions. □

From the Fig. 3.9, we observe that for every $\beta = \tilde{\beta}_i$, the Misiurewicz parameter is very close

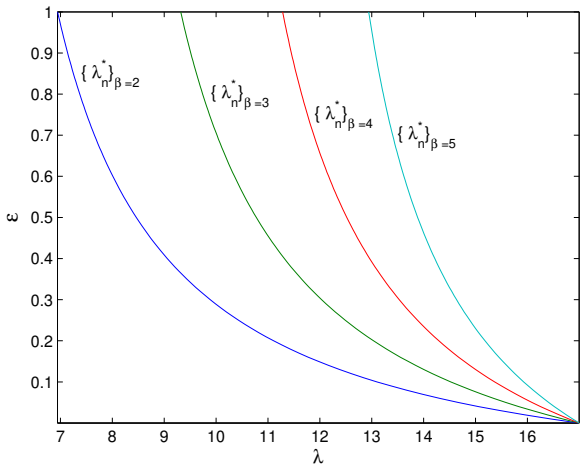


Figure 3.9. : Misiurewicz parameters λ_n^* of q-Ricker map for $\varepsilon \in (0,1)$ and for $2 \leq \beta \leq 5$.

to 16.99... for ε close to 0. As ε approaches to 1, the Misiurewicz parameter sequence $\{\lambda_n^*\}_{\beta=\tilde{\beta}_i}$ converging to the values $\lambda = \{6.9440, 9.3176, 11.2822, 12.9422\}$ for $\tilde{\beta}_i = \{2,3,4,5\}$ respectively.

From [Thunberg, 2001] the computational value of the Misiurewicz parameter in the classical Ricker map is $\lambda = 16.999$ and it is independent of the parameter β, while the Misiurewicz parameters in the q-Ricker map is dependent on the parameter β. For a fixed $\beta = 5$, the minimum value of Misiurewicz parameters $\{\lambda_n^*\}_{\beta=5}$ is 12.9422 which is obtained at $\varepsilon = 1$. As β decreases from 5 to 2, the minimum value of λ_n^* is shifting to the left which is shown in Fig. 3.9. For any $\beta = [2,3,4,5]$, the minimum value of λ_n^* is obtained at $\lambda = 6.944$ and $\beta = 2$. As a result, the q-Ricker map is able to reach stochastically stable chaos prior to the classical Ricker map.

3.5 CONCLUSIONS

In this chapter, we observed that by applying deformation on one-dimensional map, we obtained a new deformed one-dimensional map that fulfills our requirements. The obtained deformed dynamical systems must exhibit chaotic behavior in order to be used in cryptography applications. We noticed that the deformed maps have much larger key space of chaotic region than the key space of canonical maps. Therefore, the q-deformed system allows us to modify the dynamics by considering the different situations based on the dependency of deformed parameter q.

First, we showed that the deformed logistic map $F_{a,\varepsilon}(x)$ has multistability in periodic attractors. In the sense that the stable fixed point coexists with other stable periodic attractors for sufficiently higher value of ε. Further, we showed that for each $\varepsilon_n \in (-0.95, 2)$, the q-logistic map exhibits acip measure in the positive measure subsets $\Lambda_n(\varepsilon) \subset (0,4]$ containing a_n^* such that the map $F_{a,\varepsilon}$ has stochastically stable chaos.

We showed that the deformed Gaussian map $G_{c,q}(x)$ has multistability in periodic attractors. For sufficiently low value of c, the map has stable fixed point coexist with other periodic attractor. The map $G_{c,q}$ undergoes period doubling bifurcation as well as reverse period doubling bifurcation. Because of period doubling bifurcation from both sides, q-Gaussian map has two Misiurewicz

parameter values c_{1n} and c_{2n} for each $q_n \in (0, q^*)$. We showed that there exists two sets Λ_{1n} and Λ_{2n} of parameter value c for which q-Gaussian map admits an acip for each $q_n \in (0, q^*)$. The sequences c_{1n} and c_{2n} corresponding to Λ_{1n} and Λ_{2n} converge to the limit $c^* = -0.482$ as $n \to \infty$ at $q^* = 1.586$. This concludes that there exist a positive measure set λ_{c^*} containing c^* for which the map $G_{c,q}$ admits acip measure and c^* is Lebesgue density point such that $G_{c,q}$ has stochastically stable chaos.

The qualitative dynamics of q-Ricker map changes with respect to β, whereas the qualitative dynamics do not change with respect to the parameter β in case of canonical Ricker map. The Misiurewicz parameter of type $c \to c_1 \to c_2 \to c_3 \circlearrowleft$, where $c_i = R_\lambda^i(c)$ found much earlier than canonical Ricker map. This implies that the deformation on Ricker map changes the impact of β and found strong chaotic properties which can be useful in generating complexity in wide range of parameter space.