2
Signed Corona Product

In this chapter, the concept of signed corona product of graphs is introduced, with a focus on structural balance. The chapter explores how signed networks represent relationships that can be both positive and negative, making them suitable for modeling social, biological, and other complex systems. The concept of structural balance is discussed, highlighting its significance in ensuring the stability and coherence of such networks. The section sets the foundation for understanding the complexities of signed networks and their applications in various fields. It shows that by choosing an appropriate seed graph, corona graphs can be generated to preserve the properties (signed triangles and degree distribution) of real-world signed networks.

2.1 INTRODUCTION

Signed networks represent a framework to deal with binary relationship between nodes in a network that has two contradictory possibilities. For example, like and dislike, love and hate, trust and distrust are considered as measures of relationships between people, whereas alliance and antagonism between two countries can be considered as contradictory binary international relationships. Signed network model to represent social systems was first introduced by Harary and Cartwright to generalize the theory of *balanced state* of a social system developed by Heider in 1946 [Cartwright and Harary, 1956]. Heider rationalized his theory of balanced state by considering possible relationships in a system of three entities[Heider, 1946]. On the other hand, real world data can be put into the signed network setup to determine salient features of the data. For instance, in the trust network of Epinions, users establish binary relations to each other that reflect trust and distrust [Guha *et al.*, 2004; Leskovec *et al.*, 2010b]. Mathematically, a signed graph is a graph in which some edges are designated with positive sign reflecting positive relationship between the constituent pair of nodes and other edges are assigned negative sign which represents negative relationship. Thus signed graph is an ordered tuple $G^s = (V, E^s, \sigma)$ where V denotes the set of nodes, $E^s \subseteq V \times V$ the edge set, and $\sigma : E^s \to \{+, -\}$ is called the signature function [Zaslavsky, 1982]. In this Chapter I will use the terms signed graph and signed network interchangeably.

Structural balance of signed networks has always been the central topic in the study of real world signed networks. Most often it is exaggerated that real networks evolve towards balanced state [Doreian and Stokman, 2013]. See also [Doreian, 2004]. However, recently an article demonstrated that this is not the case always [Estrada, 2019]. If a signed network is not balanced it is called unbalanced. Evidently, it is unfair to call a signed network unbalanced if

only a few cycles in it are unbalanced compared to an unbalanced network in which most of the cycles are unbalanced. Thus the concept of degree of balance of a signed network has emerged in literature, and several measures are proposed to estimate it [Aref and Wilson, 2017, 2018; Estrada and Benzi, 2014; Singh and Adhikari, 2017; Aref, 2019; Kunegis *et al.*, 2010].

In this chapter, I first introduce the notion of corona product of signed graphs by generalizing the definition of corona product of unsigned graphs. Let G be an unsigned graph on n nodes, and H an unsigned graph. Then the corona product graph $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n copies of H, and then joining the ith vertex of G to every vertex in the ith copy of H [Frucht and Harary, 1970]. Note that except the existing edges of G, H in $G \circ H$ there will be nk new edges created where k denotes the number of nodes in H. For corona product of two signed graphs G^s and H^s, denoted by $G^s \circ H^s$ the job is to define the signs of these new edges so that the corona product of the corresponding underlying unsigned graphs equals the standard notion of corona product of unsigned graphs $G \circ H$. I define the signs of the new edges utilizing the formalism of marked graph which is a framework introduced by Beineke and Harary [Beineke, 1978; Harary, 1953]. I call this a marking scheme on the node set for the definition of corona product. I study structural balance of $G^s \circ H^s$, and provide a necessary and sufficient condition based on the structural properties of G^s, H^s such that $G^s \circ H^s$ becomes balanced. Obviously, the distribution of the numbers of T_is influence degree of balance of a signed network. Some of the earliest measures of degree of balance are defined in terms of the number of T_is. For example, the ratio of the number of signed to positive triangles in a signed network is considered as the degree of balance. The adjcency matrix $A = [a_{ij}]$ corresponding to a signed graph $G^s = (V, E^s, \sigma)$ is defined by $a_{ij} = 1$ if $(i, j) \in E^s$ and $\sigma((i, j)) = +$, $a_{ij} = -1$ if $\sigma((i, j)) = -$, and $a_{ij} = 0$ otherwise. The signed Laplacian matrix of G^s on n nodes is defined as $L = D - A$ where D is a diagonal matrix of order n whose ith diagonal entry is the total degree of ith node of G [Kunegis *et al.*, 2010]. The eigenvalues of A and L are called adjacency and signed Laplacian eigenvalues of G^s. The eigenvalues of these matrices inherit structural balance properties of the corresponding signed graph. For example, a signed graph G^s is balanced if and only if adjacency eigenvalues of G^s are same as the adjacency eigenvalues of the underlying unsigned graph G [Acharya, 1980]. If the graph is balanced then the least signed Laplacian eigenvalue is zero [Kunegis *et al.*, 2010]. For an unbalanced graph, the least signed Laplacian eigenvalues is considered as a measure of degree of balance of the graph and it is called algebraic conflict of the graph [Kunegis *et al.*, 2010; Aref, 2019]. I derive adjacency eigenvalues of $G^s \circ H^s$ in terms of adjacency eigenvalues of the signed graphs G^s and H^s respectively, when H^s is net-regular. I also determine signed Laplacian eigenvpairs of $G^s \circ H^s$.

Utilizing the corona product framework for unsigned graphs Sharma et al. [Sharma *et al.*, 2015, 2017] defined corona graphs as a model for generating large networks. See also [Lv *et al.*, 2015]. Given a connected graph $G = G^{(0)}$ the corona graphs are defined as

$$G^{(m)} = G^{(m-1)} \circ G,\ m \geq 1. \tag{2.1}$$

The graph G is called the seed graph for the corona graph $G^{(m)}$. Adapting this idea and

incorporating the definition of corona product introduced in this Chapter, I define signed corona graphs, denoted by $G^{s,(m)}$ by considering the seed graph as a signed graph G^s as in equation (2.1). Thus construction of signed corona graphs provides a deterministic growing signed network generative model. Obviously, the structural balance and spectral properties of $G^{s,(m)}$ depend on the choice of the seed graph and the distribution of signs of the new edges which appear due to the definition of corona product, that is, the marking scheme on the node set of the seed graph. The modelling perspective of preserving properties of real networks using $G^{s,(m)}$ raises the following question. Can I choose a seed graph G^s so that desired sign distribution, triangle distribution, degree of balance in $G^{s,(m)}$ be obtained? I show that this is indeed the case. That is, given the statistics of signs of edges and triangles of a real signed network it is possible to determine a seed graph such that the corresponding large signed corona graphs provide desired distribution of signed edges and triangles approximately. The algebraic conflict of $G^{s,(m)}$ can be obtained by iterative use of the formula of signed Laplacian eigenvalues derived for corona product of signed graphs discussed above. Besides, I determine degree sequence of $G^{s,(m)}$ from which the positive and negative degree distribution of $G^{s,(m)}$ can be computed and compared with real networks.

Throughout I assume that the seed graph is a simple connected signed graph. The main contributions of this Chapter are as follows.

(1) I introduce corona product of two signed graphs based on the framework of marked graphs and study its structural properties.

(2) I study spectra and Laplacian spectra of corona product of signed graphs, and provide computable expressions of eigenvectors corresponding to such eigenvalues.

(3) I define signed corona graphs and propose it as a signed network generative model. I provide computable formulae of number of signed edges, triangles, and positive and negative degrees of every node in a corona graph. I investigate algebraic conflict of corona graphs, which are generated by specially structured seed graphs. Finally I show that corona graphs can inherit properties of real signed networks.

The chapter is organized as follows. In Section 2.2 I define corona product of two signed graphs and determine structural and spectral properties of corona product of two graphs. In Section 2.3 I define signed corona graphs and investigate different statistical properties and algebraic conflict of signed corona graphs. Finally, I explore how a seed graph can be identified to generate a corresponding signed corona graph which can preserve certain statistical properties of real signed networks.

2.2 CORONA PRODUCT OF SIGNED GRAPHS

In this section, I introduce corona product $G^s \circ H^s$ of a pair of signed graphs G^s, H^s, and I study the structural balance and spectral properties of $G^s \circ H^s$. Note that this operation (product) is non-commutative, that is, $G^s \circ H^s$ need not be equal to $H^s \circ G^s$. First I review the notion of marked graphs as follows.

The graph whose each node is marked by either a positive or negative sign, is called as a marked graph. Mathematically, a marked graph is a 3-tuple $G_\mu = (V_\mu, E, \mu)$ where V_μ is a finite set of nodes, E the set of edges, and $\mu : V_\mu \to \{+, -\}$ is called the marking function. The the sign of an edge of the marked graph is defined as the product of signs of its adjacent vertices and is used to construct signed graphs using marked graphs[Harary and Kabell, 1981].

On the other hand, a marked graph can be defined from a signed graph $G^s = (V, E^s, \sigma)$ by defining the marking of a node $v \in V$ as

$$\mu(v) = \prod_{e \in E_v^s} \sigma(e) \tag{2.2}$$

where E_v^s is the set of signed edges adjacent at v. Such a method of marking is also known as *canonical marking* [Beineke, 1978; LOWELL W. BEINEKE, 1978]. Then the marked signed graph is denoted as $G_\mu^s = (V_\mu, E^s, \sigma, \mu)$. There can be multiple ways to define a marking function for a signed graph. I consider two marking functions, *canonical marking* and *plurality marking* in this Chapter. I define plurality marking of a node v of a signed graph $G^s = (V, E^s, \sigma)$ as

$$\mu(v) = \begin{cases} +, & \text{if } \max\{d^+(v), d^-(v)\} = d^+(v) \\ -, & \text{otherwise} \end{cases} \tag{2.3}$$

Hence a node is negatively marked in plurality marking scheme only when $d^-(v) > d^+(v)$.

Thus from now onward I denote a *marked signed graph* as a quadruple $G_\mu^s = (V_\mu, E^s, \sigma, \mu)$ where σ and μ are the signature function and marking function defined on the edge set E and node set V of the corresponding underlying unsigned graph respectively. In this Chapter, I consider a marked signed graph as the marked graph defined from a given signed graph with some appropriate marking scheme.

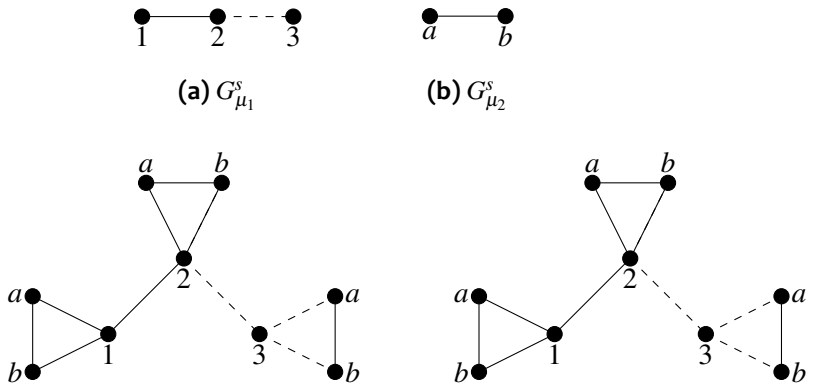
Thus I introduce the following definition for the corona product of two signed graphs.

Definition 2.2.1. *Let $G_{\mu_1}^s = (V_{\mu_1}, E^s, \sigma_1, \mu_1)$ and $G_{\mu_2}^s = (V_{\mu_2}^s, E^s, \sigma_2, \mu_2)$ be marked signed graphs on n and k nodes respectively. Then the corona product $G_{\mu_1}^s \circ G_{\mu_2}^s$ of $G_{\mu_1}^s$ and $G_{\mu_2}^s$ is a signed graph formed by taking one copy of $G_{\mu_1}^s$ and n copies of $G_{\mu_2}^s$, and then forming a signed edge from the i-th node of $G_{\mu_1}^s$ to every node of the i-th copy of $G_{\mu_2}^s$ for all i. The sign of the new edge between the i-th node of $G_{\mu_1}^s$, say u, and the j-th node in the i-th copy of $G_{\mu_2}^s$, say v, is given by $\mu_1(u)\mu_2(v)$ where μ_i is a marking scheme defined by σ_i, $i = 1, 2$.*

For instance, the corona product $G_{\mu_1}^s \circ G_{\mu_2}^s$ of signed graphs $G_{\mu_1}^s$ and $G_{\mu_2}^s$ is shown in Figure 2.1. The canonical and plurality marking are the same for the graph $G_{\mu_2}^s$. For $G_{\mu_1}^s$, the marking of the nodes 1 and 3 are the same for canonical and plurality markings, whereas the canonical and plurality markings of node 2 are $-$ and $+$ respectively. Thus the choice of the marking function results in different corona product graphs. Below I discuss structural and spectral properties of the corona product of two graphs.

2.2.1 Structural and spectral properties of $G_{\mu_1}^s \circ G_{\mu_2}^s$

I provide the statistics of the number of edges and triangles in $G^s = G_{\mu_1}^s \circ G_{\mu_2}^s = (V, E^s, \sigma)$ for a given pair of marked signed graphs $G_{\mu_1}^s = (V_{\mu_1}, E_1^s, \sigma_1, \mu_1)$ and $G_{\mu_2}^s = (V_{\mu_2}, E_2^s, \sigma_2, \mu_2)$ as



(a) $G^s_{\mu_1}$ **(b)** $G^s_{\mu_2}$

(c) $G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_1}, G^s_{\mu_2}$ are with canonical marking

(d) $G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_1}, G^s_{\mu_2}$ are with plurality marking

Figure 2.1 : The corona product $G^s_{\mu_1} \circ G^s_{\mu_2}$ is illustrated in subfigures (c) and (d). Subfigure (c) shows the product with the canonical marking function on $G^s_{\mu_1}$ and $G^s_{\mu_2}$, while subfigure (d) shows the product with the plurality marking function on $G^s_{\mu_1}$ and $G^s_{\mu_2}$. The canonical marking function assigns a unique mark to each vertex, whereas the plurality marking function assigns a mark based on the most common label among neighboring vertices.

follows. Observe that the number of nodes in $G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by $|V_{\mu_1}| + |V_{\mu_1}||V_{\mu_2}|$, where $|V^s_{\mu_i}|$ denotes the number of nodes in $G^s_{\mu_i}$, $i = 1, 2$. Let M^+_i and M^-_i denote the number of positively and negatively marked nodes in $G^s_{\mu_i}$, $i = 1, 2$, respectively. Table 2.1 describes the statistics of edges in $G^s_{\mu_1} \circ G^s_{\mu_2}$.

Edges	$G^s_{\mu_1}$	$G^s_{\mu_2}$	$G^s_{\mu_1} \circ G^s_{\mu_2}$
# of Edges	$\lvert E^s_1\rvert$	$\lvert E^s_2\rvert$	$\lvert E^s_1\rvert + \lvert V_{\mu_1}\rvert\lvert E^s_2\rvert + \lvert V_{\mu_1}\rvert\lvert V_{\mu_2}\rvert$
# of $+$ edges	$\lvert E^+_1\rvert$	$\lvert E^+_2\rvert$	$\lvert E^+_1\rvert + \lvert V_{\mu_1}\rvert\lvert E^+_2\rvert + \mathrm{M}^+_1\mathrm{M}^+_2 + \mathrm{M}^-_1\mathrm{M}^-_2$
# of $-$ edges	$\lvert E^-_1\rvert$	$\lvert E^-_2\rvert$	$\lvert E^-_1\rvert + \lvert V_{\mu_1}\rvert\lvert E^-_2\rvert + \mathrm{M}^+_1\mathrm{M}^-_2 + \mathrm{M}^-_1\mathrm{M}^+_2$

Table 2.1 : Statistics of number of nodes and edges in $G^s_{\mu_1} \circ G^s_{\mu_2}$.

Now I consider counting triangles in $G^s_{\mu_1} \circ G^s_{\mu_2}$. Let $s \in \{+, -\}$. I denote $|E^s_2|^{+}_{+}$ as the number of edges of sign s that connect two positively marked nodes in $G^s_{\mu_2}$, $|E^s_2|^{+}_{-}$ as the number of edges of sign s that connect one positively marked and one negatively marked node in $G^s_{\mu_2}$, and $|E^s_2|^{-}_{-}$ as the number of edges of sign s that connect two negatively marked nodes in $G^s_{\mu_2}$. Table 2.2 describes the count of triangles of type T_i, which denotes a triangle having i negative edges, $i = 0, 1, 2, 3$.

The proof of the formulas presented in Table 2.1 and Table 2.2 directly follows from the definition of the corona product and is easy to verify. Indeed, note that when a node i of $G^s_{\mu_1}$ gets linked with all the nodes of the ith copy of $G^s_{\mu_2}$, the total number of new triangles created is

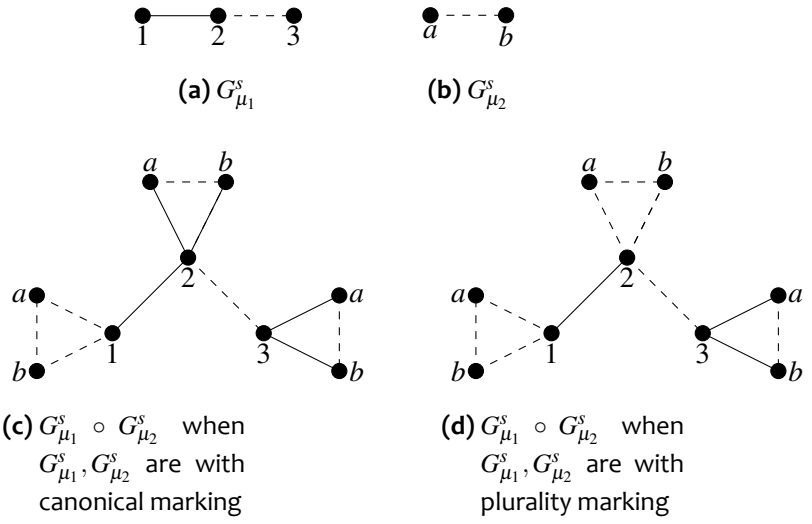
Triangles	$G^s_{\mu_1}$	$G^s_{\mu_2}$	$G^s_{\mu_1} \circ G^s_{\mu_2}$
# of T_0	$\lvert \mathrm{T}_0(G^s_{\mu_1})\rvert$	$\lvert T_0(G^s_{\mu_2})\rvert$	$\lvert \mathrm{T}_0(G^s_{\mu_1})\rvert + \lvert V_{\mu_1}\rvert \lvert \mathrm{T}_0(G^s_{\mu_2})\rvert + \mathrm{M}_1^+ \lvert E_2^+\rvert^+ + \mathrm{M}_1^- \lvert E_2^+\rvert^-$
# of T_1	$\lvert \mathrm{T}_1(G^s_{\mu_1})\rvert$	$\lvert \mathrm{T}_1(G^s_{\mu_2})\rvert$	$\lvert \mathrm{T}_1(G^s_{\mu_1})\rvert + \lvert V_{\mu_1}\rvert \lvert \mathrm{T}_1(G^s_{\mu_2})\rvert + \mathrm{M}_1^+ (\lvert E_2^+\rvert^\pm + \lvert E_2^-\rvert^+) + \mathrm{M}_1^- (\lvert E_2^+\rvert^\pm + \lvert E_2^-\rvert^-)$
# of T_2	$\lvert \mathrm{T}_2(G^s_{\mu_1})\rvert$	$\lvert \mathrm{T}_2(G^s_{\mu_2})\rvert$	$\lvert \mathrm{T}_2(G^s_{\mu_1})\rvert + \lvert V_{\mu_1}\rvert \lvert \mathrm{T}_2(G^s_{\mu_2})\rvert + \mathrm{M}_1^+ (\lvert E_2^+\rvert^- + \lvert E_2^-\rvert^\pm) + \mathrm{M}_1^- (\lvert E_2^+\rvert^+ + \lvert E_2^-\rvert^\pm)$
# of T_3	$\lvert \mathrm{T}_3(G^s_{\mu_1})\rvert$	$\lvert \mathrm{T}_3(G^s_{\mu_2})\rvert$	$\lvert \mathrm{T}_3(G^s_{\mu_1})\rvert + \lvert V_{\mu_1}\rvert \lvert \mathrm{T}_3(G^s_{\mu_2})\rvert + \mathrm{M}_1^+ \lvert E_2^-\rvert^- + \mathrm{M}_1^- \lvert E_2^-\rvert^+$

Table 2.2 : Counts of triangles in $G^s_{\mu_1} \circ G^s_{\mu_2}$.

$|E^s_2|$; that is, a triangle (i,j,l) is formed for every edge (j,l) in $G^s_{\mu_2}$. Now, the type of this triangle depends on the marking of i, j, l and the sign of the edge (j,l). The signs of all three edges are given by $\mu_1(i)\mu_2(j)$, $\mu_1(i)\mu_2(l)$, and $\sigma_2((j,l))$. The total number of triangles of $G^s_\mu = G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by

$$\mathrm{T}(G^s_\mu) = \mathrm{T}(G^s_{\mu_1}) + |V_{\mu_1}|(\mathrm{T}(G^s_{\mu_2}) + |E^s_2|), \tag{2.4}$$

It is evident that $G^s_{\mu_1} \circ G^s_{\mu_2}$ is unbalanced if one of the $G^s_{\mu_i}$, $i = 1,2$ is unbalanced. However if both $G^s_{\mu_i}$, $i = 1,2$ are balanced then it is not necessary that $G^s_{\mu_1} \circ G^s_{\mu_2}$ is balanced. For instance, the graph $G^s_{\mu_1} \circ G^s_{\mu_2}$ in Figure 2.1 is balanced while $G^s_{\mu_1}$ and $G^s_{\mu_2}$ are balanced, whereas the graph $G^s_{\mu_1} \circ G^s_{\mu_2}$ is unbalanced even if $G^s_{\mu_1}$ and $G^s_{\mu_2}$ are balanced in Figure 2.2.



(a) $G^s_{\mu_1}$ **(b)** $G^s_{\mu_2}$

(c) $G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_1}, G^s_{\mu_2}$ are with canonical marking

(d) $G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_1}, G^s_{\mu_2}$ are with plurality marking

Figure 2.2 : The corona product of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is shown in (c), (d) with different marking functions on $G^s_{\mu_i}, i = 1,2$. $(a) G^s_{\mu_1}$, $(b) G^s_{\mu_2}$, $(c) G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_1}, G^s_{\mu_2}$ are with canonical marking and (d) when $G^s_{\mu_1} \circ G^s_{\mu_2}$ are with plurality marking.

In the following theorem I classify when $G^s_{\mu_1} \circ G^s_{\mu_2}$ will be an unbalanced graph when $G^s_{\mu_i}$ is balanced, $i = 1,2$.

Theorem 2.2.1. *Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1, \mu_1)$ and $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2, \mu_2)$ be balanced marked signed graphs. Then $G^s_{\mu_1} \circ G^s_{\mu_2}$ is unbalanced if and only if $G^s_{\mu_2}$ contains one of the following types of edges.*

(i) a positive edge which connects two oppositely marked nodes

(ii) a negative edge which which connects two positively marked nodes

(iii) a negative edge which connects two negatively marked nodes.

Proof: The proof follows from the fact that any positively or negatively marked node of $G^s_{\mu_1}$ forms a triangle T_1 in $G^s_{\mu_1} \circ G^s_{\mu_2}$ when there are edges of type (i) and/or (ii) and/or (iii) in $G^s_{\mu_2}$, otherwise it forms a triangle of type T_3 when $G^s_{\mu_2}$ has an edge of type (ii) and/or (iii). □

It follows from Theorem 2.2.1 that for two balanced signed graphs $G^s_{\mu_1}$ and $G^s_{\mu_2}$, the corona product graph $G^s_{\mu_1} \circ G^s_{\mu_2}$ is balanced if and only if every positive edge of $G^s_{\mu_2}$ is incident to a pair of negatively or positively marked nodes, and every negative edge must be incidental to a pair of oppositely marked nodes.

Now I focus on the spectral properties of $G^s_{\mu_1} \circ G^s_{\mu_2}$. Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1, \mu_1)$ and $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2, \mu_2)$ be two marked signed graphs of order n and k, respectively. Suppose $V_{\mu_1} = \{u_1, u_2, \ldots, u_n\}$ and $V_{\mu_2} = \{v_1, \ldots, v_k\}$. Let us denote the marking vectors corresponding to vertex sets of $G^s_{\mu_1}$ and $G^s_{\mu_2}$. as $G^s_{\mu_2}$ as

$$\mu[V_{\mu_1}] = [\mu_1(u_1) \quad \mu_1(u_2) \quad \ldots \quad \mu_1(u_n)] \text{ and } \mu[V_{\mu_2}] = [\mu_2(v_1) \quad \mu_2(v_2) \quad \ldots \quad \mu_2(v_k)]$$

where $\mu_j(u) = 1$ if marking of $u = +$, otherwise $\mu_j(u) = -1$, $j = 1, 2$.

By selecting a suitable labeling of the nodes the adjacency matrix of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by

$$A(G^s_{\mu_1} \circ G^s_{\mu_2}) = \begin{bmatrix} A(G^s_{\mu_1}) & \mu[V_{\mu_2}] \otimes \operatorname{diag}(\mu[V_{\mu_1}]) \\ \mu[V_{\mu_2}]^T \otimes \operatorname{diag}(\mu[V_{\mu_1}]) & A(G^s_{\mu_2}) \otimes I_n \end{bmatrix} \tag{2.5}$$

where $A(G_i)$ is the adjacency matrix of graph G_i, $i = 1, 2$, $\otimes$ denotes the Kronecker product of matrices, I_n is the identity matrix of order n, and

$$\operatorname{diag}(\mu[V_{\mu_1}]) = \begin{bmatrix} \mu_1(u_1) & 0 & \ldots & 0 \\ 0 & \mu_1(u_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \mu_1(u_n) \end{bmatrix}.$$

Thus $A(G^s_{\mu_1} \circ G^s_{\mu_2})$ is a symmetric matrix of order $n + nk = n(1 + k)$. Recall that a signed graph is called net-regular if $d^+(v) - d^-(v) = d$ is same for every node v of the graph, and d is called the net-regularity of the graph. Then note that the net-regularity of a net-regular graph is always an eigenvalue of the graph and the all-one vector is the corresponding eigenvector. I denote the all-one vector of dimension k as $\mathbf{1}_k$. The following theorem provides adjacency spectra of $G^s_{\mu_1} \circ G^s_{\mu_2}$ when $G^s_{\mu_2}$ is net-regular.

Theorem 2.2.2. *Let $G^s_{\mu_1}$ be any marked signed graph of order n and $G^s_{\mu_2}$ be a net-regular marked signed graph of order k having net-regularity d. Let (λ_i, X_i) be an adjacency eigenpair of $G^s_{\mu_1}$, and (η_j, Y_j) be an eigenpair of $G^s_{\mu_2}$, $i = 1,\ldots,n$, $j = 1,\ldots,k$. Let $\eta_k = d$. Then the adjacency eigenpair of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by $(\lambda^{(i)}_{\pm}, Z^{(i)}_{\pm})$, $i = 1,\ldots,n$ where*

$$\lambda^{(i)}_{\pm} = \frac{d+\lambda_i \pm \sqrt{(d-\lambda_i)^2+4k}}{2}, Z^{(i)}_{\pm} = \begin{bmatrix} X_i \\ \frac{\mu_2(v_1)}{\lambda^{(i)}_{\pm}-d} diag(\mu[V_{\mu_1}])X_i \\ \frac{\mu_2(v_2)}{\lambda^{(i)}_{\pm}-d} diag(\mu[V_{\mu_1}])X_i \\ \vdots \\ \frac{\mu_2(v_k)}{\lambda^{(i)}_{\pm}-d} diag(\mu[V_{\mu_1}])X_i \end{bmatrix}.$$

In addition, if all the nodes in $V^s_{\mu_2}$ have either positive or negative marking, that is $\mu[V_2] = \mathbf{1}_k$ or $-\mathbf{1}_k$ then

$$\left(\eta_j, \begin{bmatrix} \mathbf{0} \\ Y_j \otimes e_i \end{bmatrix}\right)$$

is an eigenpair of $G^s_{\mu_1} \circ G^s_{\mu_2}$ where $j = 1,\ldots,k-1$, and $\{e_i : j = 1,\ldots,n\}$ the standard basis of $\mathbb{R}^n$.

Proof: Let (λ, Z) be an eigenpair of $G^s_{\mu_1} \circ G^s_{\mu_2}$ where $\lambda \in \mathbb{R}$, $Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_k \end{bmatrix}$, $Z_l \in \mathbb{R}^n$, $l = 0,1,\ldots,k$. Then setting $A(G^s_{\mu_1} \circ G^s_{\mu_2})Z = \lambda Z$ I obtain

$$\begin{bmatrix} A(G^s_{\mu_1}) & \mu[V_{\mu_2}] \otimes \mathrm{diag}(\mu[V_{\mu_1}]) \\ \mu[V_{\mu_2}]^T \otimes \mathrm{diag}(\mu[V_{\mu_1}]) & A(G^s_{\mu_2}) \otimes I_n \end{bmatrix} Z = \lambda Z$$

which yields the following system of equations.

$$\begin{aligned} A(G^s_{\mu_1})Z_0 + \mu_2(v_1)\mu[V_{\mu_1}]Z_1 + \ldots + \mu_2(v_k)\mu[V_{\mu_1}]Z_k &= \lambda Z_0 \\ \mu_2(v_1)\mu[V_{\mu_1}]Z_0 + \sum_{j=1}^{k} [A(G^s_{\mu_2})]_{1j} Z_j &= \lambda Z_1 \\ &\vdots \\ \mu_2(v_k)\mu[V_{\mu_1}]Z_0 + \sum_{j=1}^{k} [A(G^s_{\mu_2})]_{2j} Z_j &= \lambda Z_k \end{aligned}$$

where $[A(G^s_{\mu_2})]_{xj}$ denote the (x,j)th entry of $A(G^s_{\mu_2})$. Now adding the last k equations I obtain

$$\begin{aligned} & (\mu_2(v_1) + \mu_2(v_2) + \ldots + \mu_2(v_k))\mu[V_{\mu_1}]Z_0 + d(Z_1 + \ldots + Z_k) = \lambda(Z_1 + \ldots + Z_k) \\ \Rightarrow \quad & (\mu_2(v_1) + \mu_2(v_2) + \ldots + \mu_2(v_k))\mu[V_{\mu_1}]Z_0 = (\lambda - d)((Z_1 + \ldots + Z_k)). \end{aligned}$$

Set $Z_j = \frac{\mu_2(v_j)}{\lambda - d}\mu[V_{\mu_1}]Z_0$ $j = 1,\ldots,k$ and putting these in the first equation of the system I have

$$A(G^s_{\mu_1})Z_0 = \left(\lambda - \frac{k}{\lambda - d}\right) Z_0.$$

Now setting $Z_0 = X_i$ and $\lambda - \dfrac{k}{\lambda - d} = \lambda_i$ I obtain the quadratic polynomial equation $\lambda^2 - \lambda(d + \lambda_i) - (k - d\lambda_i) = 0$ solving which the roots $\lambda_{\pm}^{(i)}$ and its corresponding eigenvectors $Z_{\pm}^{(i)}$ follows.

If $\mu[V_{\mu_2}] = \mathbf{1}_k$ or $-\mathbf{1}_k$ then $\mu[V_{\mu_2}]Y_j = 0$ since $\mathbf{1}_k$ is an eigenvector of $A(G^s_{\mu_2})$, and eigenvectors of a symmetric matrix form an orthogonal set. Hence

$$\begin{bmatrix} A(G^s_{\mu_1}) & \mu[V_{\mu_2}] \otimes \mathrm{diag}(\mu[V_{\mu_1}]) \\ \mu[V_{\mu_2}]^T \otimes \mathrm{diag}(\mu[V_{\mu_1}]) & A(G^s_{\mu_2}) \otimes I_n \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ Y_j \otimes e_i \end{bmatrix} = \begin{bmatrix} \mu[V_{\mu_2}]Y_j \otimes \mathrm{diag}(\mu[V_{\mu_1}])e_j \\ \eta_j Y_j \otimes e_i \end{bmatrix} = \eta_j \begin{bmatrix} \mathbf{0} \\ Y_j \otimes e_i \end{bmatrix}.$$

This completes the proof. □

Note that if $G^s_{\mu_2}$ is net-regular with plurality marking then either $\mu_2(v_j) = +$ or $\mu_2 = -$ for all j. Hence the Theorem 2.2.2 provides all the eigenvalues of $G^s_{\mu_1} \circ G^s_{\mu_2}$ for such a signed graph $G^s_{\mu_2}$.

I mention that Barik et al. [Barik *et al.*, 2007] gave the complete description of adjacency eigenvalues of $G_1 \circ G_2$ when G_1, G_2 are unsigned and G_2 is regular. Theorem 2.2.2 generalizes their findings, and provides some insights about how signs of the edges and marking of the nodes influence the eigenpairs for corona product graphs.

Now I consider signed Laplacian spectra of corona product graphs. The signed Laplacian matrix associated with a signed graph G^s on n nodes is defined as $L(G^s) = D(G^s) - A(G^s)$ where $D(G^s)$ is a diagonal matrix of order n such that the ith diagonal entry is $d_i^+ + d_i^-$ and $A(G^s)$ is the adjacency matrix as usual [Zaslavsky, 1982]. Note that signed Laplacian matrix is a symmetric positive semi-definite matrix, and positive definite when the corresponding graph is unbalanced [Kunegis *et al.*, 2010; Hou, 2005].

First I have the following observation which will be used in sequel.

Lemma 2.2.3. *LetG^s be a signed graph onn nodes. Then $\mathbf{1}_n$ is an eigenvector corresponding to a signed Laplacian eigenvalue λ if and only if $d_i^- = d^-(= \lambda/2)$ for every node i in G^s.*

Proof: From the definition of signed Laplacian matrix, I have$L(G^s) = D(G^s) - A(G^s)$ where $[D(G^s)]_{ii} = d_i^+ + d_i^-$, the ith diagonal entry of $D(G^s)$. Then

$$L(G^s)\mathbf{1}_n = \begin{bmatrix} 2d_1^- \\ 2d_2^- \\ \vdots \\ 2d_n^- \end{bmatrix}.$$

Hence $\mathbf{1}_n$ is an eigenvector of $L(G^s)$ if and only if d_i^- equals a constant, say d^- for all i. This completes the proof. □

For a pair of marked signed graphs$G^s_{\mu_1} = (V_{\mu_1}, E_1^s, \sigma_1, \mu_1)$ and $G^s_{\mu_2} = (V_{\mu_2}, E_2^s, \sigma_2, \mu_2)$ on n and k nodes respectively, the signed matrix of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by

$$L(G^s_{\mu_1} \circ G^s_{\mu_2}) = \begin{bmatrix} L(G^s_{\mu_1}) + kI_n & -\mu[V_{\mu_2}] \otimes \mathrm{diag}(\mu[V_{\mu_1}]) \\ -\mu[V_{\mu_2}]^T \otimes \mathrm{diag}(\mu[V_{\mu_1}]) & (L(G^s_{\mu_2}) + I_k) \otimes I_n \end{bmatrix}$$

with a suitable labeling on the vertices of $G^s_{\mu_1} \circ G^s_{\mu_2}$. Then I have the following theorem.

Theorem 2.2.4. *Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1, \mu_1)$ be a marked signed graph on n nodes and $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2, \mu_2)$ be a signed graph on k nodes. Let $V_2 = \{v_1, \ldots, v_k\}$. Let (λ_i, X_i) be a signed Laplacian eigenpair of $G^s_{\mu_1}$, $i = 1, \ldots, n$. Let d_j^- denote the negative degree of a node v_j in $G^s_{\mu_2}$. Then the roots of the polynomial equations*

$$x - k - \frac{1}{x-(2d_1^-+1)} - \frac{1}{x-(2d_2^-+1)} - \ldots - \frac{1}{x-(2d_k^-+1)} = \lambda_i, \tag{2.6}$$

$i = 1, \ldots, n$, are signed Laplacian eigenvalues of $G^s_{\mu_1} \circ G^s_{\mu_2}$. An eigenvector corresponding to such an eigenvalue $x = \lambda$ corresponding to λ_i in equation (2.6) is given by

$$\begin{bmatrix} X_i \\ -\frac{\mu_2(v_1)}{\lambda-(2d_1^-+1)} diag(\mu[V_{\mu_1}])X_i \\ -\frac{\mu_2(v_2)}{\lambda-(2d_2^-+1)} diag(\mu[V_{\mu_1}])X_i \\ \vdots \\ -\frac{\mu_2(v_k)}{\lambda-(2d_k^-+1)} diag(\mu[V_{\mu_1}])X_i \end{bmatrix}.$$

where $i = 1, \ldots, n$.

Proof: The proof is similar to the proof of Theorem 2.2.2. Indeed, let $Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_k \end{bmatrix}$, $Z_j \in \mathbb{R}^n$, $j = 0, \ldots, k$ be an eigenvector corresponding to an eigenvalue λ of $L(G^s_{\mu_1} \circ G^s_{\mu_2})$. Then from $L(G^s_{\mu_1} \circ G^s_{\mu_2})Z = \lambda Z$ I have the following system of equations

$$\begin{aligned}
(L(G^s_{\mu_1}) + kI_n)Z_0 - \mu_2(v_1)\text{diag}(\mu[V_{\mu_1}])Z_1 - \ldots - \mu_2(v_k)\text{diag}(\mu[V_{\mu_1}])Z_k &= \lambda Z_0 \\
-\mu_2(v_1)\text{diag}(\mu[V_{\mu_1}])Z_0 + ([L(G^s_{\mu_2})]_{11}+1)Z_1 - \ldots + [L(G^s_{\mu_2})]_{1k}Z_k &= \lambda Z_1 \\
&\vdots \\
-\mu_2(v_k)\text{diag}(\mu[V_{\mu_1}])Z_0 + [L(G^s_{\mu_2})]_{k1}Z_1 - \ldots + ([L(G^s_{\mu_2})]_{kk}+1)Z_k &= \lambda Z_k
\end{aligned}$$

where $[L(G^s_{\mu_2})]_{xy}$ is the (x,y) entry of $L(G^s_{\mu_2})$. Then adding the last k equations I have

$$-(\mu_2(v_1) + \ldots + \mu_2(v_k))\text{diag}(\mu[V_{\mu_1}])Z_0 = \sum_{j=1}^{k} (\lambda - (2d_j^-+1))Z_j.$$

Setting $Z_j = -\frac{\mu_2(v_j)}{\lambda-(2d_j^-+1)}\text{diag}(\mu[V_{\mu_1}])Z_0$, $j = 1, \ldots, k$ and putting these in the first equation of the system I have

$$L(G^s_{\mu_1})Z_0 = \left(\lambda - k - \frac{1}{\lambda-(2d_1^-+1)} - \ldots - \frac{1}{\lambda-(2d_k^-+1)}\right)Z_0.$$

Hence the desired result follows by setting $Z_0 = X_i$. □

It is interesting to observe from Theorem 2.2.2 and Theorem 2.2.4 that adjacency and signed Laplacian eigenvalues of $G^s_{\mu_1} \circ G^s_{\mu_2}$ depend only on the degrees of the nodes of $G^s_{\mu_2}$

whereas the eigenvectors are influenced by degrees of nodes of $G^s_{\mu_2}$ and the marking scheme of $G^s_{\mu_1}$. Besides, observe that the equation (2.6) gives the complete list of eigenvalues of $G^s_{\mu_1} \circ G^s_{\mu_2}$ only when any two distinct nodes in $G^s_{\mu_2}$ have distinct negative degrees. Otherwise, the degrees of the polynomial equations (2.6) drop and it cannot produce all the eigenvalues. In the extreme case when negative degrees of nodes in $G^s_{\mu_2}$ are all equal then the polynomials become quadratic and produce only $2n$ eigenvalues. The following theorem considers this case.

Theorem 2.2.5. *Let $G^s_{\mu_1} = (V_{\mu_1}, E^s_1, \sigma_1, \mu_1)$ be a marked signed graph on n nodes and $G^s_{\mu_2} = (V_{\mu_2}, E^s_2, \sigma_2, \mu_2)$ be a signed graph on k nodes. Let $V_2 = \{v_1, \ldots, v_k\}$. Let (λ_i, X_i) be a signed Laplacian eigenpair of $G^s_{\mu_1}$, and (η_j, Y_j) be signed Laplacian eigenpairs of $G^s_{\mu_2}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$. Let $d^- = d^-_j$ denote the negative degree of node v_j in $G^s_{\mu_2}$. Then a signed Laplacian eigenpair of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is given by $(\lambda^{(i)}, Z^{(i)}_{\pm})$ where*

$$\lambda^{(i)}_{\pm} = \frac{2d^- + 1 + \lambda_i + k \pm \sqrt{[(2d^- + 1) - (\lambda_i + k)]^2 + 4k}}{2}$$

$$Z^{(i)}_{\pm} = \begin{bmatrix} X_i \\ -\dfrac{\mu_2(v_1)}{\lambda^{(i)}_{\pm} - (2d^- + 1)} diag(\mu[V_{\mu_1}])X_i \\ -\dfrac{\mu_2(v_2)}{\lambda^{(i)}_{\pm} - (2d^- + 1)} diag(\mu[V_{\mu_1}])X_i \\ \vdots \\ -\dfrac{\mu_2(v_k)}{\lambda^{(i)}_{\pm} - (2d^- + 1)} diag(\mu[V_{\mu_1}])X_i \end{bmatrix}$$

where $i = 1, \ldots, n$.

Let $\eta_k = 2d^-$. In addition, if all the nodes in $G^s_{\mu_2}$ are marked either positively or negatively then an eigenpair of $G^s_{\mu_1} \circ G^s_{\mu_2}$ is

$$\left(\eta_j + 1, \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix}\right), i = 1, \ldots, n, j = 1, \ldots, k-1$$

where $\{e_i : i = 1, \ldots, n\}$ is the standard basis of $\mathbb{R}^n$.

Proof: The first part follows from the proof of Theorem 2.2.4 by setting $d^-_j = d^-$ for $j = 1, \ldots, k$. Next, if $d^-_j = d^-$ for all j then by Lemma 2.2.3 $\mathbf{1}_k$ is an eigenvector of $G^s_{\mu_2}$ corresponding to the eigenvalue $2d^- = \eta_k$. Hence $\mathbf{1}^T_k Y_j = 0$ for $j = 1, \ldots, k-1$ since $L(G^s_{\mu_2})$ is a symmetric matrix and it has a complete set of orthogonal vectors. Further, if all the nodes of $G^s_{\mu_2}$ are positively marked then $\mu[V_{\mu_2}] = \mathbf{1}^T_k$. Finally,

$$L(G^s_{\mu_1} \circ G^s_{\mu_2}) \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix} = \begin{bmatrix} -\mu[V_{\mu_2}] Y_j \otimes e_i \\ (L(G^s_{\mu_2}) Y_j + Y_j) \otimes e_i \end{bmatrix} = (\eta_j + 1) \begin{bmatrix} 0 \\ Y_j \otimes e_i \end{bmatrix}.$$

If the nodes of $G^s_{\mu_2}$ are negatively marked, then $\mu[V_{\mu_2}] = -\mathbf{1}^T_k$ and hence the desired result follows. □

I emphasize that Theorem 2.2.5 generalizes the result on Laplacian spectra of corona product of unsigned graphs obtained in [Barik *et al.*, 2007]. Indeed, setting $d^- = 0$ in Theorem 2.2.5, it describes a complete set of eigenpairs of the corona product of two unsigned graphs.

2.3 APPLICATION TO SIGNED NETWORK MODELING

Corona product of unsigned graphs has been used as a framework to develop models for complex networks in literature [Sharma *et al.*, 2017; Sharma and Adhikari, 2015]. Inspired by this idea I propose signed corona graphs as a model for generating large signed networks. In this approach, I consider a small connected marked signed graph $G_\mu^s = G_\mu^{s,(0)}$ as a seed graph for generation of signed corona graphs given by

$$G_\mu^{s,(m)} = G_\mu^{s,(m-1)} \circ G_\mu^s, m \geq 1$$

with a suitable choice of marking function which will be used in every step $i \leq m-1$ of the formation of $G_\mu^{s,(m)}$. I explore different structural and spectral properties of $G_\mu^{s,(m)}$ which contribute to investigate how close does it preserve properties of real signed networks. Finally I consider the problem of structural reconstruction of real signed networks using $G_\mu^{s,(m)}$ by a suitable choice of seed graph that can inherit properties of real networks.

Obviously the structural properties of the seed graph G_μ^s influence the structural and spectral properties of $G_\mu^{s,(m)}$. For example, if the edges in G_μ^s are of the type mentioned in (i), (ii), (iii) of Theorem 2.2.1 and all triangles in G_μ^s are unbalanced then all the triangles in $G_\mu^{s,(m)}$ will be unbalanced. Thus in order to have balanced triangles in $G_\mu^{s,(m)}$, G_μ^s must have edges which does not fall in the category of (i), (ii), (iii) of Theorem 2.2.1 or at least one balanced triangle in G^s itself. Below I do a thorough analysis of structural and spectral properties of corona graphs generated by following the canonical marking scheme μ employed at every step of the generation of the signed corona graphs. Thus I omit the subscript μ from all the marked signed corona graphs henceforth, and I denote the signed corona graphs as

$$G^{s,(m)} = G^{s,(m-1)} \circ G^s, m \geq 1$$

with the corresponding seed graph G^s.

I mention that the analytical expression of different properties of corona graphs corresponding to plurality marking is complicated and cumbersome.

2.3.1 Statistics of signed edges and triangles in $G^{s,(m)}$

First I have the following observations about $G^{s,(m)}$, $m \geq 1$ generated by a seed graph G^s on n nodes with k edges.

1. Total number of nodes in $G^{s,(m)}$ is $n(n+1)^m$ [Sharma *et al.*, 2017].

2. The total number of copies of the seed graph in $G^{s,(m)}$ is $(n+1)^m$. This follows from that fact that at each step $1 \leq i \leq m$ the number of seed graphs added during the formation of $G^{s,(i)}$ is the number of nodes in $G^{s,(i-1)}$. Thus the total number of copies of the seed graph is
$$1 + n + n(n+1) + \ldots + n(n+1)^{m-1} = 1 + n\sum_{j=0}^{m-1}(n+1)^j = (n+1)^m.$$

3. The total number of edges added in the formation of $G^{s,(i+1)}$ from $G^{s,(i)}$, $0 \leq i \leq m-1$ due to the definition of corona product, that is, excluding the edges in $G^{s,(i)}$ and edges in the

newly appeared copies of seed graphs is given by $n^2(n+1)^i$. This follows from that fact that for each node v in $G^{s,(i)}$, n new edges will be created, and the number of nodes in $G^{s,(i)}$ is $n(n+1)^i$.

4. The total number of edges in $G^{s,(m)}$ is $k(n+1)^m + \sum_{j=0}^{m-1} n^2(n+1)^j = (k+n)(n+1)^m - n$.

I denote $n_+^{(i)}$ and $n_-^{(i)}$ the number of positive and negative nodes in $G^{s,(i)}$, $0 \leq i \leq m-1$ respectively. Thus the total number of nodes in $G^{s,(i)} = n(n+1)^i (= n_0^{(i)} + n_-^{(i)} = n^{(i)}$, say). I denote the vertex set which consists of the marked nodes in $G^{s,(i)}$ as $V^{(i)}$, $1 \leq i \leq m$. Let the number of positive and negative links in $G^{s,(0)}$ be denoted as $e_+^{(0)}$ and $e_-^{(0)}$ respectively.

Consider counting the number of signed edges in $G^{s,(m)}$. Note that the number of copies of the seed graph added during the formation of $G^{s,(i+1)}$ from $G^{s,(i)}$ is $n(n+1)^i$ (which is exactly the number of nodes in $G^{s,(i)}$), $0 \leq i \leq m-1$. Then in $G^{s,(i+1)}$ the number of new positive and negative links are created as

$$e_+^{(i+1)} = e_+^{(0)} n(n+1)^i + n_+^{(i)} n_+^{(0)} + n_-^{(i)} n_-^{(0)} \tag{2.7}$$

$$e_-^{(i+1)} = e_-^{(0)} n(n+1)^i + n_+^{(i)} n_-^{(0)} + n_-^{(i)} n_+^{(0)} \tag{2.8}$$

respectively. The first term in the expression of $e_+^{(i+1)}$ counts the number of positive links which are part of the newly added copies of seed graph, the second term is the number of links which are formed by joining positively marked nodes in $V^{(i)}$ and positively marked nodes in the copies of the seed graph, and the third term is the number of positive edges formed by existing negatively marked nodes in $G^{s,(i)}$ and the newly appeared negatively marked nodes. Similarly, the expression of $e_-^{(i+1)}$ follows.

Now let us count the number of positively and negatively marked nodes in $G^{s,(i+1)}$ after its formation from $G^{s,(i)}$. There can be two types of nodes in $G^{s,(i+1)}$, the newly appeared nodes $v \in V^{s,(i+1)} \setminus V^{s,(i)}$ and the existing set of nodes $V^{(i)}$ of $G^{s,(i)}$. First, consider the nodes in $V^{(i+1)} \setminus V^{(i)}$. The number of positively and negatively marked nodes among such nodes shall be $n_+^{(i)} n$ and $n_-^{(i)} n$ respectively. This follows from the fact that when a new node (positively/negatively marked) gets linked to an existing positively (resp. negatively) marked node in $V^{s,(i)}$, then it becomes positively (resp. negatively) marked due to the definition of canonical marking. Now let us consider the marking of nodes in $V^{(i+1)} \cap V^{(i)}$. Note that any node $v \in V^{(i+1)} \cap V^{(i)}$ was already marked during the formation of $G^{s,(i)}$. When $G^{s,(i+1)}$ is formed, each such node forms links to all the nodes in a copy of a newly appeared seed graph. If it was $\mu(v) = +$ then it remains the same marking when $n_-^{(0)}$ is even and $n_+^{(0)}$ is even or odd, whereas it changes its marking only when $n_-^{(0)}$ is odd and $n_+^{(0)}$ is either even or odd. Similarly, if it was $\mu(v) = -$ then it remains the same marking for even value of $n_+^{(0)}$ and any value of $n_-^{(0)}$, whereas it changes its marking when $n_+^{(0)}$ is odd and any value of $n_-^{(0)}$. Thus finally I have the following. The number of total positively and negatively marked nodes in $G^{s,(i+1)}$ is described in Table 2.3.

Now I are in a position to estimate the number of signed links in $G^{s,(m)}$ by utilizing equations (2.7), (2.8) and the fact that $n_+^{(i)} = n^{(i)} - n_-^{(i)}$.

Marking	Condition on G^s	in $V^{s,(i+1)} \setminus V^{s,(i)}$	in $V^{s,(i+1)} \cap V^{s,(i)}$	in $G^{s,(i+1)}$
$+$	$n_+^{(0)}$ odd $n_-^{(0)}$ odd	$n_+^{(i)} n$	$n_-^{(i)}$	$n_-^{(i)} + n n_+^{(i)}$
	$n_+^{(0)}$ even $n_-^{(0)}$ odd		0	$n_+^{(i)} n$
	$n_+^{(0)}$ even $n_-^{(0)}$ even		$n_+^{(i)}$	$n_+^{(i)}(1+n)$
	$n_+^{(0)}$ odd $n_-^{(0)}$ even		$n^{(i)}$	$n^{(i)} + n_+^{(i)} n$
$-$	$n_+^{(0)}$ odd $n_-^{(0)}$ odd	$n_-^{(i)} n$	$n_+^{(i)}$	$n_+^{(i)} + n n_-^{(i)}$
	$n_+^{(0)}$ even $n_-^{(0)}$ odd		$n^{(i)}$	$n_-^{(i)} n + n^{(i)}$
	$n_+^{(0)}$ even $n_-^{(0)}$ even		$n_-^{(i)}$	$n_-^{(i)}(1+n)$
	$n_+^{(0)}$ odd $n_-^{(0)}$ even		0	$n_-^{(i)} n$

Table 2.3 : The relation between the number of marked nodes in $G^{s,(i+1)}$ and $G^{s,(i)}$. The last column displays the values of $n_+^{(i+1)}$ and $n_-^{(i+1)}$.

$$e_+^{(m)} = e_+^{(0)} + \sum_{i=0}^{m-1} e_+^{(i+1)} = e_+^{(0)} \left(1 + n \sum_{i=0}^{m-1} (n+1)^i\right) + n_+^{(0)} \sum_{i=0}^{m-1} n_+^{(i)} + n_-^{(0)} \sum_{i=0}^{m-1} n_-^{(i)}$$
$$= e_+^{(0)} (n+1)^m + n_+^{(0)} [(n+1)^m - 1] + (n_-^{(0)} - n_+^{(0)}) \sum_{i=0}^{m-1} n_-^{(i)}$$

where the values of $n_-^{(i)}$ can be computed from the recurrence relation given in Table 2.3 for all i. Similarly,

$$e_-^{(m)} = e_-^{(0)} (n+1)^m + n_-^{(0)} [(n+1)^m - 1] + (n_+^{(0)} - n_-^{(0)}) \sum_{i=0}^{m-1} n_-^{(i)}.$$

It is evident from the above expressions that the marking of nodes in the seed graph plays the key role. Besides, the number of positive and negative edges become equal when the number of positively and negatively marked nodes are the same in the seed graph.

Now I focus on counting the number of triangles of type T_i, $i = 0,1,2,3$ in $G^{s,(m)}$. First observe that there can be six types of links in a signed graph. A positive/negative link can be incidental to positively marked nodes, oppositely marked nodes, or negatively marked nodes. When such nodes in a triangle in a copy of the seed graph get linked during the formation of

each step of the corona graph $G^{s,(i+1)}$ from $G^{s,(i)}$, different types of triangles are created. The Figure 2.3 depicts all the possible cases.

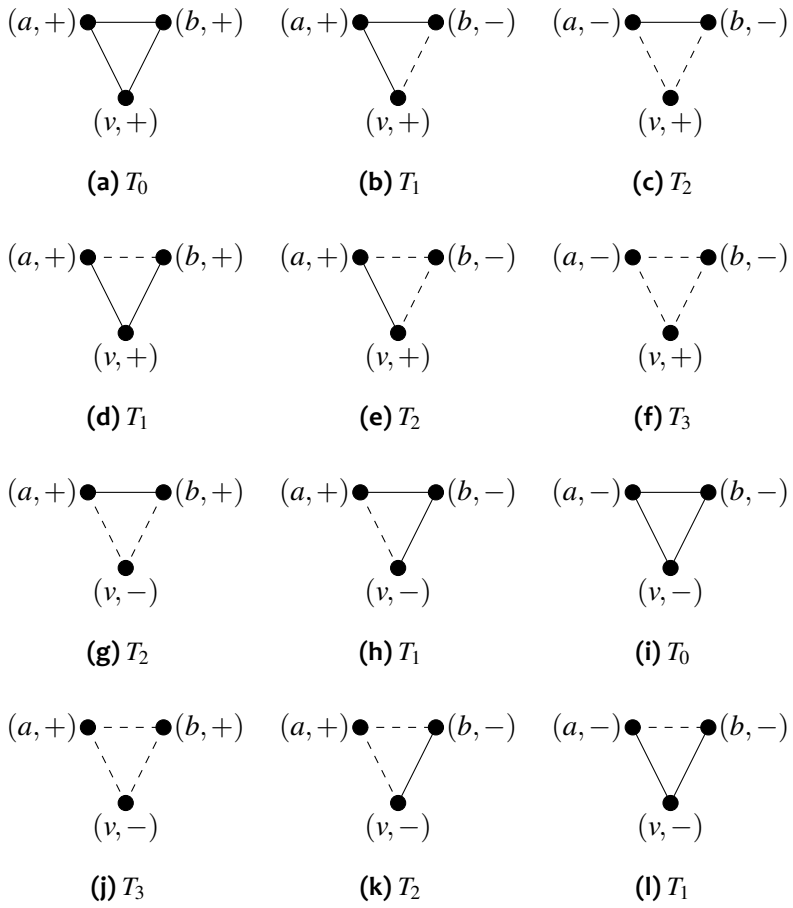


Figure 2.3 : Formation of signed triangles in every step of the corona product. Each signed edge consisiting nodes $(a,p),(b,q)$ joins an exiting node (v,r) where $p,q,r \in \{+,-\}$ denote the marking of the nodes.

I denote $|E_s^{(0)}|^{pq}$ as the number of links in $G^{s,(0)}$ having sign s that is adjacent to nodes with markings p and q. For example, $|E_-^{(0)}|^{+-}$ denotes the number of negative links in $G^{s,(0)}$ adjacent to oppositely marked nodes. Then Table 2.2 gives the number of triangles of type T_j, $j = 0,1,2,3$, added after the formation of $G^{s,(i+1)}$ from $G^{s,(i)}$ due to the new signed edges created between the nodes in $V^{s,(i)}$ and $V^{s,(i+1)} \setminus V^{s,(i)}$, $1 \leq i \leq m-1$. I denote this number as $\mathrm{T}_j^{(i+1)}$.

j	$\mathrm{T}_j^{(i+1)}$
$j=0$	$n_+^{(i)}\,\lvert E_+^{(0)}\rvert_+^+ + n_-^{(i)}\,\lvert E_+^{(0)}\rvert_-^-$
$j=1$	$n_+^{(i)}\left(\lvert E_+^{(0)}\rvert_-^+ + \lvert E_-^{(0)}\rvert_+^+\right) + n_-^{(i)}\left(\lvert E_+^{(0)}\rvert_+^- + \lvert E_-^{(0)}\rvert_-^-\right)$
$j=2$	$n_+^{(i)}\left(\lvert E_+^{(0)}\rvert_-^- + \lvert E_-^{(0)}\rvert_-^+\right) + n_-^{(i)}\left(\lvert E_+^{(0)}\rvert_+^+ + \lvert E_-^{(0)}\rvert_-^+\right)$
$j=3$	$n_+^{(i)}\,\lvert E_-^{(0)}\rvert_-^- + n_-^{(i)}\,\lvert E_-^{(0)}\rvert_+^+$

Table 2.4 : The number of T_j type triangles between the nodes in $V^{s,(i)}$ and $V^{s,(i+1)} \setminus V^{s,(i)}$ created during the formation of $G^{s,(i+1)}$ from $G^{s,(i)}$.

Note that if any type of triangle is present in a seed graph $G^s = G^{s,(0)}$, it will appear every time a copy of the seed graph gets attached to the formation of $G^{s,(m)}$. Let $\mathrm{T}_j^{(0)}$ denote the number of triangles of type T_j that exist in $G^{(0)}$. Then the total number of T_j $j=0,1,2,3$ type triangles $\mathrm{T}_j(G^{s,(m)})$ in $G^{s,(m)}$ can be calculated as follows.

$$\begin{aligned}
\mathrm{T}_0(G^{s,(m)}) &= \mathrm{T}_0^{(0)}\left(1+n\sum_{i=0}^{m-1}(n+1)^i\right)+\sum_{i=0}^{m-1}\mathrm{T}_0^{(i+1)} \\
&= \mathrm{T}_0^{(0)}(n+1)^m + \lvert E_+^{(0)}\rvert_+^+\sum_{i=0}^{m-1} n_+^{(i)} + \lvert E_+^{(0)}\rvert_-^-\sum_{i=0}^{m-1} n_-^{(i)} \\
&= \mathrm{T}_0^{(0)}(n+1)^m + \lvert E_+^{(0)}\rvert_+^+[(n+1)^m-1] + \left(\lvert E_+^{(0)}\rvert_-^- - \lvert E_+^{(0)}\rvert_+^+\right)\sum_{i=0}^{m-1} n_-^{(i)}
\end{aligned}$$

where the first term includes the number of T_0 type triangles which are in the copies of the seed graph in $G^{s,(m)}$, and the remaining terms are due to the new triangles which are born during the formation of the corona product in each step of the formation of $G^{s,(m)}$. The values of $n_-^{(i)}$ can be found from Table 2.3 for each i. Similarly I have the following.

$$\begin{aligned}
\mathrm{T}_1(G^{s,(m)}) &= \mathrm{T}_1^{(0)}(n+1)^m + \left(\lvert E_+^{(0)}\rvert_-^+ + \lvert E_-^{(0)}\rvert_+^+\right)[(n+1)^m-1] + \left(\lvert E_-^{(0)}\rvert_-^- - \lvert E_-^{(0)}\rvert_+^+\right)\sum_{i=0}^{m-1} n_-^{(i)} \\
\mathrm{T}_2(G^{s,(m)}) &= \mathrm{T}_2^{(0)}(n+1)^m + \left(\lvert E_+^{(0)}\rvert_-^- + \lvert E_-^{(0)}\rvert_-^+\right)[(n+1)^m-1] + \left(\lvert E_+^{(0)}\rvert_+^+ - \lvert E_+^{(0)}\rvert_-^-\right)\sum_{i=0}^{m-1} n_-^{(i)} \\
\mathrm{T}_3(G^{s,(m)}) &= \mathrm{T}_3^{(0)}(n+1)^m + \lvert E_-^{(0)}\rvert_-^-[(n+1)^m-1] + \left(\lvert E_-^{(0)}\rvert_+^+ - \lvert E_-^{(0)}\rvert_-^-\right)\sum_{i=0}^{m-1} n_-^{(i)}
\end{aligned}$$

where the values of $n_-^{(i)}$ can be computed by using the recurrence relation given in Table 2.3.

Note from the above formulas that the number of T_j type triangles in $G^{s,(m)}$ can be controlled by choosing the seed graph with an appropriate distribution of signed edges whose adjacent nodes are marked in a particular fashion.

2.3.2 Degrees of nodes of $G^{s,(m)}$

Note that after the appearance of a node in a particular step during the formation of $G^{s,(m)}$, from next step onward the total degree of the node increases by n at every step until the process stops. The dynamics of the change in positive and negative degrees depend on the existing marking of the node and the marking of the nodes which get linked to it at every step. Since the new nodes which get linked to an existing node are the nodes in the copy of the seed graph, the change in degrees of the existing node depends on the marking of the nodes in the seed graph. I first derive the positive and negative degrees of the nodes in $G^s = G^{s,(0)}$, and then I consider nodes in $G^{s,(i)}$ that is which appear at the ith step during the formation of $G^{s,(m)}$, $1 \leq i \leq m$. The initial marking of a node v is denoted by $\mu(v)$. By initial marking I mean once the node v joins the existing graph.

I have the following theorem which provides positive and negative degree of a node in the seed graph $G^{s,(0)}$ after the formation of $G^{s,(m)}$.

Theorem 2.3.1. *Let $G^{s,(m)}, m \geq 1$ be the signed corona graph corresponding to a seed graph $G^s = G^{s,(0)}$ on $n > 1$ nodes defined by the canonical marking μ. Let the positive and negative degree of v in $G^{s,(0)}$ be denoted by $d_0^+(v)$ and $d_0^-(v)$ respectively. Suppose $G^{s,(0)}$ has n_+ positively marked nodes and n_- negatively marked nodes. Then for any $v \in V^{(0)}$ the positive degree $d^+(v)$ and negative degree $d^-(v)$ of v in $G^{s,(m)}$ are given by the Table 2.5.*

Condition on G	$\mu(v)$	$d^+(v)$	$d^-(v)$	Condition on m
n_+ odd, n_- odd	$+$	$d_0^+(v) + \frac{m}{2}(n_+ + n_-)$	$d_0^-(v) + \frac{m}{2}(n_+ + n_-)$	even
		$d_0^+(v) + \frac{m}{2}(n_+ + n_-)$	$d_0^-(v) + \frac{m}{2}(n_+ + n_-)$	even
		$d_0^+(v) + \frac{m+1}{2}n_+ + \frac{m-1}{2}n_-$	$d_0^-(v) + \frac{m+1}{2}n_- + \frac{m-1}{2}n_+$	odd
	$-$	$d_0^+(v) + \frac{m}{2}(n_+ + n_-)$	$d_0^-(v) + \frac{m}{2}(n_+ + n_-)$	even
		$d_0^+(v) + \frac{m+1}{2}n_- + \frac{m-1}{2}n_+$	$d_0^-(v) + \frac{m+1}{2}n_+ + \frac{m-1}{2}n_-$	odd
n_+ even,	$+$	$d_0^+(v) + n_+ + (m-1)n_-$	$d_0^-(v) + n_- + (m-1)n_+$	even/odd
n_- odd	$-$	$d_0^+(v) + mn_-$	$d_0^-(v) + mn_+$	even/odd
n_+ even,	$+$	$d_0^+(v) + mn_+$	$d_0^-(v) + mn_-$	even/odd
n_- even	$-$	$d_0^+(v) + mn_-$	$d_0^-(v) + mn_+$	even/odd
n_+ odd,	$+$	$d_0^+(v) + mn_+$	$d_0^-(v) + mn_-$	even/odd
n_- even	$-$	$d_0^+(v) + n_- + (m-1)n_+$	$d_0^-(v) + n_+ + (m-1)n_-$	even/odd

Table 2.5 : Positive and negative degree of nodes of the seed graph G^s in $G^{s,(m)}$

Proof: Let n_+ and n_- be odd. If a node v in G^s is positively marked then at every step of the formation of $G^{s,(m)}$ the marking of the node will change since at every step it will generate odd number of negative links to the newly appeared nodes in a copy of the seed graph. If $\mu(v) = +$ (resp. $\mu(v) = -$) at any step then at the next step it will generate n_- (resp. n_+) negative links.

If m is even then at $m/2$ steps it will be incident to n_- and n_+ nodes. Thus the desired result follows. A similar argument can prove the case for m being odd. If $\mu(v) = -$ it's marking shall change at every step and hence the desired result follows.

Let $n+$ be even and n_- odd. If $\mu(v) = +$ then next step the v will generate n_- negative links and since n_- is odd, its marking become $-$. From next step onward it will generate even number n_+ of negative links at every step as n_+ is even. Hence $\mu(v)$ remains $-$ for all the steps till m. Thus the desired follows. If $\mu(v) = -$, it remains negatively marked for all the steps since it generates even number of negative links. This completes the proof.

Let n_+ and n_- both be even. If $\mu(v) = +$, it remains positively marked for all the steps since every time it generates n_- negative links and n_- is even. Similarly, if $\mu(v) = -$, it remains negatively marked as every times it creates even number of new negative links. Hence the proof.

Finally let n_+ be odd and n_- even. If $\mu(v) = +$ it remains $+$ since at every step it creates even number n_- of negative links. If $\mu(v) = -$ then at the next step the marking becomes $+$ since it creates n_+ number of negative links and n_+ is odd. Next step onward it creates even number n_- of negative links and hence the marking of v remains $+$ for ever until mth step. Hence the desired result follows. □

Now I consider the positive and negative degrees of a node v which appears at the ith step, that is, during the formation of $G^{s,(i)}$, $1 \leq i \leq m$ and gets linked with an existing node u in $G^{s,(i-1)}$. Obviously u is a node in a copy of the seed graph and it has a marking $\mu(v)$ as a node in the seed graph but as soon as it joins the node u in $G^{s,(i-1)}$, the marking of v may change depending on the marking of the node u. Let us denote the degree of v as $d_0^{\pm}(v)$ when I consider it as a node of the seed graph and the positive or negative degree increases by 1 when it joins u in $G^{s,(i-1)}$ depending on the marking $\mu(u)$. The following theorem provides the positive and negative degree of such a node v.

Theorem 2.3.2. *Let v be a node which joins at the ith step, $1 \leq i \leq m$ of the formation of the corona graph $G^{s,(m)}$ generated by a seed graph $G^s = G^{s,(0)}$ which contains n_+ positively marked and n_- negatively marked nodes. Let v gets attached with a node u in $G^{s,(i-1)}$ during its birth in $G^{s,(i)}$. Then the positive degree $d^+(v)$ and negative degree of $d^-(v)$ in $G^{s,(m)}$ are given by the Table 2.6. I denote the marking of v as $\mu_0(v)$ when I consider it as a node in the copy of a seed graph which joins the graph $G^{s,(i-1)}$, whereas $\mu(u)$ denotes the marking of u as a node in $G^{s,(i-1)}$.*

Proof: The proof follows by observing the change of marking of a node v due to the appearance of n new nodes which connect to v in every step after it appears in the ith step of the formation of $G^{s,(m)}$. Let v be a node which appears in $G^{s,(i)}$, $1 \leq i \leq m$. Recall that a node is positively marked if and only if it is attached to even number of negatively signed links, otherwise the marking of a node is negative.

First assume that n_- and n_+ are odd. Let $\mu(u) = +$ and $\mu_0(v) = +$. Note that sign of the edge between u and v is $\mu(u) \times \mu_0(v) = +$. Hence the marking of v in $G^{s,(i)}$ is $+$ as the number of negative edges adjacent to it does not increase by 1 and $\mu_0(v) = +$ implies it was attached

Condition on G	$(\mu(u),\mu_0(v))$	$d^+(v)$	$d^-(v)$	m	i
n_+ odd, n_- odd	$(+,+)$	$d_0^+(v)+1+\frac{m-i}{2}n$	$d_0^-(v)+\frac{m-i}{2}n$	odd	odd
				even	even
		$d_0^+(v)+1+\frac{m-i+1}{2}n_+ + \frac{m-i-1}{2}n_-$	$d_0^-(v)+\frac{m-i+1}{2}n_- + \frac{m-i-1}{2}n_+$	odd	even
				even	odd
	$(+,-)$	$d_0^+(v)+\frac{m-i}{2}n$	$d_0^-(v)+1+\frac{m-i}{2}n$	odd	odd
				even	even
		$d_0^+(v)+\frac{m-i+1}{2}n_+ + \frac{m-i-1}{2}n_-$	$d_0^-(v)+1+\frac{m-i+1}{2}n_- + \frac{m-i-1}{2}n_+$	odd	even
				even	odd
	$(-,+)$	$d_0^+(v)+\frac{m-i}{2}n$	$d_0^-(v)+1+\frac{m-i}{2}n$	odd	odd
				even	even
		$d_0^+(v)+\frac{m-i+1}{2}n_- + \frac{m-i-1}{2}n_+$	$d_0^-(v)+1+\frac{m-i+1}{2}n_+ + \frac{m-i-1}{2}n_-$	odd	even
				even	odd
	$(-,-)$	$d_0^+(v)+1+\frac{m-i}{2}n$	$d_0^-(v)+\frac{m-i}{2}n$	odd	odd
				even	even
		$d_0^+(v)+1+\frac{m-i+1}{2}n_- + \frac{m-i-1}{2}n_+$	$d_0^-(v)+\frac{m-i+1}{2}n_+ + \frac{m-i-1}{2}n_-$	odd	even
				even	odd
n_+ even, n_- odd	$(+,+)$	$d_0^+(v)+1+n_+ +(m-i-1)n_-$ if $m>i$ $d_0^+(v)+1$ if $m=i$	$d_0^-(v)+n_- +(m-i-1)n_+$ if $m>i$ $d_0^-(v)$ if $m=i$	any	any
	$(+,-)$	$d_0^+(v)+n_+ +(m-i-1)n_-$ if $m>i$ $d_0^+(v)$ if $m=i$	$d_0^-(v)+1+n_- +(m-i-1)n_+$ if $m>i$ $d_0^-(v)+1$ if $m=i$	any	any
	$(-,+)$	$d_0^+(v)+(m-i)n_-$	$d_0^-(v)+1+(m-i)n_+$	any	any
	$(-,-)$	$d_0^+(v)+1+(m-i)n_-$	$d_0^-(v)+(m-i)n_+$	any	any
n_+ even, n_- even	$(+,+)$	$d_0^+(v)+1+(m-i)n_+$	$d_0^-(v)+(m-i)n_-$	any	any
	$(+,-)$	$d_0^+(v)+(m-i)n_+$	$d_0^-(v)+1+(m-i)n_-$	any	any
	$(-,+)$	$d_0^+(v)+(m-i)n_-$	$d_0^-(v)+1+(m-i)n_+$	any	any
	$(-,-)$	$d_0^+(v)+1+(m-i)n_-$	$d_0^-(v)+(m-i)n_+$	any	any
n_+ odd, n_- even	$(+,+)$	$d_0^+(v)+1+(m-i)n_+$	$d_0^-(v)+(m-i)n_-$	any	any
	$(+,-)$	$d_0^+(v)+(m-i)n_+$	$d_0^-(v)+1+(m-i)n_-$	any	any
	$(-,+)$	$d_0^+(v)+n_- +(m-i-1)n_+$ if $m>i$ $d_0^+(v)$ if $m=i$	$d_0^-(v)+1+n_+ +(m-i-1)n_-$ if $m>i$ $d_0^-(v)+1$ if $m=i$	any	any
	$(-,-)$	$d_0^+(v)+1+n_- +(m-i-1)n_+$ if $m>i$ $d_0^+(v)+1$ if $m=i$	$d_0^-(v)+n_+ +(m-i-1)n_-$ if $m>i$ $d_0^-(v)$ if $m=i$	any	any

Table 2.6 : Positive and negative degree of a node which appears at the ith step during the formation of $G^{s,(m)}$, $1 \le i \le m$

to even number of negative edges. At the $(i+1)$th step v forms n_- negative edges and n_- is odd. Hence $\mu(v) = -$ in $G^{s,(i+1)}$. Proceeding similarly, the sequence of marking of v will be $+,-,+,-,\ldots$. Thus the positive degree of v increases as follows. At $G^{s,(i)}$ the degree of v becomes $d_0^+(v)+1$. At $(i+1)$th step the positive degree of v becomes $d_0^+(v)+1+n_+$, and the negative degree becomes $d_0^-(v)+n_-$. Since marking of v is $-$ at $(i+1)$th step, it will create n_- positive edges and n_+ negative edges at the next step. Thus the positive and negative degree will be $(d_0^+(v)+1)+n_++n_-+n_++\ldots$ and the negative degree will be $d_0^-(v)+n_-+n_++n_-+\ldots$, and both the series stop after $m-i$ steps. The desired result follows by considering different cases of m and i being even and/ odd.

If $\mu(u) = +$ and $\mu_0(v) = -$ the pattern of change of markings of v starting from the i th step is $+,-,+,-,\ldots$. The increase of positive and negative degree are given by $d_0^+(v)+n_++n_-+n_++n_-+\ldots$ and $(d_0^-+1)+n_-+n_++n_-+n_++\ldots$. Thus the result follows by considering different cases for m and i.

Similarly it can be easily verified that the positive and negative degree of v will be $d_0^+(v)+n_-+n_++n_-+\ldots$ and $(d_0^-(v)+1)+n_++n_-+n_++\ldots$ respectively when $\mu(u) = -$ and $\mu_0(v) = +$. Finally if $\mu(u) = -$ and $\mu_0(v) = -$ then the positive and negative degree of v will be $(d_0^+(v)+1)+n_-+n_++n_-+\ldots$ and $d_0^-+n_++n_-+n_++\ldots$ respectively. Thus the desired result follows.

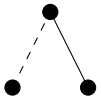
Following a similar procedure for different values of n_- and n_+ all the desired result can be verified. This completes the proof. □

It is evident from the expressions of the positive and negative degree from Table 2.5 and Table 2.6 that as i increases the degree decreases. This means a degree of node is higher than the degree of another node if it joins the network before than the another one. Finally the degrees of the nodes in $G^{s,(0)}$ are the highest ones.

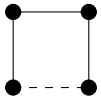
For the computation of degree distribution of corona graphs the frequency or total number of occurrence of a node with a particular degree need to be determined. Thus the question is how many nodes do have a particular degree given in the tables above? It is obvious from the construction of the corona graph that the number of nodes which appear at step i with a given degree $d^{\pm}(v)$ with $(\mu(u),\mu_0(v)) = (+,+)$ is $n_+^{(i)}n_+$; $(\mu(u),\mu_0(v)) = (+,-)$ is $n_+^{(i)}n_-$; $(\mu(u),\mu_0(v)) = (-,+)$ is $n_-^{(i)}n_+$; and $(\mu(u),\mu_0(v)) = (-,-)$ is $n_-^{(i)}n_-$, for $1 \le i \le m$. Below I compute positive and negative degree distributions of corona graphs $G^{s,(i)}, 0 \le i \le 5$ in Figure 2.4 for the seed graphs G_1^s, G_2^s.

2.3.3 Algebraic conflict of corona graphs

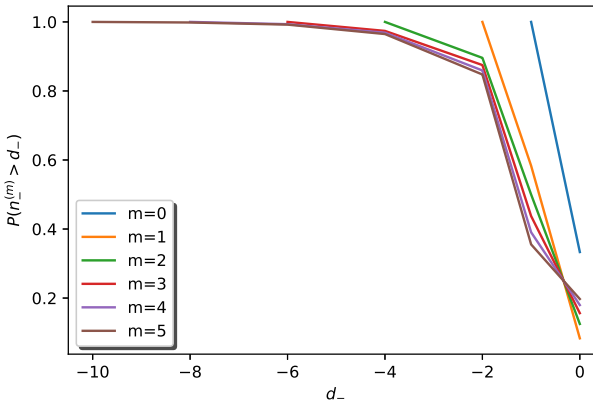
Recall that algebraic conflict of a signed graph is the least signed Laplacian eigenvalue of the graph [Aref, 2019; Kunegis *et al.*, 2010]. Theorem 2.2.4 gives the complete list of signed Laplacian eigenvalues only when distinct nodes of the corresponding graph have distinct negative degrees. However for a large graph the computation of these eigenvalues by calculating roots of high degree polynomials is computationally challenging. Due to the exponential growth of corona graphs, withing a few steps the corona graphs contain huge



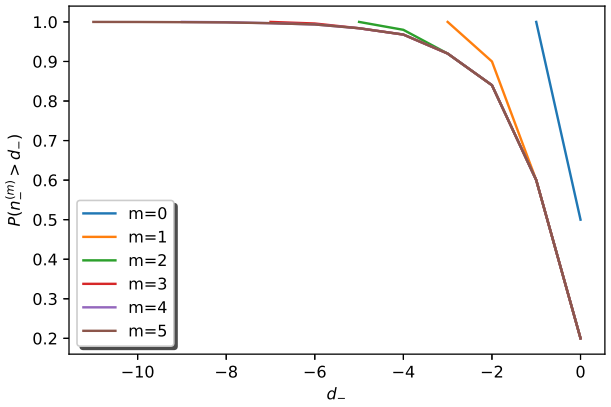
(a) G_1^s



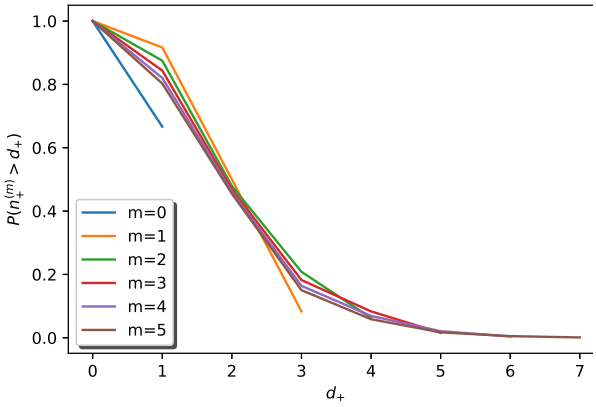
(b) G_2^s



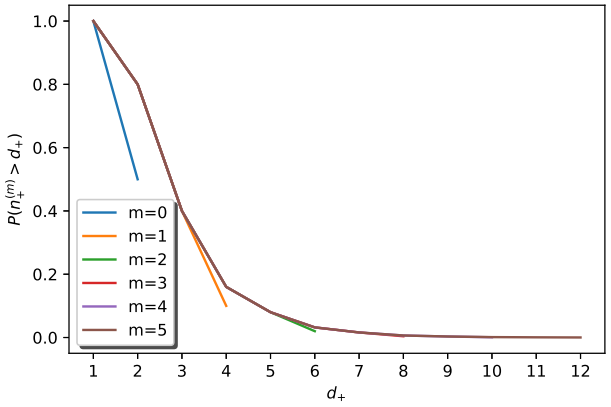
(c) Negative degree distribution of $G_1^{s,(m)}$



(d) Negative degree distribution of $G_2^{s,(m)}$



(e) Positive degree distribution of $G_1^{s,(m)}$



(f) Positive degree distribution of $G_2^{s,(m)}$

Figure 2.4 : Degree distributions of $G_i^{s,(m)}$, $i = 1, 2$.

number of nodes even if the seed graph has considerably small number of nodes. Thus I consider Theorem 2.2.5 which gives the collection of all signed Laplacian eigenvalues of the signed graph $G^s_{\mu_1} \circ G^s_{\mu_2}$ when marking of all the nodes of $G^s_{\mu_2}$ are either $+$ or $-$, and negative degree of all the nodes in $G^s_{\mu_2}$ are same. Hence setting a seed graph $G^s = G^{s,(0)}$ having these properties I provide expression of signed Laplacian eigenvalues of $G^{s,(m)}$ as follows by iterative use of Theorem 2.2.5.

First I have the following lemma which will be used in sequel.

Lemma 2.3.3. *Let n, m be positive integers. Then*

$$2^m n + \sum_{i=1}^{m-1} 2^i n(n-1)(n+1)^{m-i-1} + n(n-1)(n+1)^{m-1} = n(n+1)^m.$$

Proof: The proof follows from Lemma 1 in [Sharma *et al.*, 2017]. □

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{2d^- + 1 + x + n \pm \sqrt{[(2d^- + 1) - (x+n)]^2 + 4n}}{2} \tag{2.9}$$

where d^- is a nonnegative integer and n is a positive integer. Denote f^i as i number of compositions of f with itself. For example, $f^1 = f$ and $f^2 = f \circ f$. Then I have the following theorem.

Theorem 2.3.4. *Let G^s be a signed graph on n nodes such that d^- is the fixed negative degree of each node of G^s and every node of G^s is either positively or negatively marked. Suppose $\eta_j, j = 1, \ldots, n$ are the signed Laplacian eigenvalues of G^s. Let $\eta_n = 2d^-$. Then signed Laplacian eigenvalues of $G^{s,(m)}, m \geq 1$ defined by the seed graph G^s are given by*

(a) $f^m(\eta_j)$ *with multiplicity* 1, $j = 1, \ldots, n$

(b) $f^i(\eta_j + 1)$ *with multiplicity* $n(n+1)^{m-i-1}$, $1 \leq i \leq m-1$, $j = 1, \ldots, n-1$, *and* $m > 1$

(c) $\eta_j + 1$ *with multiplicity* $n(n+1)^{m-1}$, $j = 1, \ldots, n-1$.

Proof: First note that $2d^-$ is a signed Laplacian eigenvalue of G^s due to Lemma 2.2.3. The expression of signed laplacian eigenvalues of $G^{s,(m)}$ follows from iterative use of Theorem 2.2.5. Using Lemma 2.3.3 it can be easily verified that this is the complete list of signed Laplacian eigenvalues of $G^{s,(m)}$. □

The algebraic conflict of corona graphs defined by a seed graph satisfying the properties of Theorem 2.3.4 can be calculated by taking the minimum of the signed Laplacian eigenvalues of $G^{s,(m)}$ as given in Theorem 2.3.4. However it is interesting to verify that $2d^-$ is a fixed point of the function

$$f(x) = \frac{2d^- + 1 + x + n \pm \sqrt{[(2d^- + 1) - (x+n)]^2 + 4n}}{2}$$

for any n. This implies that if $2d^-$ is the algebraic conflict of a seed graph G^s with every node having negative degree d^-, $2d^-$ remains the algebraic conflict of $G^{s,(m)}$ for any $m \geq 1$. An example of a seed graph having algebraic conflict $2d^-$ is given by 2.5 where $n = 4$ and $d^- = 1$.

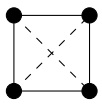


Figure 2.5 : Seed graph G^s with algebraic conflict 2

Otherwise, for seed graph $G^{s,(0)}$ if $2d^-$ is not a signed Laplacian eigenvalue , I conjecture that $\eta_{\min} + 1 \leq f^i(\eta_j + 1), 1 \leq i \leq m-1, 1 \leq j \leq n-1$ and $\eta_{\min} + 1 \leq f^m(\eta_j), 1 \leq j \leq n$, and hence algebraic conflict of $G^{s,(m)}$ is algebraic conflict of $G^{(0)}$ plus one. I verify the same for all possible signed graphs on $3,4,5$ nodes satisfying the conditions in Theorem 2.3.4. In that case it would be a remarkable feature that degree of balance of corona graphs $G^{s,(m)}$ is not affected by values of m and large signed corona graphs can have small algebraic conflict which is at the order of the algebraic conflict of the seed graph. I consider a few seed graphs on $3,4,5$ nodes that are given in Figure 2.6 and compute the algebraic conflict of $G^{s,(m)}, 1 \leq m \leq 5$ in Figure 2.7.

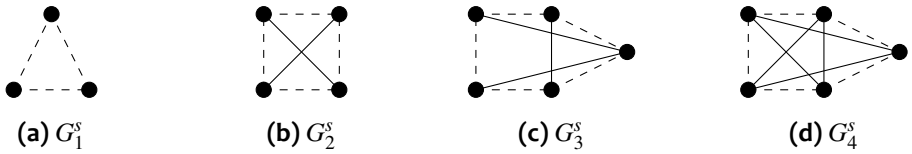


Figure 2.6 : Sample seed graphs on $3,4,5$ nodes.

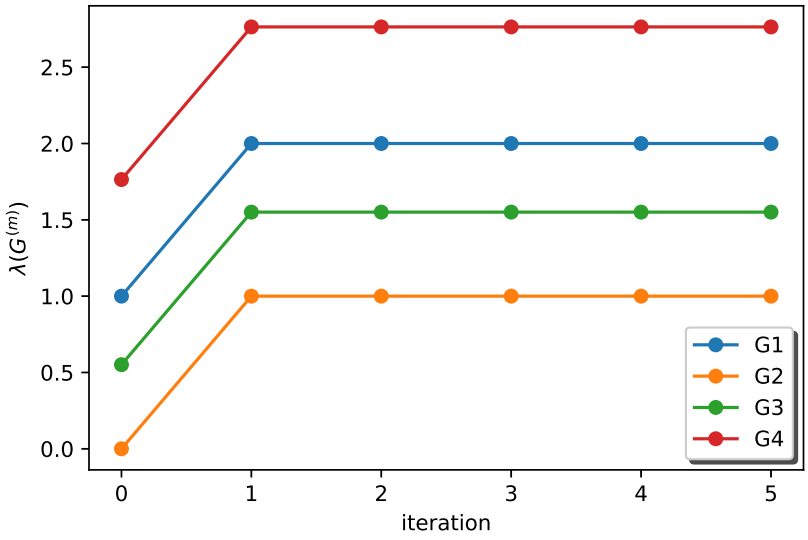


Figure 2.7 : Algebraic conflicts of corona graphs generated by seed graphs listed in Figure 2.6. $\lambda(G^{s,(m)})$ denotes algebraic conflict.

$1 \leq m \leq 5$ and observe that it is always $\eta_{\min} + 1$ where $\eta_{\min}$ is the least signed eigenvalue of the seed graph which I consider all possible signed graphs on $3, 4, 5$ nodes satisfying the conditions in Theorem 2.3.4. I verify that algebraic conflict remains constant for corona graphs for large values of m defined by these seed graphs.

2.3.4 Real signed networks and signed corona graphs

As mentioned in the introduction, I now explore how signed corona graphs can gain us to preserve properties of real signed networks. I consider all the data sets of real signed networks, see Table 2.7, that are reported in [Leskovec *et al.*, 2010b] and [Derr *et al.*, 2018]. One remarkable characteristic about these real networks is that each data set contains very high percentage of positive links than negative links, and balance triangles dominates in number in all these networks. In particular, the number of T_0 is almost $70-90\%$ among all other type of triangles in all the networks.

Network	**Nodes**	**Edges**	$p(E_+)$	**triangle**	$p(T)$
Epinions	119,217	814,200	0.85	T_0 T_1 T_2 T_3	0.870 0.052 0.071 0.007
Slashdot	82,114	549,202	0.77	T_0 T_1 T_2 T_3	0.840 0.077 0.072 0.011
wiki-Elec	7,118	103,747	0.78	T_0 T_1 T_2 T_3	0.702 0.080 0.207 0.011
Bitcoin-OTC	5,881	35,592	0.89	T_0 T_1 T_2 T_3	0.724 0.122 0.145 0.009
Bitcoin-Alpha	3,783	24,186	0.93	T_0 T_1 T_2 T_3	0.793 0.134 0.069 0.004

Table 2.7 : Data sets of real signed networks. Epinions: A signed social network, Slashdot: Slashdot Zoo signed social network from February 21 2009, wiki-Elec: Wikipedia adminship election data, Bitcoin OTC: Bitcoin OTC web of trust network, Bitcoin Alpha: Bitcoin Alpha web of trust network. The original data sets are signed and directed/weighted, but I consider the undirected/unweighted version of the same data. $p(E_+)$ denotes the proportion of positive edges in the entire network. $p(T)$ is the fraction of triangles of type T in the entire network.

It can be observed from the the expression of number of each type of triangles and signed edges in $G^{s,(m)}$ obtained in Section 2.3.1 that the distribution of T_js and signed edges in $G^{s,(m)}$ are highly influenced by number of T_j, signed links, number of nodes and marking of nodes in the

seed graph. Besides, the number of nodes in $G^{s,(m)}$ only can be of the form $n(n+1)^m$ where n denotes the number of nodes in the seed graph. Hence it is evident that $G^{s,(m)}$ can not have any desired number of nodes in it, but given a total number of nodes in a synthetic network to be produced, one can determine suitable choices of n, m such that the number of nodes in $G^{s,(m)}$ approximately matches with the desired number of nodes. Then the goal will be to choose a seed graph on n nodes that can approximately catch up the desired number of signed edges and triangles T_j in $G^{s,(m)}$. An another guide from Theorem 2.2.1 is that the marking of nodes which are adjacent to a positive/negative link in the seed graph must follow some conditions in order to produce all types of signed triangles in the network.

Structural properties of corona graphs corresponding to some seed graphs on 4 or 5 nodes for which all type of triangles exist, are described in Table 2.8. It follows that these corona graphs fail to preserve properties of real networks. Evidently, a corona graph $G^{s,(m)}$ for some $m \geq 1$ can preserve structural properties of real networks only when an appropriate seed graph should be chosen. Note that in real networks $\mathrm{T}_0 \gg \mathrm{T}_j, j = 1,2,3$ and $e_+ \gg e_-$. Thus seed graph must contain high number of positive edges and T_0, and negative edges in the seed graph must be connected in such a way that the generation of $T_j, j = 1,2,3$ can be controlled during the formation of resulting corona graph.

m	$G^{(0)}$	N	E^s	$p(E_+)$	triangle	$p(T_l)$
6		62,500	1,24,996	0.625	T_0 T_1 T_2 T_3	0.125 0.625 0.125 0.125
6		62,500	1,56,246	0.700	T_0 T_1 T_2 T_3	0.250 0.650 0.049 0.049
5		38,800	1,01,083	0.754	T_0 T_1 T_2 T_3	0.450 0.433 0.050 0.066
5		38,800	1,16,653	0.786	T_0 T_1 T_2 T_3	0.470 0.460 0.030 0.039
5		38,800	1,16,635	0.586	T_0 T_1 T_2 T_3	0.180 0.500 0.220 0.100

Table 2.8 : Statistics of signed links and signed triangles of Corona graphs $G^{s,(m)}$ defined by seed graph $G^{s,(0)}$. N and E denote the total number of nodes and links in $G^{s,(m)}$. $p(E_+)$ and $p(T_j)$ denote the fraction of positive links and triangles of type T_j in $G^{s,(m)}$.

Recall that the total number of triangles T_j of type $T_j, j = 0,1,2,3$ in $G^{s,(m)}$ generated by

a seed graph $G = G^{s,(0)}$ is given by

$$\begin{aligned}
\mathrm{T}_0 &= \mathrm{T}_0^{(0)}k_1 + |E_+^{(0)}|^{+}k_2 + \left(|E_+^{(0)}|^{-} - |E_+^{(0)}|^{+}\right)k_3 \\
\mathrm{T}_1 &= \mathrm{T}_1^{(0)}k_1 + \left(|E_+^{(0)}|^{+} + |E_-^{(0)}|^{+}\right)k_2 + \left(|E_-^{(0)}|^{-} - |E_-^{(0)}|^{+}\right)k_3 \\
\mathrm{T}_2 &= \mathrm{T}_2^{(0)}k_1 + \left(|E_+^{(0)}|^{-} + |E_-^{(0)}|^{+}\right)k_2 + \left(|E_+^{(0)}|^{+} - |E_+^{(0)}|^{-}\right)k_3 \\
\mathrm{T}_3 &= \mathrm{T}_3^{(0)}k_1 + |E_-^{(0)}|^{-}k_2 + \left(|E_-^{(0)}|^{+} - |E_-^{(0)}|^{-}\right)k_3,
\end{aligned}$$

and the total number of positive and negative edges in $G^{s,(m)}$ are given by

$$\begin{aligned}
e_+ &= e_+^{(0)}k_1 + n_+^{(0)}k_2 + (n_-^{(0)} - n_+^{(0)})k_3 \\
e_- &= e_-^{(0)}k_1 + n_-^{(0)}k_2 + (n_+^{(0)} - n_-^{(0)})k_3
\end{aligned}$$

respectively, where $k_1 = (n+1)^m, k_2 = (n+1)^m - 1, k_3 = \sum_{i=0}^{m-1} n_-^{(i)}$, the values of $n_-^{(i)}$ can be found from Table 2.3. $\mathrm{T}_j^{(0)}$ denotes the number of triangles of type T_j in $G^{s,(0)}$. $n_-^{(0)}$ and $n_+^{(0)}$ denote the number of negatively and positively marked nodes in G, and $e_-^{(0)}$ and $e_+^{(0)}$ denote the number of negative and positive edges in $G^{s,(0)}$ respectively. However there are six types of edges in the seed graph based on the marking of the adjacent nodes. In order to obtain a compact expression of T_js in the corona graph I set some of the values of the parameters to be zero that is I exclude certain types of edges in G. This helps to set values of the other parameters so that finally the desired distribution of T_j can be obtained.

Setting $|E_+^{(0)}|^{+} = |E_+^{(0)}|^{-} = |E_-^{(0)}|^{-} = 0$ and $\mathrm{T}_3^{(0)} = 0$, I obtain

$$\begin{aligned}
\mathrm{T}_0 &= \mathrm{T}_0^{(0)}k_1 + |E_+^{(0)}|^{+}(k_2 - k_3) \\
\mathrm{T}_1 &= \mathrm{T}_1^{(0)}k_1 + |E_-^{(0)}|^{+}(k_2 - k_3) \\
\mathrm{T}_2 &= \mathrm{T}_2^{(0)}k_1 + |E_-^{(0)}|^{+}k_2 + |E_+^{(0)}|^{+}k_3 \\
\mathrm{T}_3 &= |E_-^{(0)}|^{+}k_3.
\end{aligned}$$

The reason behind choosing $\mathrm{T}_3^{(0)} = 0$ is that I want the number of T_3 to be minimum among all triangles. Besides, note that $k_2 \gg k_3$. Finally considering $e_+^{(0)} \gg e_-^{(0)}, \mathrm{T}_0^{(0)} \gg \mathrm{T}_j^{(0)}, j = 1,2$ the desired statistics of real networks can be obtaind. In Table 2.9 I consider certain seed graphs under these conditions and show that the corresponding corona graphs can preserve the signed edge and triangle distribution of real networks.

2.4 CONCLUSION

I defined corona product of two signed graphs and study their structural and spectral properties. A signed network generative model is proposed based on corona product by taking

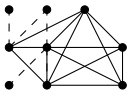
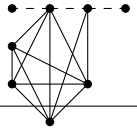
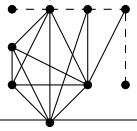
Graph	m	$G^{s,(0)}$	N	E^s	$p(E_+)$	triangle	$p(T_l)$
G_1^s	3		9,000	25,991	0.734	T_0 T_1 T_2 T_3	0.762 0.053 0.173 0.012
G_2^s	3		5,832	16,759	0.739	T_0 T_1 T_2 T_3	0.714 0.098 0.178 0.009
G_3^s	4		90,000	2,59,991	0.738	T_0 T_1 T_2 T_3	0.705 0.151 0.133 0.010

Table 2.9 : Corona graph $G^{s,(m)}$ which preserves characteristics of signed links and triangles of real networks. N and E^s denote the total number of nodes and links in $G^{s,(m)}$. $p(E_+)$ and $p(T_j)$ denote the fraction of positive links and triangles of type T_j in $G^{s,(m)}$ respectively.

the corona product of a small graph with itself iteratively. This small graph is called seed graph for the resulting signed graphs, which are called corona graphs. I derived fundamental properties of corona graphs that include number of signed links, signed triangles, degree distribution and algebraic conflict of the corona graphs. Finally I show that a seed graph can be chosen for which the corresponding corona graphs can preserve properties of real signed networks.

I believe that the framework of corona product introduced in this Chapter can be extended to define corona product of gain graphs and weighted graphs. Besides, generalized corona product of signed graphs can be defined alike generalized corona product for unsigned graphs. [Laali *et al.*, 2016].