2

Entanglement, nonlocal correlations and efficiencies of two qubit states in real conditions

2.1 INTRODUCTION

The question of quantum theory being a complete physical theory- raised by Einstein, Podolsky and Rosen (EPR) [Einstein *et al.*, 1935] during an era of academic turbulence, excitement and creative reasoning- channeled an ever ending intellectual discussion to understand the assumptions laid in the foundations of quantum theory as against the local hidden variable (LHV) theories. Borrowing the reasoning of locality and realism proposed by EPR, John S. Bell [Bell, 1964] designed an inequality to distinguish correlations that are local and thus can be explained by assumptions of locality and realism, in comparison to correlations which are not local and clearly cannot be explained by principles of locality and realism. The long-range correlations which are found to be *not local* are simply termed as *nonlocal* correlations. These correlations establish an amalgamation between foundations of quantum mechanics and information theory thereby evolving computing into a paradigm of quantum realm. For a two-qubit pure state, a violation of the Bell inequality is considered as a trademark for the presence of nonlocal correlations between qubits [Gisin, 1991]. All entangled two-qubit pure states, $|\psi\rangle = \sin\theta\,|00\rangle + e^{\imath\phi}\cos\theta\,|11\rangle; 0 < \theta \leq \frac{\pi}{4}$, violate the Bell inequality with the maximum violation obtained for $\theta = \frac{\pi}{4}$, i.e., for a maximally entangled two-qubit *Bell* state. Clearly, for pure states, maximum violation coincides with maximum entanglement and hence there are no signs of nonlocality anomaly [Méthot and Scarani, 2007]. Interestingly, not all two-qubit entangled mixed states violate the Bell Inequality [Werner, 1989; Aravind, 1995; Horedecki *et al.*, 1996; Gisin, 1996a; Peres, 1996a; Munro *et al.*, 2001a]. A seminal example is the two-qubit *Werner* state which though entangled for $a > \frac{1}{3}$, where a is the state parameter, violates the Bell inequality only for $\frac{1}{\sqrt{2}} < a \leq 1$ [Miranowicz, 2004; Bartkiewicz *et al.*, 2013]. Therefore, the *Werner* state does not violate the Bell inequality for complete range of the state parameter where it is entangled. Similar to the *Werner* state, there are several examples of other classes of two-qubit mixed entangled states in literature which are entangled but still do not violate the Bell inequality [Horedecki *et al.*, 1996; Munro *et al.*, 2001b]. Moreover, in certain cases, one also witnesses nonlocality anomaly where the maximum entanglement does not coincide with the maximum nonlocality for some seemingly good measures of entanglement and nonlocality [Brunner *et al.*, 2005; Acín *et al.*, 2005]. The discussions following mixed states and nonlocality are much more important as pure states evolve into mixed states in real experimental conditions, and thus even though the evolved states may be entangled, they may not violate the Bell inequality. Further, the presence of noise may destroy the necessary coherence between entangled qubits such that the resource may not be useful for quantum information. This further raises the question of usefulness of the quantum resource for quantum information and computation. Therefore, the question of identification and usefulness of correlations in real conditions still looms large at the horizons of quantum information and computation.

As discussed in Chapter 1, the Bell-CHSH inequality fails to identify nonlocal correlations in two-qubit mixed states for certain ranges of state parameters. This chapter readdresses the question of analyzing nonlocal correlations in pure and mixed two-qubit entangled states by modifying the existing Bell-CHSH inequality [Bell, 1964] using statistical correlation coefficients- such coefficients indicate the extent of correlations between two qubits and also provide a general

definition of bipartite entanglement between qubits [Fano, 1983; Huang, 1987; Watson, 1956; Osborn, 1977; Schlienz and Mahler, 1995; Kumar and Krishnan, 2009b; Audenaert and Plenio, 2006]. For this, we define a Bell-cumulant operator or inequality, and show that the expectation value of this operator for product states is always zero. Clearly, a violation of the modified inequality confirms the existence of nonlocal correlations in two qubit pure as well as mixed states. Our results demonstrate nonlocality in pure and mixed two-qubit states by establishing an analytical relation between the maximum expectation value of the modified operator and extent of entanglement as measured by concurrence [Wootters, 2001, 1998]. Interestingly, our results show that the Bell-cumulant operator identifies nonlocal correlations in certain classes of two-qubit mixed states, where the Bell-CHSH operator fails to capture non-classical correlations. The chapter further extends the analysis to understand the robustness of nonlocal correlations under noisy conditions [Chuang and Nielsen, 2010]. In addition, we further demonstrate the significance of weak measurement and its applications to protect nonlocal correlations from decohering very fast [Korotkov and Keane, 2010; Lee *et al.*, 2011a; Kim *et al.*, 2012]. The analysis shows that applications of weak measurement operations help the state to regain lost nonlocal correlations identified using the modified Bell operator. In all the cases discussed in this chapter, analytical results completely agree with numerical results except for the *MEMS* state at lower concurrence. These nonlocal correlations are useful for many interesting applications in quantum information and computation, such as quantum computing [Steane, 1998], quantum cryptography [Gisin *et al.*, 2002], quantum teleportation [Bennett *et al.*, 1993], dense coding [Bennett and Wiesner, 1992], secure direct communication [Boström and Felbinger, 2002], quantum secret sharing [Hillery *et al.*, 1999], secure quantum communication technologies [Cavaliere *et al.*, 2020], quantum internet [Wang *et al.*, 2020], quantum machine learning [Canabarro *et al.*, 2019] and quantum illumination [Lloyd, 2008]. Among these applications, this chapter focuses on the usefulness of partially entangled two-qubit states for teleportation protocol in the presence of noise and weak measurements. Our results show that quantum systems for which the Bell inequality fails to identify nonlocal correlations, but for which quantum correlations are identified by our operators, can also be used for the successful quantum teleportation protocol. The chapter also shows that all these results are consistent with the quantum discord.

The rest of the chapter is organized as follows: Section 2.2 analyzes the entanglement properties of two-qubit pure as well as mixed states. Section 2.3 describes the notion of statistical correlations and the modified Bell-CHSH inequality using the formal definition of correlation coefficients. The analysis of relationship between nonlocality and entanglement of two-qubit pure and mixed states is discussed in Section 2.4. Section 2.5 discusses the teleportation fidelities of two-qubit partially entangled states in the presence of noise. Finally, Section 2.6 summarizes the chapter.

2.2 ENTANGLEMENT PROPERTIES OF TWO QUBIT STATES

In this section, we readdress entanglement properties of two-qubit pure and mixed states in presence of noise and weak measurement and its reversal operations. For this, we first analyze few standard measures of entanglement for two qubit states. For example, the degree of entanglement in a pure two-qubit state is measured by its entropy of entanglement (Eof) [Bennett *et al.*, 1996a; Wootters, 1998, 2001], which is defined as the entropy of either of the subsystems considered alone, such that

$$E(|\psi\rangle) = S(\rho_A) = S(\rho_B) \tag{2.1}$$

where $S(\rho) = -Tr[\rho log_2 \rho]$ is the von-Neumann entropy and $\rho_A = Tr_B |\psi\rangle\langle\psi|$ ($\rho_B = Tr_A |\psi\rangle\langle\psi|$) is the reduced density operator obtained by tracing the joint density operator $|\psi\rangle\langle\psi|$ with respect to Bob's (Alice's) subsystem. As an entanglement measure, entropy of entanglement ranges from 0 for a product state to 1 for a maximally entangled two-qubit pure state. The Eof for mixed states is

defined as the average entanglement of pure states forming the decomposition minimized over all ensembles of pure states. Unlike pure states, the computation of Eof for mixed states is extremely difficult due to the optimization involved therein [Bennett *et al.*, 1996c; Hill and Wootters, 1997; Wootters, 1998; Vidal and Werner, 2002; Horodecki *et al.*, 2009]. As one considers the presence of noise and mechanisms to protect correlations from decoherence, number of parameters also increases which further result in a much harder computational problem to evaluate the degree of entanglement. Therefore, here, we use the two widely celebrated entanglement measures concurrence [Hill and Wootters, 1997; Wootters, 1998, 2001] and negativity [Miranowicz and Grudka, 2004; Vidal and Werner, 2002]- briefly discussed in Chapter 1 to characterize entanglement properties in two qubit pure and mixed states.

For generalized two-qubit states, $|\psi\rangle = \cos\theta\,|00\rangle + e^{\iota\phi}\sin\theta\,|11\rangle$, concurrence [Hill and Wootters, 1997; Wootters, 1998, 2001] and negativity [Vidal and Werner, 2002; Miranowicz and Grudka, 2004] reveal same values such that $C(\rho_{Bell}) = N(\rho_{Bell}) = \sin 2\theta$. Both measures show that degree of entanglement in the $|\psi\rangle$ state monotonically increases from 0 to 1 for a product state ($\theta = 0$) to a maximally entangled state ($\theta = \frac{\pi}{4}$); states lying between $0 < \theta < \frac{\pi}{4}$ are clearly non-maximally entangled pure states. Similar to a generalized two-qubit state, a *Werner* state (Eq. (1.60)) also reflects same concurrence and negativity as measures of two-qubit mixed entanglement, i.e., $C(\rho_w) = N(\rho_w) = Max\left[0, \frac{1}{2}(-1+3a)\right]$. Thus, a *Werner* state is entangled for $a > \frac{1}{3}$, and is considered as a separable state for $0 < a \leq \frac{1}{3}$. Interestingly, discord [Ollivier and Zurek, 2001] demonstrates that a *Werner* state exhibits quantum correlations for the whole range of the state parameter a which suggests that a *Werner* state in the range $0 < a \leq \frac{1}{3}$, though separable, is still useful for quantum information and computation [Popescu, 1994; Lee and Kim, 2000]. Unlike a generalized two-qubit state or a *Werner* state, *Horodecki* [Horedecki *et al.*, 1996] and *MEMS* [Munro *et al.*, 2001b] states show different values for concurrence and negativity. For *Horodecki* states, concurrence and negativity can be expressed as $C(\rho_H) = b$ and $N(\rho_H) = Max\left[0, -1+b+\sqrt{b^2+(1-b)^2}\right]$, respectively. Therefore, *Horodecki* states are also entangled for the complete range of the state parameter, i.e., for all $b > 0$. Similarly, concurrence for *MEMS* is also a function of its state parameter, i.e., $C(\rho_{MEMS}) = \gamma$. However, negativity for *MEMS* depends on whether $\gamma < \frac{2}{3}$ or $\gamma \geq \frac{2}{3}$ such that negativity is either $N(\rho_{MEMS}^{(\gamma<\frac{2}{3})}) = Max\left[0, \frac{1}{3}(-1+\sqrt{1+9\gamma^2})\right]$ or $N(\rho_{MEMS}^{(\gamma\geq\frac{2}{3})}) = Max\left[0, -1+\gamma+\sqrt{1+2(-1+\gamma)\gamma}\right]$, respectively.

Considering that the chapters in this thesis discuss entanglement, nonlocal properties and efficiencies of states in real conditions, we further characterize entanglement properties of two qubit states in presence of a noisy channel with and without applications of weak measurement and its reversal operations.

Figures 2.1(a)-(d) describe effects of amplitude damping channels on concurrences of generalized-*Bell* states, *Werner* states, *Horodecki* states and *MEMS*, respectively. Clearly, decoherence adversely affects the degree of entanglement in two qubit entangled states, be it pure or mixed. Interestingly, for pure states, the degree of entanglement of the evolved mixed state is maximum if one starts with a partially entangled input state instead of a maximally entangled input state in the presence of mild or high decoherence- to be precise for $d \geq 0.2$. However, in case of mixed states, i.e., *Werner*, *Horodecki* and *MEMS* states, concurrences for respective evolved states are maximum if the input state is also a maximally entangled state. A similar observation can be made for pure and mixed states in presence of weak measurement and its reversal operations, considering $d = 0.7$ in Figures 2.2(a)-(d). This clearly demonstrates the complexity of nature of entanglement in pure versus mixed states. Further, applications of weak measurement and its reversal operations with an optimal condition for nonlocal correlations ensure that correlations are

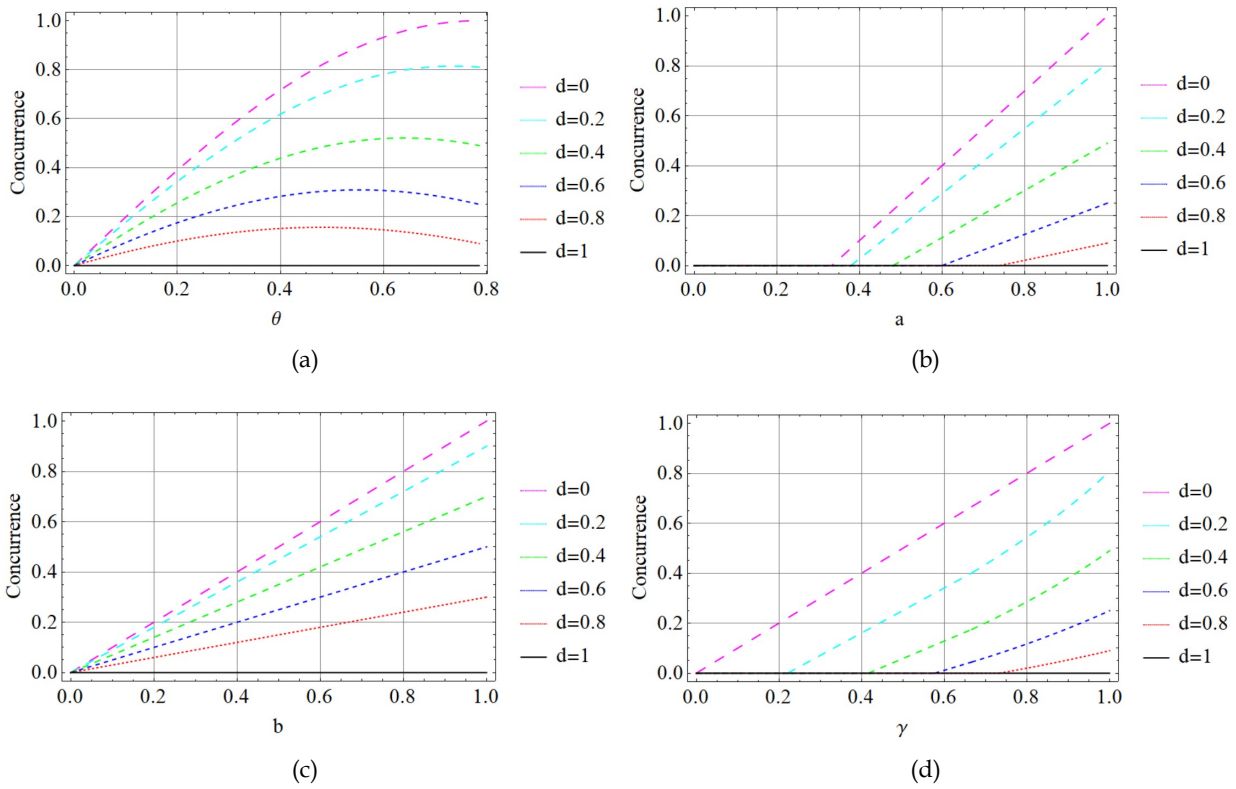


Figure 2.1 : Concurrence versus state parameter at different values of decoherence parameter d, for (a) generalized-*Bell* states, (b) *Werner* states, (c) *Horodecki* states and (d) *MEMS*

protected against adverse effects of decoherence; and in case of higher values of weak measurement parameter, correlations can be completely restored with a certain probability.

2.3 THE MODIFIED INEQUALITY USING CLUSTER COEFFICIENTS

Classifying correlations in an underlying quantum state as classical or quantum is imperative not only for a better interpretation of fundamentals imbibed in the theory but also to identify efficient and optimal resources for quantum information and computation. The fundamental principle to distinguish between classical and quantum correlations probably lies in the intrinsic properties of measurements and correlations induced therein. For example, performing a classical measurement essentially reveals the outcome without disturbing the system as such, however, quantum measurements lead to much more restricted information with a collapse of the original wave function to one of the eigen states of the system. This difference becomes even more prominent for multiqubit systems, e.g., a three-qubit system, as identified by monogamy of quantum correlations [Koashi and Winter, 2004; Toner, 2009; Devi *et al.*, 2012; Prabhu *et al.*, 2012; Dhar *et al.*, 2017]. Such correlations exhibiting quantum nature- as identified by entanglement measures or information theoretic measures- have kindled the required spark responsible for a paradigm shift in computing from classical to quantum.

As discussed in the previous chapter, entanglement- though the most important metonym of nonlocal correlations- is not the only resource leading to nonlocal correlations, e.g., there may be separable states witnessing quantum correlations useful for tasks which are otherwise not possible using classical resources [Meyer, 2000; Laflamme *et al.*, 2002; Kenigsberg *et al.*, 2006; Datta and Vidal, 2007; Lanyon *et al.*, 2008; Melo-Luna *et al.*, 2017]. One such measure that advocates the importance of studying correlations from a context different from separability versus entanglement paradigm is quantum discord [Ollivier and Zurek, 2001; Henderson and Vedral, 2001; Modi

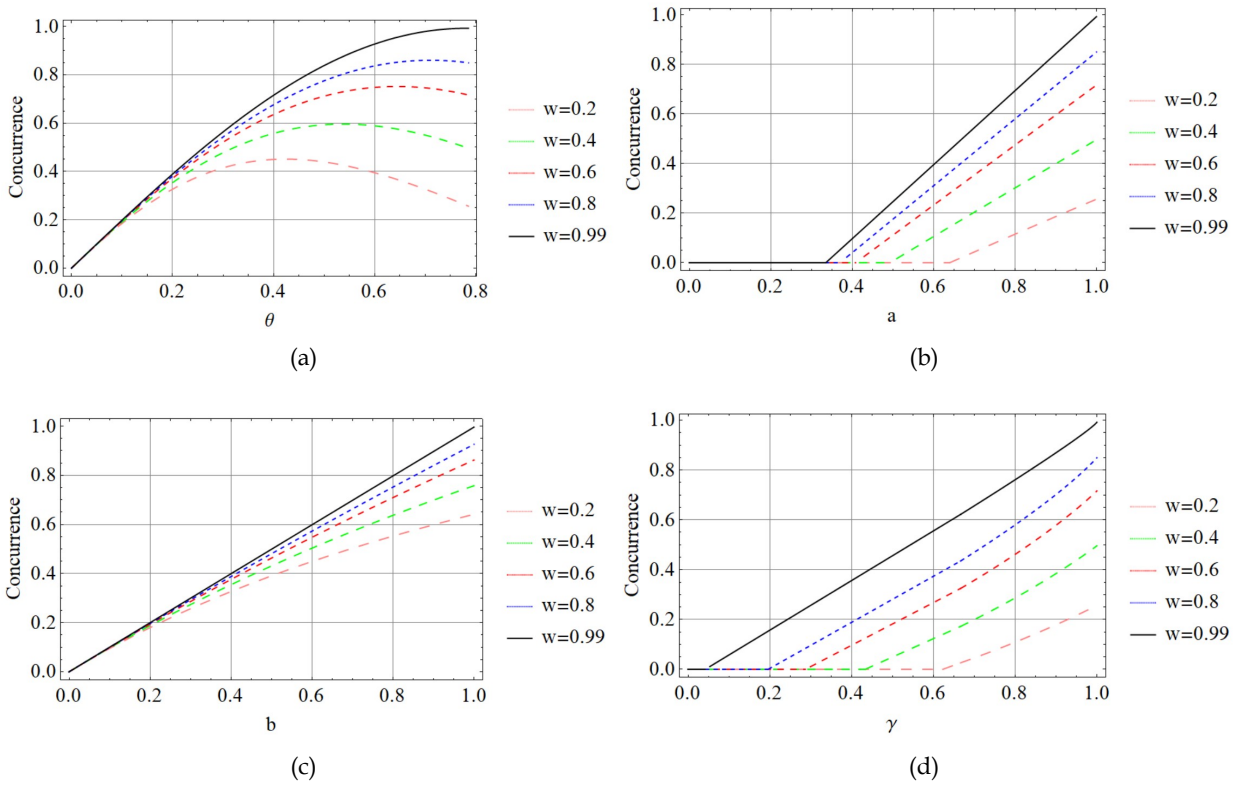


Figure 2.2 : Concurrence versus state parameter at different values of weak measurement parameter w, considering decoherence parameter $d = 0.7$, for (a) generalized-*Bell* states, (b) *Werner* states, (c) *Horodecki* states and (d) *MEMS*

et al., 2012]- the measure arrests quantum correlations in two-qubit pure and mixed entangled states, as well as in separable states. Considering its importance, discord has found its way to many important concepts and applications [Knill and Laflamme, 1998; Datta *et al.*, 2008; Vedral, 2010; Datta and Shaji, 2011; Passante *et al.*, 2011; Datta and Madhok, 2017]. Although quantum discord clearly distinguishes the quantum world from its classical analogue, the optimization procedure involved in evaluation of discord makes it not-so-handy tool for capturing quantumness of correlations. Therefore, we build on defining a quantifier of nonlocal correlation consistent with discord but one that can be obtained analytically as well as numerically with a relative ease.

This chapter focuses on analyzing the extent of correlation between qubits using standard Ursell-Mayer type expansions based on the principles of many body statistical mechanics [Fano, 1983; Huang, 1987; Watson, 1956; Osborn, 1977; Schlienz and Mahler, 1995; Kumar and Krishnan, 2009b; Audenaert and Plenio, 2006]. Before discussing the results obtained in this chapter, we briefly describe correlations existing between qubits using cluster coefficients. For this, we first define a two-qubit density operator $\rho^{(12)}$ representing a product state with lack of quantum correlation between qubits, such that

$$\rho^{(12)}_{uncorr} = \frac{1}{4}\left[\left(I^{(1)} + P^{(1)}.\sigma^{(1)}\right)\left(I^{(2)} + P^{(2)}.\sigma^{(2)}\right)\right] \tag{2.2}$$

where I's are identity operators for individual qubits, P's are polarization vectors of respective qubits where magnitude of $P^i = \langle\sigma^i\rangle$, and σ's are Pauli spin operators. For example, a two-qubit pure state $|01\rangle$ can be represented as [Fano, 1983]

$$\rho^{(12)} = \frac{1}{4}\left[\left(I^{(1)} + \sigma_z^{(1)}\right)\left(I^{(2)} - \sigma_z^{(2)}\right)\right] \tag{2.3}$$

However, if a two-qubit general state is a correlated one then the density operator can be represented as

$$\rho_{corr}^{(12)} = \frac{1}{4}\left[\left(I^{(1)} + P^{(1)}.\sigma^{(1)}\right)\left(I^{(2)} + P^{(2)}.\sigma^{(2)}\right) + \sigma^{(1)}\mathscr{C}_{12}\sigma^{(2)}\right] \tag{2.4}$$

where $\mathscr{C}_{12}$ represents a second-rank tensor representing correlations between qubits given by $\mathscr{C}_{ij} = \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$; here σ's are Pauli operators associated with respective qubits and $i, j = x, y, z$. The approach presented here can be applied to qubits or other spin-$\frac{1}{2}$ systems as well as photons. Interactions between qubits do result in correlations of the type described here [Fano, 1983]. Based on the above discussion, we propose to modify the original Bell-CHSH inequality using correlation coefficients, which is represented as

$$|B_{cum}| = |\langle AB\rangle_{cum} + \langle AB'\rangle_{cum} + \langle A'B\rangle_{cum} - \langle A'B'\rangle_{cum}| \tag{2.5}$$

where

$$\begin{aligned}\langle AB\rangle_{cum} =& \langle AB\rangle - \langle A\rangle\langle B\rangle \\ =& \iint ab f_{AB}(a,b)\,da\,db - \int a f_A(a)\,da \int b f_B(b)\,db,\end{aligned} \tag{2.6}$$

Here $f_A(a)$, $f_B(b)$, $f_{AB}(a,b)$ represent probability densities and A, B are measure spaces. Now, if A and B are independent, i.e., the case of a product state, we can factorize probability density $f_{AB}(a,b)$ into a product of probability density $f_A(a)$ for A and probability density $f_B(b)$ for B using Fubini-Tonelli theorem [Fubini, 1907; Tonelli, 1909]. Therefore,

$$\begin{aligned}\langle AB\rangle_{cum} &= \iint ab f_{AB}(a,b)\,da\,db - \int a f_A(a)\,da \int b f_B(b)\,db \\ &= \iint ab f_A(a) f_B(b)\,da\,db - \int a f_A(a)\,da \int b f_B(b)\,db \\ &= \int a f_A(a)\,da \int b f_B(b)\,db - \int a f_A(a)\,da \int b f_B(b)\,db \\ &= 0\end{aligned} \tag{2.7}$$

The remaining terms $\langle AB'\rangle_{cum}$, $\langle A'B\rangle_{cum}$ and $\langle A'B'\rangle_{cum}$ in Eq. (2.5) can also be evaluated to be 0 using the similar approach. Combining all these terms, we get the upper bound of $|B_{cum}|$ operator consistent with the local correlations as 0. Alternately, one can also consider the example of a general two-qubit product state, $|\psi_{A-B}\rangle$ given as

$$|\psi_{A-B}\rangle = \{Cos\theta_1\,|0\rangle + e^{i\phi_1}Sin\theta_1\,|1\rangle\}_A \otimes \{Cos\theta_2\,|0\rangle + e^{i\phi_2}Sin\theta_2\,|1\rangle\}_B \tag{2.8}$$

For the above two-qubit state, the expectation values of operators $\langle AB\rangle$, $\langle A\rangle$ and $\langle B\rangle$ are

$$\begin{aligned}\langle AB\rangle =& (Cos\theta_a Cos2\theta_1 + (Cos\phi_1 Cos\phi_a + Sin\phi_1 Sin\phi_a)Sin\theta_a Sin2\theta_1) \\ & (Cos\theta_b Cos2\theta_2 + (Cos\phi_2 Cos\phi_b + Sin\phi_2 Sin\phi_b)Sin\theta_b Sin2\theta_2),\end{aligned} \tag{2.9}$$

$$\langle A\rangle = (Cos\theta_a Cos2\theta_1 + (Cos\phi_1 Cos\phi_a + Sin\phi_1 Sin\phi_a)Sin\theta_a Sin2\theta_1) \tag{2.10}$$

and

$$\langle B\rangle = (Cos\theta_b Cos2\theta_2 + (Cos\phi_2 Cos\phi_b + Sin\phi_2 Sin\phi_b)Sin\theta_b Sin2\theta_2), \tag{2.11}$$

respectively. Using Eqs. (2.9)- (2.11), we evaluate the first term in Eq. (2.5), i.e., $\langle AB\rangle_{cum} = 0$. Similarly, expectation values $\langle AB'\rangle_{cum}$, $\langle A'B\rangle_{cum}$ and $\langle A'B'\rangle_{cum}$ are also evaluated to be 0. Hence, expectation value of the Bell-cumulant operator for the state defined in Eq. (2.8) is 0. Therefore, for a product state, expectation value of the Bell-cumulant operator is always 0. On a similar note, expectation value of the proposed Bell-cumulant operator for a maximally mixed state, i.e., $\frac{I}{4} = \frac{1}{4}\left[|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\right]$ is also 0. Clearly, a two-qubit state will possess nonlocal correlations if $|B_{cum}| > 0$, i.e., the violation of Bell-cumulant inequality is a signature of quantum correlations in an underlying state.

2.4 NONLOCALITY IN TWO-QUBIT PARTIALLY ENTANGLED STATES IN NOISY ENVIRONMENT

In this section, we proceed to analyze nonlocal properties of two-qubit pure and mixed states in presence of noise. For pure states, we analyze a set of generalized-*Bell* states, and for mixed states we characterize *Werner* states, *Horodecki* states, and Maximally entangled mixed states (*MEMS*). In order to analyze nonlocal correlations in presence of noise, we optimize expectation value of the Bell-cumulant operator for respective states and thus try to establish a relation between the maximum expectation value of our operator, noise parameter, and degrees of entanglement for states under consideration.

2.4.1 Partially entangled two-qubit pure states

We first consider the most important class of two-qubit entangled states- the generalized-*Bell* states- where the generalized class of two-qubit pure states can be represented as

$$|\psi\rangle = Cos\theta\,|00\rangle + e^{\iota\phi}Sin\theta\,|11\rangle \tag{2.12}$$

For $\theta = \frac{\pi}{4}$ and $\phi = 0$, the state represented in Eq. (2.12) is one of the *Bell* states. The other *Bell* states can be mapped to the one represented in Eq. (2.12) by performing single qubit local transformations on any one of the qubits. As discussed in section 2.1, concurrence for the generalized class of *Bell* states can be evaluated as $C(\rho_{Bell}) = Sin2\theta$. Evidently, the state is entangled for $0 < \theta \leq \dfrac{\pi}{4}$ with the degree of entanglement being maximum for $\theta = \dfrac{\pi}{4}$.

Analytical Solution: For analytical optimization of the $|B_{cum}|$ operator, we begin with the first term representing expectation value $\langle AB\rangle_{cum}$ in Eq. (2.5) such that

$$\langle AB\rangle_{cum} = Sin2\theta\,[Cos\theta_a Cos\theta_b Sin2\theta + Sin\theta_a Sin\theta_b(Cos\phi Cos(\phi_a+\phi_b) + Sin\phi Sin(\phi_a+\phi_b))] \tag{2.13}$$

The expectation value of $\langle AB\rangle_{cum}$ can be maximized with respect to $(\phi_a+\phi_b)$,i.e.,

$$\begin{aligned}\langle AB\rangle_{cum} &\leq Sin2\theta\left[Cos\theta_a Cos\theta_b Sin2\theta + Sin\theta_a Sin\theta_b\left(Cos^2\phi + Sin^2\phi\right)^{1/2}\right]\\ &\leq Sin2\theta(Cos\theta_a Cos\theta_b Sin2\theta + Sin\theta_a Sin\theta_b)\end{aligned} \tag{2.14}$$

where we have used the fact that the maximum value of $pSin\theta + qCos\theta$ is $\sqrt{p^2+q^2}$, and $Cos^2\phi + Sin^2\phi = 1$. Eq. (2.14) can be further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq Sin2\theta[Cos^2\theta_b Sin^2 2\theta + Sin^2\theta_b]^{1/2} \tag{2.15}$$

Similarly, other terms in Eq. (2.5) representing expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ can be optimized as

$$\begin{aligned}\langle A'B\rangle_{cum} &= Sin2\theta(Cos\theta_{a'}Cos\theta_b Sin2\theta + Sin\theta_{a'}Sin\theta_b(Cos\phi Cos(\phi_{a'}+\phi_b) + Sin\phi Sin(\phi_{a'}+\phi_b)))\\ &\leq Sin2\theta[Cos^2\theta_b Sin^2 2\theta + Sin^2\theta_b]^{1/2}\end{aligned} \tag{2.16}$$

$$\begin{aligned}\langle AB'\rangle_{cum} &= Sin2\theta(Cos\theta_a Cos\theta_{b'}Sin2\theta + Sin\theta_a Sin\theta_{b'}(Cos\phi Cos(\phi_a+\phi_{b'}) + Sin\phi Sin(\phi_a+\phi_{b'})))\\ &\leq Sin2\theta[Cos^2\theta_{b'}Sin^2 2\theta + Sin^2\theta_{b'}]^{1/2}\end{aligned} \tag{2.17}$$

$$\begin{aligned}\langle A'B'\rangle_{cum} &= Sin2\theta(Cos\theta_{a'}Cos\theta_{b'}Sin2\theta + Sin\theta_{a'}Sin\theta_{b'}(Cos\phi Cos(\phi_{a'}+\phi_{b'}) + Sin\phi Sin(\phi_{a'}+\phi_{b'})))\\ &\leq Sin2\theta[Cos^2\theta_{b'}Sin^2 2\theta + Sin^2\theta_{b'}]^{1/2}\end{aligned} \tag{2.18}$$

We now consider a pair of mutually orthogonal unit vectors d, d' such that $b + b' = 2dCos\chi$, and $b - b' = 2d'Sin\chi$, and use Eq. (2.15) for $\langle AB\rangle_{cum}$ (and other similar expressions) to rewrite Eq. (2.5) as

$$|B_{cum}| = 2|\langle AD\rangle_{cum}Cos\chi + \langle A'D'\rangle_{cum}Sin\chi| \tag{2.19}$$

where $\langle AD\rangle_{cum}$ and $\langle A'D'\rangle_{cum}$ can be defined in a similar fashion as Eq. (2.15). Thus, the maximum value of $|B_{cum}|$ operator for the set of states in Eq. (2.12) is

$$|B_{cum}| \leq 2Sin2\theta|[Cos^2\theta_d Sin^2 2\theta + Sin^2\theta_d]^{1/2}Cos\chi + [Cos^2\theta_{d'}Sin^2 2\theta + Sin^2\theta_{d'}]^{1/2}Sin\chi| \tag{2.20}$$

We now optimize Eq. (2.20) with respect to χ, to get

$$\begin{aligned}|B_{cum}| &\leq 2Sin2\theta|([Cos^2\theta_d Sin^2 2\theta + Sin^2\theta_d] + [Cos^2\theta_{d'}Sin^2 2\theta + Sin^2\theta_{d'}])^{1/2}| \\ &\leq 2Sin2\theta|([(1-Sin^2\theta_d)Sin^2 2\theta + Sin^2\theta_d] + [(1-Sin^2\theta_{d'})Sin^2 2\theta + Sin^2\theta_{d'}])^{1/2}| \\ &\leq 2Sin2\theta|([(1-Sin^2 2\theta)(Sin^2\theta_d + Sin^2\theta_{d'}) + 2Sin^2 2\theta]^{1/2}| \end{aligned} \tag{2.21}$$

Since the maximum value of $Sin^2\theta_d + Sin^2\theta_{d'}$ is 2 [Ghose *et al.*, 2009], we have

$$\begin{aligned}|B_{cum}| &\leq 2Sin2\theta[2(1-Sin^2 2\theta) + 2Sin^2 2\theta]^{1/2} \\ &\leq 2\sqrt{2}Sin2\theta = 2\sqrt{2}C(\rho_{Bell}) \end{aligned} \tag{2.22}$$

Hence, in ideal conditions, the optimum value of our operator monotonically increases from 0 to $2\sqrt{2}$ for a product state to a maximally entangled state, respectively.

Nonlocality Robustness of generalized-Bell states in presence of Noise

We now proceed to analyze effects of noise and weak measurement operations on the maximum expectation value of the Bell-cumulant operator. For this, we consider both qubits of the state represented in Eq. (2.12) to pass through an amplitude damping channel during the entanglement distribution. For convenience, we assume strengths of amplitude damping channels for both qubits to be equal, i.e, $d_1 = d_2 = d$.

Once qubits are subjected to the amplitude damping noise, the joint state of two-qubits evolves as

$$\rho_{Bell}^{AD} = \begin{bmatrix} Cos^2\theta + d^2 Sin^2\theta & 0 & 0 & \bar{d}e^{-\iota\phi}Cos\theta Sin\theta \\ 0 & \bar{d}dSin^2\theta & 0 & 0 \\ 0 & 0 & \bar{d}dSin^2\theta & 0 \\ \bar{d}e^{\iota\phi}Cos\theta Sin\theta & 0 & 0 & \bar{d}^2 Sin^2\theta \end{bmatrix} \tag{2.23}$$

where d is the noise parameter, and $\bar{d} = (1-d)$. For analytical optimization of the $|B_{cum}|$ operator, we first consider the expectation value $\langle AB\rangle_{cum}$ in Eq. (2.5), such that

$$\langle AB\rangle_{cum} = \bar{d}Sin2\theta\left[\bar{d}Cos\theta_a Cos\theta_b Sin2\theta + Sin\theta_a Sin\theta_b\{Cos\phi\, Cos(\phi_a+\phi_b) + Sin\phi\, Sin(\phi_a+\phi_b)\}\right] \tag{2.24}$$

The maximization of Eq. (2.24) with respect to $(\phi_a + \phi_b)$ leads to

$$\langle AB\rangle_{cum} \leq \bar{d}Sin2\theta(\bar{d}Cos\theta_a Cos\theta_b Sin2\theta + Sin\theta_a Sin\theta_b) \tag{2.25}$$

Further maximizing Eq. (2.25) with respect to θ_a results in

$$\langle AB\rangle_{cum} \leq \bar{d}Sin2\theta\left(\bar{d}^2 Cos^2\theta_b Sin^2 2\theta + Sin^2\theta_b\right)^{1/2} \tag{2.26}$$

The other terms in Eq. (2.5) representing expectation values $\langle A'B_{cum}\rangle$, $\langle AB'_{cum}\rangle$, and $\langle A'B'_{cum}\rangle$ are also evaluated in a similar manner such that

$$\begin{aligned}\langle A'B\rangle_{cum} &= \bar{d}Sin2\theta\left[\bar{d}Cos\theta_{a'}Cos\theta_b Sin2\theta + Sin\theta_{a'}Sin\theta_b\{Cos\phi\, Cos(\phi_{a'}+\phi_b) + Sin\phi\, Sin(\phi_{a'}+\phi_b)\}\right] \\ &\leq \bar{d}Sin2\theta[\bar{d}^2 Cos^2\theta_b Sin^2 2\theta + Sin^2\theta_b]^{1/2} \end{aligned} \tag{2.27}$$

$$\begin{aligned}\langle AB'\rangle_{cum} &= \bar{d}Sin2\theta\left[\bar{d}Cos\theta_a Cos\theta_{b'} Sin2\theta + Sin\theta_a Sin\theta_{b'}\{Cos\phi\, Cos(\phi_a+\phi_{b'}) + Sin\phi\, Sin(\phi_a+\phi_{b'})\}\right] \\ &\leq \bar{d}Sin2\theta[\bar{d}^2 Cos^2\theta_{b'} Sin^2 2\theta + Sin^2\theta_{b'}]^{1/2} \end{aligned} \tag{2.28}$$

$$\begin{aligned}\langle A'B'\rangle_{cum} &= \bar{d}Sin2\theta\left[\bar{d}Cos\theta_{a'}Cos\theta_{b'} Sin2\theta + Sin\theta_{a'}Sin\theta_{b'}\{Cos\phi\, Cos(\phi_{a'}+\phi_{b'}) + Sin\phi\, Sin(\phi_{a'}+\phi_{b'})\}\right] \\ &\leq \bar{d}Sin2\theta[\bar{d}^2 Cos^2\theta_{b'} Sin^2 2\theta + Sin^2\theta_{b'}]^{1/2} \end{aligned} \tag{2.29}$$

Therefore, Eq. (2.19) for ρ_{Bell}^{AD} can now be expressed as

$$|B_{cum}| \leq 2\bar{d}Sin2\theta|[\bar{d}^2Cos^2\theta_d Sin^2 2\theta + Sin^2\theta_d]^{1/2}Cos\chi + [\bar{d}^2Cos^2\theta'_d Sin^2 2\theta + Sin^2\theta'_d]^{1/2}Sin\chi| \quad (2.30)$$

which when maximized with respect to χ, gives

$$|B_{cum}| \leq 2\bar{d}Sin2\theta|((1-\bar{d}^2Sin^2 2\theta)(Sin^2\theta_d + Sin^2\theta'_d) + 2\bar{d}^2Sin^2 2\theta)^{1/2}| \quad (2.31)$$

Again considering the maximum value of $Sin^2\theta_d + Sin^2\theta'_d$ as 2, we have

$$|B_{cum}| \leq 2\sqrt{2}\bar{d}Sin2\theta = 2\sqrt{2}\bar{d}C(\rho_{Bell}) \quad (2.32)$$

Figure 2.3(a) and Figure 2.3(c) show effects of decoherence on nonlocality of two-qubit general *Bell* states. As expected, nonlocal correlations decrease with the strength of amplitude damping noise. Nevertheless, even for higher values of decoherence, the nonlocal correlations are highest for a maximally entangled input state. This result is in line with the violation of the Bell inequality where the maximum expectation value of the Bell operator in presence of noise is more for a maximally entangled input state in comparison to partially entangled input states, if the inequality is violated. Further, the analytical results obtained here are in complete agreement with the numerical results as shown in Figure 2.3(a).

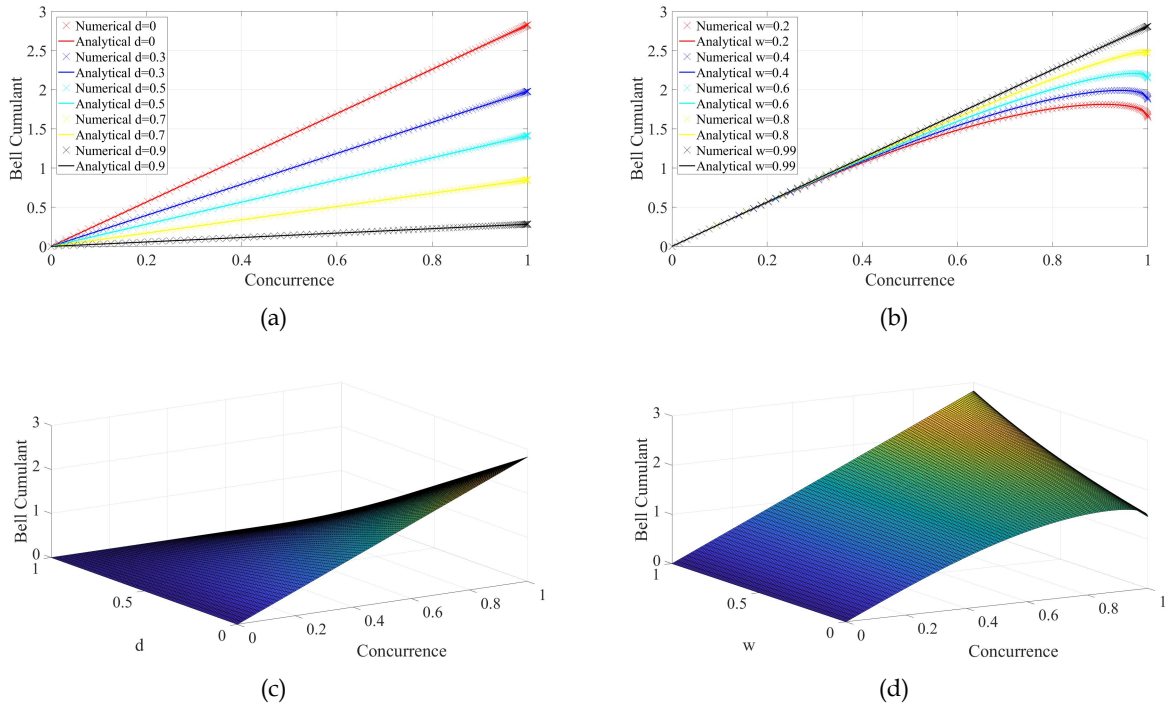


Figure 2.3 : Maximum expectation value of the Bell cumulant operator vs concurrence for the generalized-*Bell* state: (a) comparison of analytical and numerical results at different values of decoherence parameter d, (b) comparison of analytical and numerical results at different values of weak measurement parameter w, considering $d = 0.7$, (c) in the presence of amplitude damping noise and (d) with applications of weak measurements at $d = 0.7$

Clearly, with the increasing strength of the amplitude damping noise, the correlations-responsible for efficient speed and memory advantage in quantum information and computation-experience a detrimental effect, and for a very strong noise the correlations will eventually die to zero rendering the original state to be only as good as a classical resource. Thus, one further needs to employ mechanisms to protect nonlocal correlations between qubits in the presence

of noise. We, therefore, analyze applications of weak measurement and its operations on both qubits of generalized-*Bell* states to analyze and study effects of decoherence in presence of weak measurements and its reversal operations. As discussed in the first chapter, weak measurement and its reversal operations are performed on qubits before and after passing through the noisy channels, respectively and then the correlations between qubits are optimized. Therefore, the final evolved state after passing through noisy channels and weak measurement operations is given as

$$\rho_{Bell}^{WM} = \frac{1}{A}\begin{bmatrix} Cos^2\theta + d^2Sin^2\theta & 0 & 0 & \bar{d}e^{-\iota\phi}Cos\theta Sin\theta \\ 0 & \bar{w}dSin^2\theta & 0 & 0 \\ 0 & 0 & \bar{w}dSin^2\theta & 0 \\ \bar{d}e^{\iota\phi}Cos\theta Sin\theta & 0 & 0 & Sin^2\theta \end{bmatrix} \tag{2.33}$$

where $A = Cos^2 2\theta + (1+d\bar{w})^2 Sin^2 2\theta$ and $\bar{w} = (1-w)$; w being the weak measurement strength parameter. In line with the discussion of optimizing expectation value of the proposed operator in presence of noise, we first evaluate the expectation value $\langle AB\rangle_{cum}$ for ρ_{Bell}^{WM} as

$$\begin{aligned}\langle AB\rangle_{cum} =& Sin2\theta\Bigg[\frac{4Cos\theta_a Cos\theta_b Sin2\theta}{(2+2d\bar{w}+d^2\bar{w}^2-d(2+d\bar{w})\bar{w}Cos2\theta)^2} \\ &+ \frac{Sin\theta_a Sin\theta_b(Cos\phi Cos(\phi_a+\phi_b)+Sin\phi Sin(\phi_a+\phi_b))}{Cos^2\theta+(1+d\bar{w})^2Sin^2 2\theta}\Bigg]\end{aligned} \tag{2.34}$$

For convenience, we consider $P = \frac{4}{(2+2d\bar{w}+d^2\bar{w}^2-d(2+d\bar{w})\bar{w}Cos2\theta)^2}$ and $Q = \frac{1}{Cos^2\theta+(1+d\bar{w})^2Sin^2 2\theta}$, so that $\langle AB\rangle_{cum}$ can be rewritten as

$$\langle AB\rangle_{cum} = Sin2\theta\,[PCos\theta_a Cos\theta_b Sin2\theta + QSin\theta_a Sin\theta_b\{Cos\phi Cos(\phi_a+\phi_b)+Sin\phi Sin(\phi_a+\phi_b)\}] \tag{2.35}$$

As above, we first maximize Eq. (2.35) with respect to $(\phi_a+\phi_b)$ such that

$$\langle AB\rangle_{cum} \leq Sin2\theta\,(PCos\theta_a Cos\theta_b Sin2\theta + QSin\theta_a Sin\theta_b) \tag{2.36}$$

$\langle AB\rangle_{cum}$ in Eq. (2.36) when maximized with respect to θ_a, results in

$$\langle AB\rangle_{cum} \leq Sin2\theta\left(P^2Cos\theta_b^2Sin^2 2\theta + Q^2Sin\theta_b^2\right)^{1/2} \tag{2.37}$$

Similar to $\langle AB\rangle_{cum}$ in Eq. (2.37), other terms in Eq. (2.5) representing expectation values $\langle A'B_{cum}\rangle$, $\langle AB'_{cum}\rangle$, and $\langle A'B'_{cum}\rangle$ are also calculated as

$$\begin{aligned}\langle A'B\rangle_{cum} &= Sin2\theta\,[PCos\theta_{a'}Cos\theta_b Sin2\theta + QSin\theta_{a'}Sin\theta_b\{Cos\phi Cos(\phi_{a'}+\phi_b)+Sin\phi Sin(\phi_{a'}+\phi_b)\}] \\ &\leq Sin2\theta\left(P^2Cos\theta_b^2Sin^2 2\theta + Q^2Sin\theta_b^2\right)^{1/2} \qquad (2.38)\\ \langle AB'\rangle_{cum} &= Sin2\theta\,[PCos\theta_a Cos\theta_{b'}Sin2\theta + QSin\theta_a Sin\theta_{b'}\{Cos\phi Cos(\phi_a+\phi_{b'})+Sin\phi Sin(\phi_a+\phi_{b'})\}] \\ &\leq Sin2\theta\left(P^2Cos^2\theta_{b'}Sin^2 2\theta + Q^2Sin^2\theta_{b'}\right)^{1/2} \qquad (2.39)\\ \langle A'B'\rangle_{cum} &= Sin2\theta\,[PCos\theta_{a'}Cos\theta_{b'}Sin2\theta + QSin\theta_{a'}Sin\theta_{b'}\{Cos\phi Cos(\phi_{a'}+\phi_{b'})+Sin\phi Sin(\phi_{a'}+\phi_{b'})\}] \\ &\leq Sin2\theta\left(P^2Cos^2\theta_{b'}Sin^2 2\theta + Q^2Sin^2\theta_{b'}\right)^{1/2} \qquad (2.40)\end{aligned}$$

Hence, using Eq. (2.19) $|B_{cum}|$ for ρ_{Bell}^{WM} can therefore be expressed as

$$|B_{cum}| \leq 2Sin2\theta|\left[P^2Cos^2\theta_d Sin^2 2\theta + Q^2Sin^2\theta_d\right]^{1/2}Cos\chi + \left[P^2Cos^2\theta_{d'}Sin^2 2\theta + Q^2Sin^2\theta_{d'}\right]^{1/2}Sin\chi| \tag{2.41}$$

A further optimization of Eq. (2.41) with respect to χ, leads to

$$|B_{cum}| \leq 2\sqrt{2}Sin2\theta Q \tag{2.42}$$

One can now replace the expression for Q as discussed above and simplify Eq. (2.42) to obtain

$$|B_{cum}| \leq 2\sqrt{2}Sin2\theta \left(\frac{1}{Cos^2\theta + \eta^2 Sin^2\theta}\right) = 2\sqrt{2}C(\rho_{Bell}) \left(\frac{1}{1+\eta^2+\left(1-\eta^2\sqrt{1-C(\rho_{Bell})^2}\right)}\right) \tag{2.43}$$

where $\eta = (1 + d\bar{w})$. A plot establishing a relation between the maximum value of $|B_{cum}|$ and degree of entanglement of the original state for different values of weak measurement operation is presented in Figure 2.3(b) or Figure 2.3(d), considering the value of noise parameter to be 0.7. Interestingly, for lower values of the weak measurement parameter, maximum violation of the $|B_{cum}|$ operator is obtained for a partially entangled input state instead of a maximally entangled input state, hence, introducing the nonlocality anomaly in the system [Méthot and Scarani, 2007] where the anomaly is described with respect to the entanglement of the original input state. The nonlocality in the state increases with increasing values of the weak measurement strength parameter, and the anomaly also reduces to zero for w values very close to 1. For example, at $w = .99$, the proposed inequality is maximally violated even if the value of decoherence parameter is as high as 0.7. A comparison between Figure 2.3(a) and Figure 2.3(b) shows the importance of weak measurement and its reversal operations for preserving nonlocal correlations under noisy conditions.

In next subsections, we analyze few classes of two-qubit mixed states for characterizing nonlocal correlations in noisy conditions.

2.4.2 Werner states

As examples of mixed states, we start our discussion with the two-qubit *Werner* state [Werner, 1989] represented as

$$\rho_w = a\,|\Phi^+\rangle\langle\Phi^+| + \frac{1-a}{4}I_4 \tag{2.44}$$

where $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and I_4 is a 4×4 Identity matrix. As discussed in section 2.1, concurrence of the *Werner* state is $C(\rho_w) = Max[0, \frac{1}{2}(-1+3a)]$, i.e., the state is entangled only when the state parameter $a > \frac{1}{3}$. Interestingly, even though the state is entangled for $\frac{1}{3} < a \leq \frac{1}{\sqrt{2}}$, the original Bell inequality is violated only for $a > \frac{1}{\sqrt{2}}$ [Miranowicz, 2004; Bartkiewicz *et al.*, 2013]. However, geometric discord [Dakić *et al.*, 2010; Luo and Fu, 2010] of the *Werner* state is always nonzero irrespective of the state parameter a, indicating that the state possesses nonlocal correlations for the whole range of state parameter a (i.e. $0 < a \leq 1$) [Ollivier and Zurek, 2001; Luo, 2008]. Clearly, the *Werner* state is useful for quantum information and computation even when the state is considered to be a separable state ($a < \frac{1}{3}$). Further, even though the state is entangled for $\frac{1}{3} < a \leq \frac{1}{\sqrt{2}}$, the original Bell inequality fails to identify nonlocal correlations in the state for the above mentioned range. The properties of *Werner* states have been extensively studied in literature to analyze the foundations of quantum theory and to demonstrate states potential as a useful resource in quantum information and computation [Lee and Kim, 2000; Hiroshima and Ishizaka, 2000; Mišta Jr *et al.*, 2002; Zhang *et al.*, 2002; Azuma and Ban, 2006; Jin *et al.*, 2010a; Adhikari *et al.*, 2010; Tang *et al.*, 2012; Lyons *et al.*, 2012; Li-Hui *et al.*, 2015; Qiang *et al.*, 2019; Sanchez *et al.*, 2020; Han *et al.*, 2021]. Therefore, it is important to analyze correlations in *Werner* states using our proposed correlation measure.

Analytical Solution: We start our discussion by analyzing correlations in *Werner* states in absence of noise, i.e., in ideal conditions. For this, we first evaluate the term $\langle AB\rangle_{cum}$ in Eq. (2.5)

with respect to the *Werner* state such that

$$\langle AB\rangle_{cum} = a\,[Cos\theta_a Cos\theta_b + Cos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b] \tag{2.45}$$

Following the discussions for optimizing expectation value of the $|B_{cum}|$ operator for generalized-*Bell* states in the previous subsection, Eq. (2.45) can be further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq a[Cos^2\theta_b + Cos^2(\phi_a+\phi_b)Sin^2\theta_b]^{1/2} \tag{2.46}$$

Since, the maximum value of $Cos^2(\phi_a+\phi_b) = 1$, we have

$$\langle AB\rangle_{cum} \leq a \tag{2.47}$$

Similarly, the expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ can also be calculated to be

$$\langle A'B\rangle_{cum} = a\,[Cos\theta_{a'}Cos\theta_b + Cos(\phi_{a'}+\phi_b)Sin\theta_{a'}Sin\theta_b] \leq a \tag{2.48}$$
$$\langle AB'\rangle_{cum} = a\,[Cos\theta_a Cos\theta_{b'} + Cos(\phi_a+\phi_{b'})Sin\theta_a Sin\theta_{b'}] \leq a \tag{2.49}$$
$$\langle A'B'\rangle_{cum} = a\,[Cos\theta_{a'}Cos\theta_{b'} + Cos(\phi_{a'}+\phi_{b'})Sin\theta_{a'}Sin\theta_{b'}] \leq a \tag{2.50}$$

Hence, using Eq. (2.19), the maximum value of $|B_{cum}|$ operator for ρ_w can be calculated as

$$|B_{cum}| \leq 2a|Cos\chi + Sin\chi| \tag{2.51}$$

which when maximized with respect to χ, gives

$$|B_{cum}| \leq 2\sqrt{2}a = 2\sqrt{2}\left(\frac{2C(\rho_w)+1}{3}\right) \tag{2.52}$$

For $a = 1$, the *Werner* state transforms into one of the maximally entangled *Bell* states and therefore our operator leads to the maximum violation, i.e., $|B_{cum}| = 2\sqrt{2}$. On the other hand, for $a = 0$, where the state is a maximally mixed state, our operator indicates complete lack of nonlocal correlations, i.e, $|B_{cum}|$ is 0. Interestingly, unlike the original Bell inequality which is violated for $a > \frac{1}{sqrt2}$, the optimized expectation value of proposed operator monotonically increases from 0 to 1 for the whole range of state parameter. Therefore, the Werner state always exhibits nonlocal correlations in ideal conditions irrespective of the nature of entanglement. Hence, in accordance with the geometrical discord, our operator also identifies nonlocal correlations in all entangled as well as separable *Werner* states.

Nonlocality Robustness of Werner states in presence of Noise

In this subsection, we consider both qubits of the *Werner* state to pass through an amplitude damping channel during entanglement distribution and then analyze effects of noise and weak measurement operations on the maximum expectation value of the Bell-cumulant operator. As earlier, for simplicity, we assume the strength of amplitude damping channels to be the same.

After the applications of amplitude damping noise, the evolved two-qubit *Werner* state is represented as

$$\rho_w^{AD} = \frac{1}{4}\begin{bmatrix} (\nu^2 + a\bar{d}^2) & 0 & 0 & 2a\bar{d} \\ 0 & \bar{d}(\nu - a\bar{d}) & 0 & 0 \\ 0 & 0 & \bar{d}(\nu - a\bar{d}) & 0 \\ 2a\bar{d} & 0 & 0 & (1+a)\bar{d}^2 \end{bmatrix} \tag{2.53}$$

where $\bar{d} = (1-d)$ and $\nu = (1+d)$. In order to optimize the expectation value of $|B_{cum}|$ for the *Werner* state in noisy conditions, we first evaluate the term $\langle AB\rangle_{cum}$ such that

$$\langle AB\rangle_{cum} = a\bar{d}\left[\bar{d}Cos\theta_a Cos\theta_b + Cos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b\right] \tag{2.54}$$

Following our discussions for optimization of $\langle AB\rangle_{cum}$ in absence of noise, Eq. (2.54) can be optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq a\bar{d}[\bar{d}^2 Cos^2\theta_b + Cos^2(\phi_a+\phi_b)Sin^2\theta_b]^{1/2} \tag{2.55}$$

Using the optimal value of $Cos^2(\phi_a+\phi_b)=1$, the above equation can be rewritten as

$$\langle AB\rangle_{cum} \leq a\bar{d}[\bar{d}^2 Cos^2\theta_b + Sin^2\theta_b]^{1/2} \tag{2.56}$$

Similarly, we also evaluate other terms in Eq. (2.5) representing expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ as

$$\begin{aligned}
\langle A'B\rangle_{cum} &= a\bar{d}\left[\bar{d}Cos\theta_{a'}Cos\theta_b + Cos(\phi_{a'}+\phi_b)Sin\theta_{a'}Sin\theta_b\right] \\
&\leq a\bar{d}[\bar{d}^2 Cos^2\theta_b + Sin^2\theta_b]^{1/2} \qquad (2.57)\\
\langle AB'\rangle_{cum} &= a\bar{d}\left[\bar{d}Cos\theta_a Cos\theta_{b'} + Cos(\phi_a+\phi_{b'})Sin\theta_a Sin\theta_{b'}\right] \\
&\leq a\bar{d}[\bar{d}^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.58)\\
\langle A'B'\rangle_{cum} &= a\bar{d}\left[\bar{d}Cos\theta_{a'}Cos\theta_{b'} + Cos(\phi_{a'}+\phi_{b'})Sin\theta_{a'}Sin\theta_{b'}\right] \\
&\leq a\bar{d}[\bar{d}^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.59)
\end{aligned}$$

Further using Eq. (2.19), $|B_{cum}|$ for ρ_w^{AD} can thus be expressed as

$$|B_{cum}| \leq 2a\bar{d}|[\bar{d}^2 Cos^2\theta_d + Sin^2\theta_d]^{1/2}Cos\chi + [\bar{d}^2 Cos^2\theta_{d'} + Sin^2\theta_{d'}]^{1/2}Sin\chi| \tag{2.60}$$

which when maximized with respect to χ, gives

$$|B_{cum}| \leq 2a\bar{d}|[2\bar{d}^2 + (1-\bar{d}^2)(Sin^2\theta_d + Sin^2\theta_{d'})]^{1/2}| \tag{2.61}$$

Since the maximum value of $Sin^2\theta_d + Sin^2\theta_{d'}$ is 2, we have

$$|B_{cum}| \leq 2\sqrt{2}\bar{d}a = 2\sqrt{2}(1-d)\left(\frac{2C(\rho_w)+1}{3}\right) \tag{2.62}$$

For an ideal channel, i.e., for $d=0$, Eq. (2.62) transforms to Eq. (2.52) representing the ideal case. Also, Eq. (2.62) shows that nonlocal correlations decrease to zero for very high decoherence value. Now, we proceed to analyze effects of weak measurements on nonlocal correlations of the *Werner* state in presence of the amplitude damping noise. The final state after qubits pass through noisy channels under the influence of weak measurement and its reversal operations is evaluated as

$$\rho_w^{WM} = \frac{1}{A}\begin{bmatrix} \eta^2+a\kappa^2 & 0 & 0 & 2a \\ 0 & \eta-a\kappa & 0 & 0 \\ 0 & 0 & \eta-a\kappa & 0 \\ 2a & 0 & 0 & 1+a \end{bmatrix} \tag{2.63}$$

where $A = 4+4d\bar{w}+(1+a)d^2\bar{w}^2$, $\eta=(1+d\bar{w})$, $\kappa=(1-d\bar{w})$ and $\bar{w}=(1-w)$. Considering the above scenario, $\langle AB\rangle_{cum}$ is evaluated as

$$\langle AB\rangle_{cum} = \frac{4a}{A^2}[4Cos\theta_a Cos\theta_b + ACos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b] \tag{2.64}$$

Eq. (2.64) can be further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq \frac{4a}{A^2}[16Cos^2\theta_b + A^2Cos^2(\phi_a+\phi_b)Sin^2\theta_b]^{1/2} \tag{2.65}$$

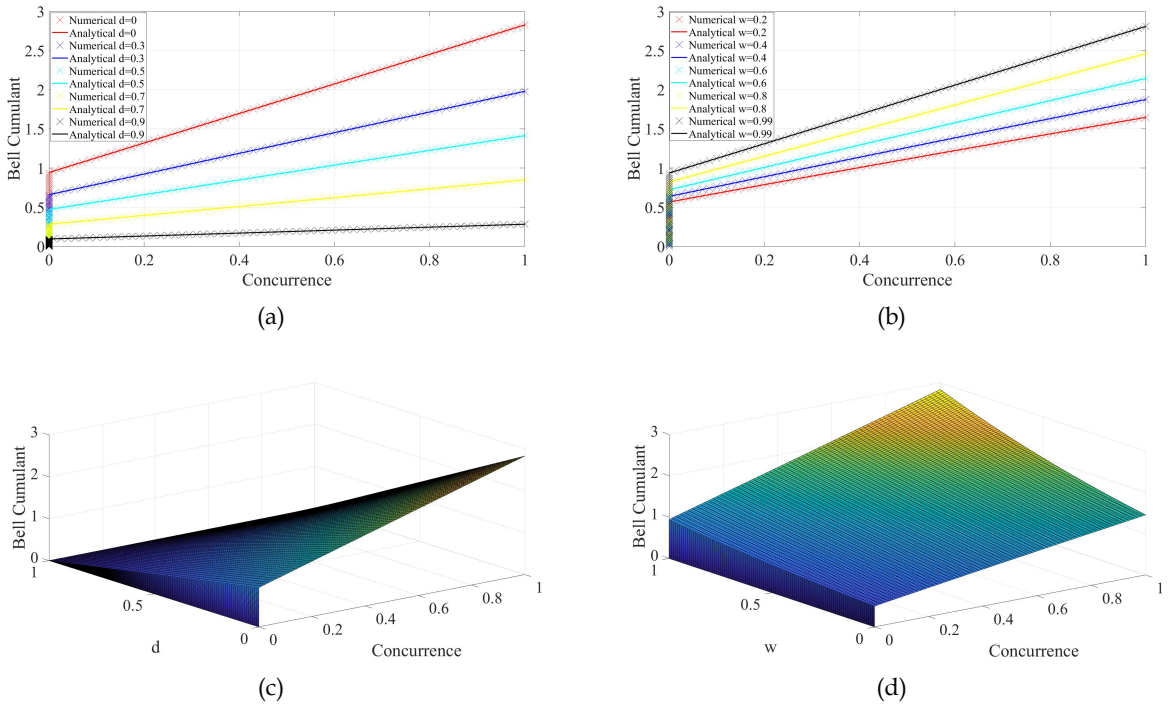


Figure 2.4 : Maximum expectation value of the Bell cumulant operator vs concurrence for *Werner* states: (a) comparison of analytical and numerical results at different values of decoherence parameter d, (b) comparison of analytical and numerical results at different values of weak measurement parameter w, considering $d = 0.7$, (c) in the presence of amplitude damping noise and (d) with applications of weak measurements at $d = 0.7$

Considering $Cos^2(\phi_a + \phi_b) = 1$, the above equation can be rewritten as

$$\langle AB\rangle_{cum} \leq \frac{4a}{A^2}[16Cos^2\theta_b + A^2Sin^2\theta_b]^{1/2} \tag{2.66}$$

Similar to the previous discussions, the expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ are also evaluated as

$$\begin{aligned}\langle A'B\rangle_{cum} &= \frac{4a}{A^2}[4Cos\theta_{a'}Cos\theta_b + ACos(\phi_{a'} + \phi_b)Sin\theta_{a'}Sin\theta_b] \\ &\leq \frac{4a}{A^2}[16Cos^2\theta_b + A^2Sin^2\theta_b]^{1/2} \qquad (2.67)\\ \langle AB'\rangle_{cum} &= \frac{4a}{A^2}[4Cos\theta_aCos\theta_{b'} + ACos(\phi_a + \phi_{b'})Sin\theta_aSin\theta_{b'}] \\ &\leq \frac{4a}{A^2}[16Cos^2\theta_{b'} + A^2Sin^2\theta_{b'}]^{1/2} \qquad (2.68)\\ \langle A'B'\rangle_{cum} &= \frac{4a}{A^2}[4Cos\theta_{a'}Cos\theta_{b'} + ACos(\phi_{a'} + \phi_{b'})Sin\theta_{a'}Sin\theta_{b'}] \\ &\leq \frac{4a}{A^2}[16Cos^2\theta_{b'} + A^2Sin^2\theta_{b'}]^{1/2} \qquad (2.69)\end{aligned}$$

Using Eq. (2.19) for ρ_w^{WM}, the expectation value, $|B_{cum}|$, can be further expressed as

$$|B_{cum}| \leq \frac{8a}{A^2}|[16Cos^2\theta_d + A^2Sin^2\theta_d]^{1/2}Cos\chi + [16Cos^2\theta_{d'} + A^2Sin^2\theta_{d'}]^{1/2}Sin\chi| \tag{2.70}$$

Eq. (2.70) is further maximized with respect to χ to get

$$|B_{cum}| \leq \frac{8a}{A^2}|[32 + (A^2 - 16)(Sin^2\theta_d + Sin^2\theta_{d'})]^{1/2}| \tag{2.71}$$

Since the maximum value of $Sin^2\theta_d + Sin^2\theta_{d'}$ is 2, we have

$$|B_{cum}| \leq \frac{8\sqrt{2}a}{A} = \frac{8\sqrt{2}\left(\frac{2C(\rho_w)+1}{3}\right)}{4\eta + \left(1+\frac{2C(\rho_w)+1}{3}\right)d^2\bar{w}^2} \tag{2.72}$$

Again for $w = d = 0$, Eq. (2.72) is same as Eq. (2.52) indicating the case of an ideal channel. The effects of decoherence channels and weak measurements are further elaborated in Figure 2.4(a) or 2.4(c) and Figure 2.4(b) or 2.4(d), respectively. From Figures 2.4(a) to 2.4(d), it is evident that analytical and numerical results are in excellent agreement.

Moreover, our results are also consistent with discord or geometric discord capturing the quantumness of correlations for the entire range of state parameter a. The analysis further confirms utility of cluster coefficients as a measure of correlations in an underlying state. Therefore, as against the Bell inequality which is only violated for $a > \frac{1}{\sqrt{2}}$, the proposed measure is violated by the *Werner* state irrespective of the value of state parameter. The analysis further reflects properties of nonlocal correlations in presence of noise. Figures 2.4(a) and 2.4(c) show that as the decoherence increases the extent of nonlocal correlations decreases to a smaller value. Nevertheless, for mild or moderate noise, correlations still exist between qubits even if the input state is a separable state. Furthermore, our results suggest that in case of the *Werner* state, maximum nonlocality corresponds to maximum entanglement of the original state indicating the absence of nonlocality anomaly. The applications of weak measurement operations clearly prevent the degradation of nonlocal correlations as shown in the Figures 2.4(b) and 2.4(d). In fact for larger values of weak measurement parameter, the extent of correlation between qubits can be very close to the one exhibited by a pure two-qubit entangled state. Interestingly, unlike the generalized pure two-qubit entangled state where nonlocality anomaly is feasible for lower values of weak measurement parameter; in this case, even for lower values of weak measurement parameter strength, the maximally entangled initial state violates the inequality maximally. Therefore, for a Werner state, entanglement of the input state coincides with nonlocal correlations for the Bell-cumulant operator.

2.4.3 Horodecki states

In this subsection, we extend our discussion by analyzing nonlocal properties of another two-qubit mixed state, namely the *Horodecki* state [Horedecki *et al.*, 1996], represented as

$$\rho_H = b\,|\Psi^+\rangle\langle\Psi^+| + (1-b)\,|00\rangle\langle 00| \tag{2.73}$$

where $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, and b is the state parameter. As per our discussions in section 2.1, concurrence for the *Horodecki* state is $C(\rho_H) = b$, i.e., the state is entangled for the entire range of state parameter. Although the state is entangled for all values of b, the Bell inequality is only violated by the state for $\frac{1}{\sqrt{2}} < b \leq 1$ [Horst *et al.*, 2013]. The case is very similar to the *Werner* state where the Bell inequality is violated only for a certain range of the state parameter. On the other hand, discord confirms the existence of nonlocal correlations in *Horodecki* states for complete range of the state parameter [Zhu *et al.*, 2018]. As nonlocal correlations in the *Horodecki* state are not identified by the Bell inequality for the complete range of state parameter- but are confirmed by discord- in this subsection, we analyze nonlocal correlations in the *Horodecki* state using our proposed operator.

Analytical Solution: Similar to our discussions in previous subsections, $\langle AB\rangle_{cum}$ in Eq. (2.5) is evaluated as

$$\langle AB\rangle_{cum} = b\,[-bCos\theta_a Cos\theta_b + Cos(\phi_a - \phi_b)Sin\theta_a Sin\theta_b] \tag{2.74}$$

As earlier, Eq. (2.74) can be further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq b[b^2Cos^2\theta_b + Cos^2(\phi_a+\phi_b)Sin^2\theta_b]^{1/2} \tag{2.75}$$

Using the optimal value of $Cos^2(\phi_a+\phi_b)=1$, Eq. (2.75) can be rewritten as

$$\langle AB\rangle_{cum} \leq b[b^2Cos^2\theta_b + Sin^2\theta_b]^{1/2} \tag{2.76}$$

Clearly, the other terms in Eq. (2.5) representing expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ can be expressed as

$$\begin{aligned}
\langle A'B\rangle_{cum} &= b\left[-bCos\theta_{a'}Cos\theta_b + Cos(\phi_{a'}-\phi_b)Sin\theta_{a'}Sin\theta_b\right] \\
&\leq b[b^2Cos^2\theta_b + Sin^2\theta_b]^{1/2} \qquad (2.77)\\
\langle AB'\rangle_{cum} &= b\left[-bCos\theta_aCos\theta_{b'} + Cos(\phi_a-\phi_{b'})Sin\theta_aSin\theta_{b'}\right] \\
&\leq b[b^2Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.78)\\
\langle A'B'\rangle_{cum} &= b\left[-bCos\theta_{a'}Cos\theta_{b'} + Cos(\phi_{a'}-\phi_{b'})Sin\theta_{a'}Sin\theta_{b'}\right] \\
&\leq b[b^2Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.79)
\end{aligned}$$

Therefore, using Eq. (2.19) and Eqs. (2.76) - (2.79), expectation value of the $|B_{cum}|$ operator for ρ_H is obtained as

$$|B_{cum}| \leq 2b|[b^2Cos^2\theta_d + Sin^2\theta_d]^{1/2}Cos\chi + [b^2Cos^2\theta_{d'} + Sin^2\theta_{d'}]^{1/2}Sin\chi| \tag{2.80}$$

A further optimization with respect to χ results in

$$|B_{cum}| \leq 2b|[2b^2 + (1-b^2)(Sin^2\theta_d + Sin^2\theta_{d'})]^{1/2}| \tag{2.81}$$

Now, using the maximum value of $Sin^2\theta_d + Sin^2\theta_{d'}$ as 2, the optimal value for $|B_{cum}|$ is

$$|B_{cum}| \leq 2\sqrt{2}b = 2\sqrt{2}C(\rho_H) \tag{2.82}$$

From Eq. (2.82), nonlocal correlations using our proposed operator increase monotonically with an increase in the state parameter or concurrence from 0 to 1. The maximally entangled state corresponding to unit concurrence violates the Bell-cumulant operator maximally. The results further suggest that entanglement and nonlocality measures coincide in the ideal case for *Horodecki* states.

Nonlocality Robustness of Horodecki states in presence of Noise

Considering the presence of noise inevitable, here, we analyze effects of amplitude damping noise and weak measurement operations on the maximum expectation value of the Bell-cumulant operator. For this, we further consider both qubits of the *Horodecki* state to pass through amplitude damping channels of equal strengths.

The final state, after qubits pass through amplitude damping channels, evolves as

$$\rho_H^{AD} = \frac{1}{2}\begin{bmatrix} 2(1+b\bar{d}) & 0 & 0 & 0 \\ 0 & b\bar{d} & b\bar{d} & 0 \\ 0 & b\bar{d} & b\bar{d} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.83}$$

where $\bar{d}=1-d$. Similar to the ideal case, in order to obtain the optimized value of $|B_{cum}|$ for the *Horodecki* state in noisy conditions, we first evaluate the expectation value $\langle AB\rangle_{cum}$, such that

$$\langle AB\rangle_{cum} = b\bar{d}\left[-b\bar{d}Cos\theta_aCos\theta_b + Cos(\phi_a+\phi_b)Sin\theta_aSin\theta_b\right] \tag{2.84}$$

Eq. (2.84) is further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq b\bar{d}[b^2\bar{d}^2Cos^2\theta_b + Cos^2(\phi_a+\phi_b)Sin^2\theta_b]^{1/2} \tag{2.85}$$

Clearly, $Cos^2(\phi_a+\phi_b)=1$, therefore Eq. (2.85) can be rewritten as

$$\langle AB\rangle_{cum} \leq b\bar{d}[b^2\bar{d}^2Cos^2\theta_b + Sin^2\theta_b]^{1/2} \tag{2.86}$$

The expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ are also evaluated in a similar fashion so that

$$\begin{aligned}
\langle A'B\rangle_{cum} &= b\bar{d}\left[-b\bar{d}Cos\theta_{a'}Cos\theta_b + Cos(\phi_{a'}+\phi_b)Sin\theta_{a'}Sin\theta_b\right] \\
&\leq b\bar{d}[b^2\bar{d}^2Cos^2\theta_b + Sin^2\theta_b]^{1/2} \qquad (2.87)\\
\langle AB'\rangle_{cum} &= b\bar{d}\left[-b\bar{d}Cos\theta_aCos\theta_{b'} + Cos(\phi_a+\phi_{b'})Sin\theta_aSin\theta_{b'}\right] \\
&\leq b\bar{d}[b^2\bar{d}^2Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.88)\\
\langle A'B'\rangle_{cum} &= b\bar{d}\left[-b\bar{d}Cos\theta_{a'}Cos\theta_{b'} + Cos(\phi_{a'}+\phi_{b'})Sin\theta_{a'}Sin\theta_{b'}\right] \\
&\leq b\bar{d}[b^2\bar{d}^2Cos^2\theta_{b'} + Sin^2\theta_{b'}]^{1/2} \qquad (2.89)
\end{aligned}$$

The optimized value of the proposed operator for ρ_H^{AD} is therefore

$$|B_{cum}| \leq\ 2b\bar{d}|[b^2\bar{d}^2Cos^2\theta_d + Sin^2\theta_d]^{1/2}Cos\chi + [b^2\bar{d}^2Cos^2\theta_{d'} + Sin^2\theta_{d'}]^{1/2}Sin\chi| \tag{2.90}$$

Following the previous subsections, Eq. (2.90) can be further optimized with respect to χ such that

$$|B_{cum}| \leq 2b\bar{d}|[2b^2\bar{d}^2 + (1-b^2\bar{d}^2)(Sin^2\theta_d + Sin^2\theta_{d'})]^{1/2}| \tag{2.91}$$

or

$$|B_{cum}| \leq 2\sqrt{2}\bar{d}b = 2\sqrt{2}(1-d)C(\rho_H) \tag{2.92}$$

as earlier.

For $d=0$, i.e. for an ideal channel without any noise, Eq. (2.92) is same as Eq. (2.82). For the limiting value $d=1$, correlations seize to zero and the state is no longer useful for quantum information and computation.

Moreover, applications of weak measurement and its reversal operations on qubits subjected to the amplitude damping noise lead to the final state as

$$\rho_H^{WM} = \frac{1}{2A}\begin{bmatrix} 1-b(1-d\bar{w}) & 0 & 0 & 0 \\ 0 & b & b & 0 \\ 0 & b & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.93}$$

where $A = 1+bd\bar{w}$ and $\bar{w}=(1-w)$. Using the final evolved state, we calculate the expectation value $\langle AB\rangle_{cum}$ as

$$\langle AB\rangle_{cum} = \frac{b}{A^2}\left[-bCos\theta_aCos\theta_b + ACos(\phi_a+\phi_b)Sin\theta_aSin\theta_b\right] \tag{2.94}$$

As discussed above, Eq. (2.94) can be further optimized with respect to θ_a to obtain

$$\langle AB\rangle_{cum} \leq \frac{b}{A^2}(-b^2Cos^2\theta_b + A^2Sin^2\theta_b)^{1/2} \tag{2.95}$$

where for optimization $Cos^2(\phi_a+\phi_b)=1$. On similar lines, we further evaluate the expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ as

$$\begin{aligned}\langle A'B\rangle_{cum} &= \frac{b}{A^2}\left[-bCos\theta_{a'}Cos\theta_b + ACos(\phi_{a'}+\phi_b)Sin\theta_{a'}Sin\theta_b\right]\\ &\leq \frac{b}{A^2}(-b^2Cos^2\theta_b + A^2Sin^2\theta_b)^{1/2} \qquad (2.96)\\ \langle AB'\rangle_{cum} &= \frac{b}{A^2}\left[-bCos\theta_aCos\theta_{b'} + ACos(\phi_a+\phi_{b'})Sin\theta_aSin\theta_{b'}\right]\\ &\leq \frac{b}{A^2}(-b^2Cos^2\theta_{b'} + A^2Sin^2\theta_{b'})^{1/2} \qquad (2.97)\\ \langle A'B'\rangle_{cum} &= \frac{b}{A^2}\left[-bCos\theta_{a'}Cos\theta_{b'} + ACos(\phi_{a'}+\phi_{b'})Sin\theta_{a'}Sin\theta_{b'}\right]\\ &\leq \frac{b}{A^2}(-b^2Cos^2\theta_{b'} + A^2Sin^2\theta_{b'})^{1/2} \qquad (2.98)\end{aligned}$$

Therefore, $|B_{cum}|$ for ρ_H^{WM} can be expressed as

$$|B_{cum}| \leq 2\frac{b}{A^2}|[-b^2Cos^2\theta_d + A^2Sin^2\theta_d]^{1/2}Cos\chi + [-b^2Cos^2\theta_{d'} + A^2Sin^2\theta_{d'}]^{1/2}Sin\chi| \qquad (2.99)$$

which when maximized with respect to χ, leads to

$$|B_{cum}| \leq \frac{2\sqrt{2}b}{A} = \frac{2\sqrt{2}C(\rho_H)}{1+C(\rho_H)d\bar{w}} \qquad (2.100)$$

where $Sin^2\theta_d + Sin^2\theta_{d'} = 2$.

Similar to Eq. (2.92), Eq. (2.100) is also equivalent to Eq. (2.82) for an ideal channel. Figure 2.5(a) or 2.5(c) and Figure 2.5(b) or 2.5(d) describe effects of amplitude damping channels and weak measurement and its reversal operations on the Bell Cumulant operator, respectively. The figures show that the numerical results are in complete agreement with the analytical results. Our results further demonstrate that the *Horodecki* state exhibits nonlocal correlations for the entire range of state parameter which is consistent with geometric discord for the *Horodecki* state. In fact, the state exhibits nonlocal properties even at higher values of the noise parameter. Figure 2.5(b) shows the efficiency of weak measurement operations in protecting and enhancing correlations in the state considering $d = 0.7$. In general, a comparison of Figure 2.4 and Figure 2.5 indicates that the nonlocal correlations in *Werner* states are more prominent in comparison to the nonlocal correlations in *Horodecki* states. Further, similar to the *Werner* state and unlike generalized-*Bell* states, weak measurements and reversal operations do not result in any anomaly as far as nonlocal correlations are concerned, i.e., the maximal violation coincides with the maximally entangled input state.

2.4.4 MEMS

We finally consider another important class of two-qubit mixed states [Munro *et al.*, 2001b], termed as maximally entangled mixed states (MEMS), and represented as

$$\rho_{MEMS} = \begin{bmatrix} g(\gamma) & 0 & 0 & \gamma/2 \\ 0 & 1-2g(\gamma) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & g(\gamma) \end{bmatrix} \text{ where } g(\gamma) = \begin{cases} 1/3, & \gamma < \frac{2}{3} \\ \gamma/2, & \gamma \geq \frac{2}{3} \end{cases} \qquad (2.101)$$

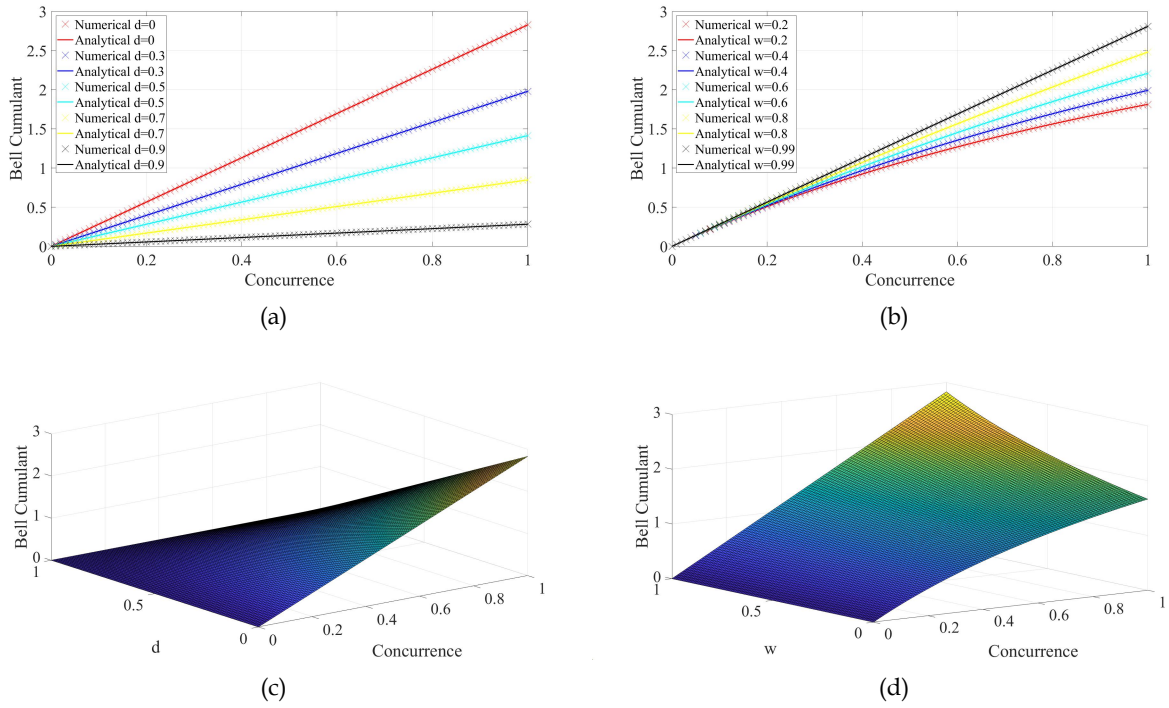


Figure 2.5 : Maximum expectation value of the Bell cumulant operator vs concurrence for *Horodecki* states: (a) comparison of analytical and numerical results at different values of decoherence parameter d, (b) comparison of analytical and numerical results at different values of weak measurement parameter w, considering $d = 0.7$, (c) in the presence of amplitude damping noise and (d) with applications of weak measurements at $d = 0.7$

i.e. for $\gamma < \frac{2}{3}$, the *MEMS* state is

$$\rho_{MEMS} = \begin{bmatrix} 1/3 & 0 & 0 & \gamma/2 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & 1/3 \end{bmatrix}, \tag{2.102}$$

and for $\gamma \geq \frac{2}{3}$, the *MEMS* state is

$$\rho_{MEMS} = \begin{bmatrix} \gamma/2 & 0 & 0 & \gamma/2 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma/2 & 0 & 0 & \gamma/2 \end{bmatrix} \tag{2.103}$$

As an entanglement measure for the *Horodecki* state, we follow concurrence, such that $C(\rho_{MEMS}) = \gamma$, indicating that the state is entangled for complete range of state parameter γ. Similar to *Werner* and *Horodecki* states, *MEMS* also violate the Bell inequality only for $\gamma > \frac{1}{\sqrt{2}}$. However, discord as a measure demonstrates the presence of nonlocal correlation even in the region defined by $0 < \gamma \leq \frac{1}{\sqrt{2}}$ [Batle *et al.*, 2011; Sharma and Pandey, 2015]. We, therefore, proceed to analyze nonlocality in *MEMS* using our proposed operator.

Analytical Solution: For evaluating the maximum expectation value of the $|B_{cum}|$ operator, as earlier, we first calculate the term $\langle AB \rangle_{cum}$ in Eq. (2.5) with respect to *MEMS*, such that

$$\begin{aligned} \langle AB \rangle_{cum}^{(\gamma < \frac{2}{3})} &= \frac{4}{9} Cos\theta_a Cos\theta_b + \gamma Cos(\phi_a + \phi_b) Sin\theta_a Sin\theta_b \text{ , and} \\ \langle AB \rangle_{cum}^{(\gamma \geq \frac{2}{3})} &= \gamma(\gamma Cos\theta_a Cos\theta_b + Cos(\phi_a + \phi_b) Sin\theta_a Sin\theta_b) \end{aligned} \tag{2.104}$$

We further optimize Eq. (2.104) with respect to θ_a to obtain

$$\langle AB\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_b + \gamma^2 Cos^2(\phi_a+\phi_b)Sin^2\theta_b\right]^{\frac{1}{2}}, \text{ and}$$
$$\langle AB\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \gamma\left[\gamma^2 Cos^2\theta_b + Cos^2(\phi_a+\phi_b)Sin^2\theta_b\right]^{1/2} \tag{2.105}$$

Using $Cos^2(\phi_a+\phi_b)=1$, the above equation can be rewritten as

$$\langle AB\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_b + \gamma^2 Sin^2\theta_b\right]^{\frac{1}{2}}, \text{ and}$$
$$\langle AB\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \gamma\left[\gamma^2 Cos^2\theta_b + Sin^2\theta_b\right]^{1/2} \tag{2.106}$$

Similar to previous subsections, $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ are equivalently evaluated as

$$\langle A'B\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_b + \gamma^2 Sin^2\theta_b\right]^{\frac{1}{2}}, \text{ and}$$
$$\langle A'B\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \gamma\left[\gamma^2 Cos^2\theta_b + Sin^2\theta_b\right]^{1/2} \tag{2.107}$$
$$\langle AB'\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_{b'} + \gamma^2 Sin^2\theta_{b'}\right]^{\frac{1}{2}}, \text{ and}$$
$$\langle AB'\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \gamma\left[\gamma^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}\right]^{1/2} \tag{2.108}$$
$$\langle A'B'\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_{b'} + \gamma^2 Sin^2\theta_{b'}\right]^{\frac{1}{2}}, \text{ and}$$
$$\langle A'B'\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \gamma\left[\gamma^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}\right]^{1/2} \tag{2.109}$$

Using, Eqs. (2.19) and (2.106), we have

$$|B_{cum}|^{(\gamma<\frac{2}{3})} \leq 2\left|\left[\left(\frac{4}{9}\right)^2 Cos^2\theta_d + \gamma^2 Sin^2\theta_d\right]^{\frac{1}{2}} Cos\chi + \left[\left(\frac{4}{9}\right)^2 Cos^2\theta_{d'} + \gamma^2 Sin^2\theta_{d'}\right]^{\frac{1}{2}} Sin\chi\right|, \text{ and}$$
$$|B_{cum}|^{(\gamma\geq\frac{2}{3})} \leq 2\gamma|\left[\gamma^2 Cos^2\theta_d + Sin^2\theta_d\right]^{1/2} Cos\chi + \left[\gamma^2 Cos^2\theta_{d'} + Sin^2\theta_{d'}\right]^{1/2} Sin\chi| \tag{2.110}$$

which when maximized with respect to χ, gives

$$|B_{cum}|^{(0\leq\gamma\leq1)} \leq 2\sqrt{2}\gamma = 2\sqrt{2}C(\rho_{MEMS}) \tag{2.111}$$

Nonlocality Robustness of $MEMS$ in presence of Noise

Similar to our discussions for pure and mixed states above, in this case also we consider both qubits of *MEMS* to pass through amplitude damping channels of equal strength, i.e. $d_1 = d_2 = 1$. Once qubits are subjected to the amplitude damping noise, *MEMS* under the decoherence channel evolve as

$$\rho_{MEMS}^{AD} = \begin{bmatrix} \frac{1}{3}(1+d+d^2) & 0 & 0 & \frac{1}{2}\bar{d}\gamma \\ 0 & \frac{1}{3}(1-d^2) & 0 & 0 \\ 0 & 0 & \frac{1}{3}d\bar{d} & 0 \\ \frac{1}{2}\bar{d}\gamma & 0 & 0 & \frac{1}{3}\bar{d}^2 \end{bmatrix} \quad \text{for } \gamma < \frac{2}{3}, \text{ and}$$

$$\rho_{MEMS}^{AD} = \frac{1}{2}\begin{bmatrix} 2d(1-\gamma)+\gamma(1+d^2) & 0 & 0 & \bar{d}\gamma \\ 0 & d\bar{d}\gamma & 0 & 0 \\ 0 & 0 & d\bar{d}\gamma & 0 \\ \bar{d}\gamma & 0 & 0 & \bar{d}^2\gamma \end{bmatrix} \quad \text{for } \gamma \geq \frac{2}{3} \tag{2.112}$$

In order to optimize the expectation value of the proposed operator using ρ_{MEMS}^{AD}, we start our calculation by evaluating $\langle AB\rangle_{cum}^{(\gamma<\frac{2}{3})}$ and $\langle AB\rangle_{cum}^{(\gamma\geq\frac{2}{3})}$ such that

$$\begin{aligned} \langle AB\rangle_{cum}^{(\gamma<\frac{2}{3})} &= \bar{d}\left[\frac{4}{9}\bar{d}Cos\theta_a Cos\theta_b + \gamma Cos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b\right], \text{ and} \\ \langle AB\rangle_{cum}^{(\gamma\geq\frac{2}{3})} &= \bar{d}\gamma(\bar{d}\gamma Cos\theta_a Cos\theta_b + Cos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b) \end{aligned} \tag{2.113}$$

Following the ideal case, we can further optimize Eq. (2.113) with respect to θ_a to obtain

$$\begin{aligned} \langle AB\rangle_{cum}^{(\gamma<\frac{2}{3})} &\leq \bar{d}\left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_b + \gamma^2 Sin^2\theta_b\right]^{1/2}, \text{ and} \\ \langle AB\rangle_{cum}^{(\gamma\geq\frac{2}{3})} &\leq \bar{d}\gamma\left[\bar{d}^2\gamma^2 Cos^2\theta_b + Sin^2\theta_b\right]^{1/2} \end{aligned} \tag{2.114}$$

where we consider $Cos^2(\phi_a+\phi_b)=1$. In a similar fashion, we further evaluate expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ such that

$$\langle A'B\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \bar{d}\left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_b + \gamma^2 Sin^2\theta_b\right]^{1/2}, \text{ and}$$

$$\langle A'B\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \bar{d}\gamma\left[\bar{d}^2\gamma^2 Cos^2\theta_b + Sin^2\theta_b\right]^{1/2} \tag{2.115}$$

$$\langle AB'\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \bar{d}\left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_{b'} + \gamma^2 Sin^2\theta_{b'}\right]^{1/2}, \text{ and}$$

$$\langle AB'\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \bar{d}\gamma\left[\bar{d}^2\gamma^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}\right]^{1/2} \tag{2.116}$$

$$\langle A'B'\rangle_{cum}^{(\gamma<\frac{2}{3})} \leq \bar{d}\left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_{b'} + \gamma^2 Sin^2\theta_{b'}\right]^{1/2}, \text{ and}$$

$$\langle A'B'\rangle_{cum}^{(\gamma\geq\frac{2}{3})} \leq \bar{d}\gamma\left[\bar{d}^2\gamma^2 Cos^2\theta_{b'} + Sin^2\theta_{b'}\right]^{1/2} \tag{2.117}$$

Now, using Eqs. (2.112) and (2.114), we get

$$\begin{aligned} |B_{cum}|^{(\gamma<\frac{2}{3})} &\leq 2\bar{d}\left|\left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_d + \gamma^2 Sin^2\theta_d\right]^{\frac{1}{2}} Cos\chi + \left[\left(\frac{4}{9}\right)^2 \bar{d}^2 Cos^2\theta_{d'} + \gamma^2 Sin^2\theta_{d'}\right]^{\frac{1}{2}} Sin\chi\right|, \text{and} \\ |B_{cum}|^{(\gamma\geq\frac{2}{3})} &\leq 2\bar{d}\gamma\left|\left[\bar{d}^2\gamma^2 Cos^2\theta_d + Sin^2\theta_d\right]^{1/2} Cos\chi + \left[\bar{d}^2\gamma^2 Cos^2\theta_{d'} + Sin^2\theta_{d'}\right]^{1/2} Sin\chi\right| \end{aligned} \tag{2.118}$$

Eq. (2.118) can be further maximized with respect to χ, to obtain

$$\begin{aligned} |B_{cum}|^{(\gamma<\frac{2}{3})} &\leq 2\bar{d}\left|\left[2\left(\frac{4}{9}\right)^2 \bar{d}^2 + \left(\gamma^2 - \left(\frac{4}{9}\right)^2 \bar{d}^2\right)(Sin^2\theta_d + Sin^2\theta_{d'})\right]^{1/2}\right|, \text{ and} \\ |B_{cum}|^{(\gamma\geq\frac{2}{3})} &\leq 2\bar{d}\gamma\left|\left[2\bar{d}^2\gamma^2 + (1-\bar{d}^2\gamma^2)(Sin^2\theta_d + Sin^2\theta_{d'})\right]^{1/2}\right| \end{aligned} \tag{2.119}$$

Similar to previous cases, the maximization of $Sin^2\theta_d + Sin^2\theta_{d'} = 2$, allows us to rewrite Eq. (2.119) as

$$|B_{cum}|^{(0\leq\gamma\leq 1)} \leq 2\sqrt{2}\bar{d}\gamma = 2\sqrt{2}\bar{d}C(\rho_{MEMS}) \tag{2.120}$$

One can easily see that for an ideal channel Eq. (2.120) is actually the same as Eq. (2.111). Moreover, Eq. (2.120) clearly demonstrates adverse effects of decoherence. We finally analyze effects of weak measurement and its reversal operations on nonlocal correlations of $MEMS$. Similar to the strength of noisy channels, for weak measurement operations also we assume weak measurement strengths for both qubits to be the same. The final evolved state after weak measurement-noise-weak measurement reversal operations can be represented as

$$\rho^{WM}_{MEMS} = \frac{1}{M}\begin{bmatrix} 2d\bar{w}+(1-d\bar{w})^2\gamma & 0 & 0 & \gamma \\ 0 & 2-(2-d\bar{w})\gamma & 0 & 0 \\ 0 & 0 & d\bar{w}\gamma & 0 \\ \gamma & 0 & 0 & \gamma \end{bmatrix} \quad \text{for } \gamma < \frac{2}{3}\text{ , and}$$

$$\rho^{WM}_{MEMS} = \frac{1}{N}\begin{bmatrix} 1+d\bar{w}+(d\bar{w})^2 & 0 & 0 & \frac{3\gamma}{2} \\ 0 & 1+d\bar{w} & 0 & 0 \\ 0 & 0 & d\bar{w} & 0 \\ \frac{3\gamma}{2} & 0 & 0 & 1 \end{bmatrix} \quad \text{for } \gamma \geq \frac{2}{3} \tag{2.121}$$

where $M = 3+3d\bar{w}+d^2\bar{w}^2$, $N = 2+2d\bar{w}+d^2\bar{w}^2\gamma$ and $\bar{w} = (1-w)$. For demonstrating effects of weak measurement and it reversal operations in presence of noise, we now proceed to evaluate $\langle AB\rangle_{cum}$ using ρ^{WM}_{MEMS}, such that

$$\begin{aligned} \langle AB\rangle^{(\gamma<\frac{2}{3})}_{cum} &= \frac{1}{M^2}\left[4Cos\theta_a Cos\theta_b + 3M\gamma Cos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b\right]\text{ , and} \\ \langle AB\rangle^{(\gamma\geq\frac{2}{3})}_{cum} &= \frac{2\gamma}{N^2}(2\gamma Cos\theta_a Cos\theta_b + NCos(\phi_a+\phi_b)Sin\theta_a Sin\theta_b) \end{aligned} \tag{2.122}$$

Following our discussion for the ideal case, we can further maximize the expectation value $\langle AB\rangle_{cum}$ with respect to θ_a such that for $Cos^2(\phi_a+\phi_b) = 1$, we have

$$\begin{aligned} \langle AB\rangle^{(\gamma<\frac{2}{3})}_{cum} &\leq \frac{1}{M^2}\left[16Cos^2\theta_b + 9M^2\gamma^2 Sin^2\theta_b\right]^{1/2}\text{ , and} \\ \langle AB\rangle^{(\gamma\geq\frac{2}{3})}_{cum} &\leq \frac{2\gamma}{N^2}\left[4\gamma^2 Cos^2\theta_b + N^2 Sin^2\theta_b\right]^{1/2} \end{aligned} \tag{2.123}$$

Other relevant expectation values $\langle A'B\rangle_{cum}$, $\langle AB'\rangle_{cum}$, and $\langle A'B'\rangle_{cum}$ can aslo be evaluated in a similar fashion, so that

$$\begin{aligned} \langle A'B\rangle^{(\gamma<\frac{2}{3})}_{cum} &\leq \frac{1}{M^2}\left[16Cos^2\theta_b + 9M^2\gamma^2 Sin^2\theta_b\right]^{1/2}\text{ , and} \\ \langle A'B\rangle^{(\gamma\geq\frac{2}{3})}_{cum} &\leq \frac{2\gamma}{N^2}\left[4\gamma^2 Cos^2\theta_b + N^2 Sin^2\theta_b\right]^{1/2} \end{aligned} \tag{2.124}$$

$$\begin{aligned} \langle AB'\rangle^{(\gamma<\frac{2}{3})}_{cum} &\leq \frac{1}{M^2}\left[16Cos^2\theta_{b'} + 9M^2\gamma^2 Sin^2\theta_{b'}\right]^{1/2}\text{ , and} \\ \langle AB'\rangle^{(\gamma\geq\frac{2}{3})}_{cum} &\leq \frac{2\gamma}{N^2}\left[4\gamma^2 Cos^2\theta_{b'} + N^2 Sin^2\theta_{b'}\right]^{1/2} \end{aligned} \tag{2.125}$$

$$\begin{aligned} \langle A'B'\rangle^{(\gamma<\frac{2}{3})}_{cum} &\leq \frac{1}{M^2}\left[16Cos^2\theta_{b'} + 9M^2\gamma^2 Sin^2\theta_{b'}\right]^{1/2}\text{ , and} \\ \langle A'B'\rangle^{(\gamma\geq\frac{2}{3})}_{cum} &\leq \frac{2\gamma}{N^2}\left[4\gamma^2 Cos^2\theta_{b'} + N^2 Sin^2\theta_{b'}\right]^{1/2} \end{aligned} \tag{2.126}$$

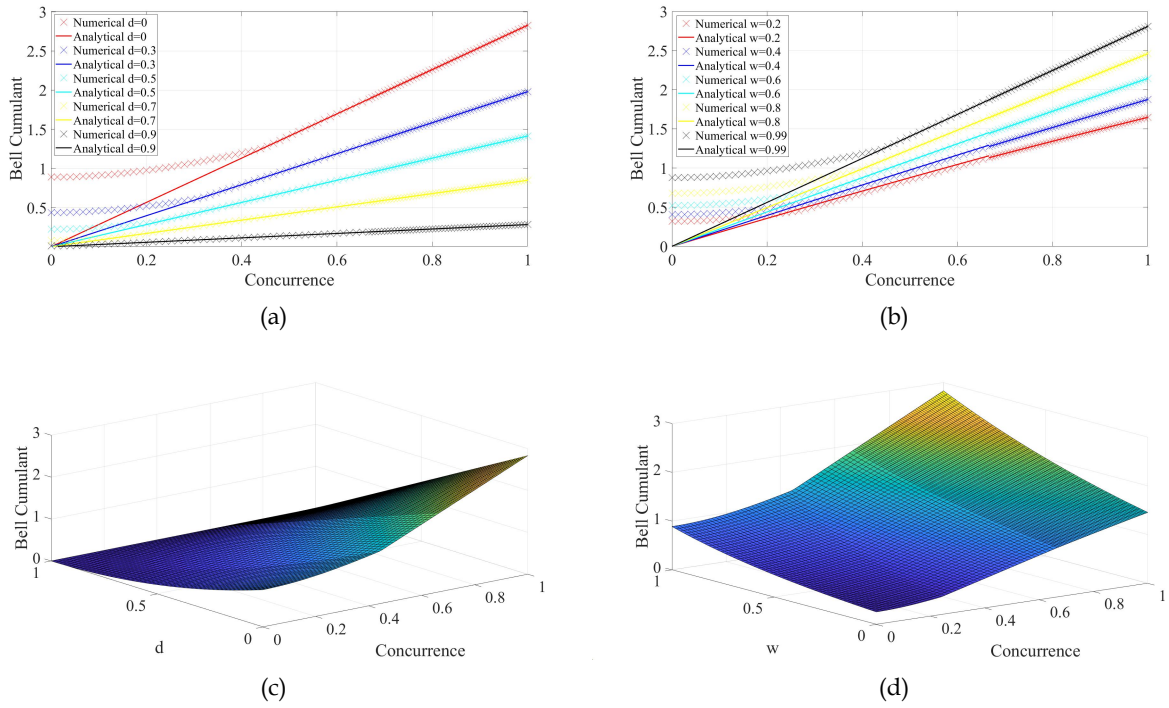


Figure 2.6 : Maximum expectation value of the Bell cumulant operator vs concurrence for *MEMS*: (a) comparison of analytical and numerical results at different values of decoherence parameter d, (b) comparison of analytical and numerical results at different values of weak measurement parameter w, considering $d = 0.7$, (c) in the presence of amplitude damping noise and (d) with applications of weak measurements at $d = 0.7$

Using Eq. (2.19), $|B_{cum}|$ for states given by Eq. (2.121) can be written as

$$|B_{cum}|^{(\gamma<\frac{2}{3})} \leq \frac{2}{M^2}\left|\left[16Cos^2\theta_d + 9M^2\gamma^2Sin^2\theta_d\right]^{1/2}Cos\chi + \left[16Cos^2\theta_{d'} + 9M^2\gamma^2Sin^2\theta_{d'}\right]^{1/2}Sin\chi\right|, \text{and}$$
$$|B_{cum}|^{(\gamma\geq\frac{2}{3})} \leq \frac{4\gamma}{N^2}\left|\left[4\gamma^2Cos^2\theta_d + N^2Sin^2\theta_d\right]^{1/2}Cos\chi + \left[4\gamma^2Cos^2\theta_{d'} + N^2Sin^2\theta_{d'}\right]^{1/2}Sin\chi\right| \quad (2.127)$$

respectively. One can further optimize Eq. (2.127) with respect to χ

$$|B_{cum}|^{(\gamma<\frac{2}{3})} \leq \frac{6\sqrt{2}\gamma}{M} = \frac{6\sqrt{2}C(\rho_{MEMS})}{3+3d\bar{w}+d^2\bar{w}^2}, \text{ and}$$
$$|B_{cum}|^{(\gamma\geq\frac{2}{3})} \leq \frac{4\sqrt{2}\gamma}{N} = \frac{4\sqrt{2}C(\rho_{MEMS})}{2+2d\bar{w}+d^2\bar{w}^2C(\rho_{MEMS})} \quad (2.128)$$

A comparison between results obtained for an ideal channel or a noisy channel for *Horodecki* states and *MEMS* shows that expectation value of the $|B_{cum}|$ operator is exactly the same. Moreover, in case of MEMS, for an ideal channel Eq. (2.128) are also same as Eq. (2.111). Figure 2.6(a) or 2.6(c) and Figure 2.6(b) or 2.6(d) depict effects of amplitude damping channel and weak measurement and its reversal operations, respectively on Bell Cumulant operator for *MEMS*. A comparison of Figure 2.6(a) and Figure 2.6(b) shows the importance of weak measurements for improving nonlocal correlations in the noisy environment. Figure 2.6 also shows the correspondence in numerical and analytical results. Interestingly, a closer observation shows that after applications of weak measurements, the extent of nonlocal correlations in *Horodecki* states are better than the extent of nonlocal correlations in *MEMS*, for higher concurrence. Nevertheless, the proposed inequality is violated by all MEMS, i.e., for the complete range of the state parameter γ.

2.5 TELEPORTATION FIDELITY

In this section, we analyze the operational interpretation of the proposed operator for quantum information processing. For this, we establish a relation between the proposed operator and teleportation fidelity. Quantum teleportation is one of the most celebrated example used for identifying the efficiency of a quantum mechanical resource. On these lines, the usefulness of a two-qubit entangled state for teleportation protocol is measured by fidelity of quantum teleportation [Horodecki *et al.*, 1996], which is defined as

$$F_{max} = \frac{1}{2}\left(1 + \frac{1}{3}Tr\sqrt{T^{\dagger}T}\right) \tag{2.129}$$

where T is a real matrix of the state used as a quantum channel for teleportation protocol whose matrix elements are defined as $t_{ij} = Tr(\rho.\sigma_i \otimes \sigma j)$. The underlying state is termed as a useful resource for quantum teleportation if $F_{max} > \frac{2}{3}$, i.e., if teleportation fidelity is greater than the classical bound of $\frac{2}{3}$ [Popescu, 1994; Massar and Popescu, 1995; Gisin, 1996b].

The teleportation fidelities for generalized *Bell* and *Werner* states in the ideal conditions are calculated as

$$F_{max}(\rho_{Bell}) = \frac{1}{3}(2 + Sin2\theta) \tag{2.130}$$

and

$$F_{max}(\rho_w) = \frac{1}{2}(1 + a), \tag{2.131}$$

respectively which implies that generalized-*Bell* states are useful resource for quantum teleportation for $Sin2\theta > 0$ (or $\theta > 0$); whereas, *Werner* states are useful for teleportation protocol for $a > \frac{1}{3}$. To understand the usefulness of states for teleportation protocol in real conditions, we also analyzed fidelities in the presence of noise. The teleportation fidelities of the above mentioned states in the presence of amplitude damping noise are evaluated as

$$F_{max}^{(AD)}(\rho_{Bell}) = \frac{1}{3}\left(2 - 2d\bar{d}Sin^2\theta + \bar{d}Sin2\theta\right) \tag{2.132}$$

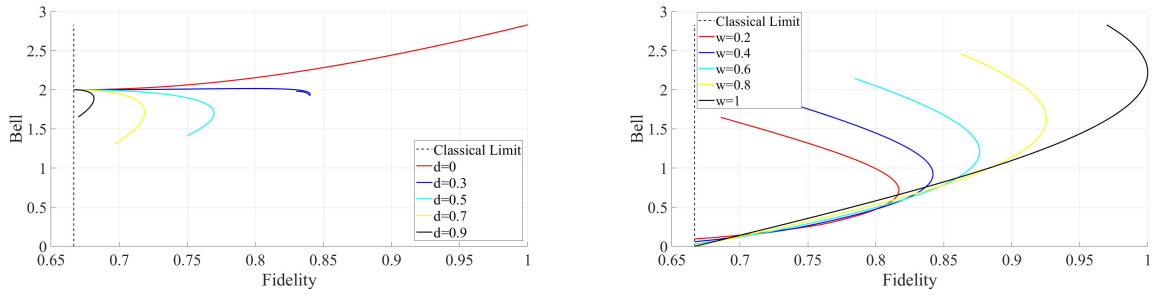
and

$$F_{max}^{(AD)}(\rho_w) = \frac{1}{6}\left(3 + d^2 + a(3 - 4d + d^2)\right), \tag{2.133}$$

respectively. For any value of d except $d = 1$, the fidelity in Eq. (2.132) is greater than $\frac{2}{3}$, which is the classical bound of fidelity. Hence, generalized-*Bell* states are useful for quantum teleportation even when the value of noise parameter is very high. However, for Werner states, teleportation fidelity is greater than the classical bound if $d \leq \frac{-1+3a}{1+a}$ only given that the state parameter a is greater than $\frac{1}{3}$.

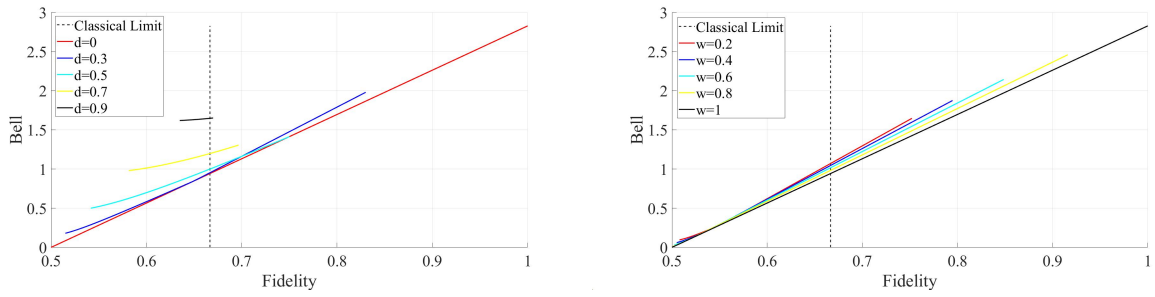
As discussed in previous sections the degree of entanglement of a state can be preserved with applications of weak measurements in presence of decoherence. Therefore, here, we further analyze teleportation fidelities of generalized *Bell* and *Werner* states in the presence of amplitude damping noise with applications of weak measurement operations. We evaluate the expressions of fidelities as

$$F_{max}^{(ADWM)}(\rho_{Bell}) = \frac{2 + Sin2\theta + 2d\bar{w}(1 + d\bar{w})Sin^2\theta}{3((1 + d\bar{w})^2Sin^2\theta + Cos^2\theta)} \tag{2.134}$$



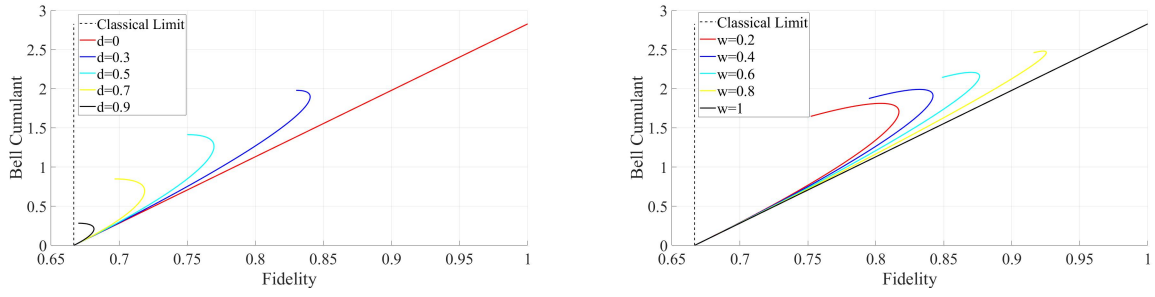
(a) (b)

Figure 2.7 : Maximum expectation value of the Bell operator vs Fidelity for the generalized-*Bell* state (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering $d = 0.7$



(a) (b)

Figure 2.8 : Maximum expectation value of the Bell operator vs Fidelity for *Werner* states (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering $d = 0.7$



(a) (b)

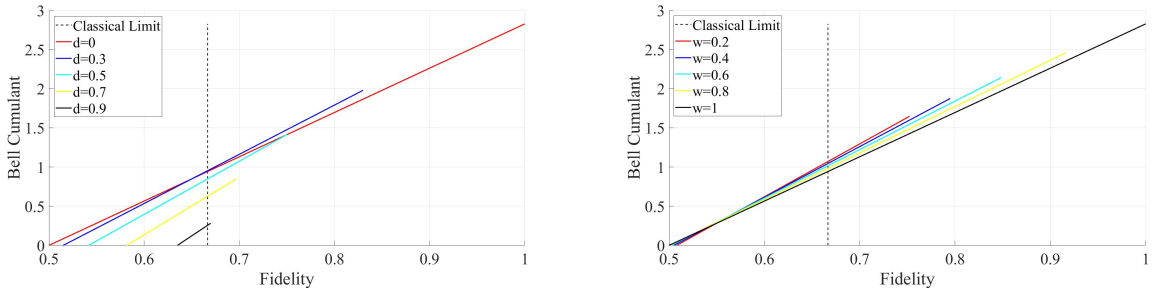
Figure 2.9 : Maximum expectation value of the Bell cumulant operator vs Fidelity for the generalized-*Bell* state (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering $d = 0.7$

and

$$F_{max}^{(ADWM)}(\rho_w) = \frac{2(3+a(3-d\bar{w}+d^2\bar{w}^2)+3d\bar{w}+d^2\bar{w}^2)}{3(4+4d\bar{w}+(1+a)d^2\bar{w}^2)}, \tag{2.135}$$

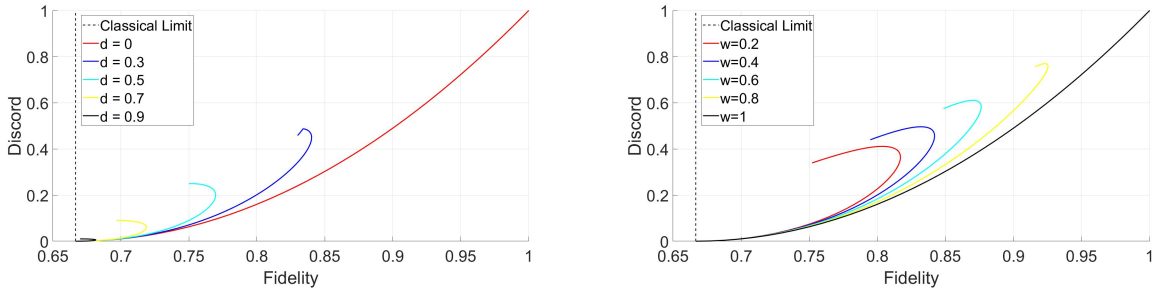
respectively. Since the fidelity using Bell states in presence of noise is always greater than the classical bound except for the maximum value of the noise parameter, applications of weak measurements further lead to increase in fidelity with increasing strengths of weak measurement

parameter. However, for *Werner* states fidelity to exceed the lower bound of $\frac{2}{3}$ weak measurement parameter w must be greater than $\frac{1-3a+d+ad}{d+ad}$, for $d > \frac{-1+3a}{1+a}$. Interestingly, Figure 2.7 shows



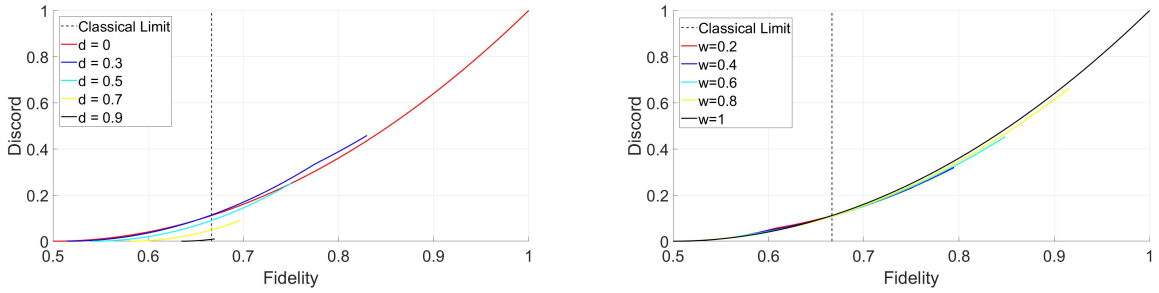
(a) (b)

Figure 2.10 : Maximum expectation value of the Bell cumulant operator vs Fidelity for *Werner* states (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering $d = 0.7$



(a) (b)

Figure 2.11 : Discord vs Fidelity for the generalized-*Bell* state (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering decoherence parameter $d = 0.7$



(a) (b)

Figure 2.12 : Discord vs Fidelity for *Werner* states (a) at different values of decoherence parameter d and (b) at different values of weak measurement w, considering decoherence parameter $d = 0.7$

that the Bell inequality does not show any violation by generalized-*Bell* states for mild to higher decoherence, however, for the same values of decoherence fidelity is greater than the classical

limit indicating the usefulness of nonlocal correlations even for higher values of decoherence parameter. Similarly, Figure 2.8 shows that *Werner* states do not violate the Bell inequality even for decoherence as small as 0.3; whereas even for $d = 0.7$, there exist states resulting in fidelity greater than the classical bound. We further characterize fidelity of generalized Bell and Werner states using our modified Bell operator and discord in Figures 2.9 and 2.11 for generalized Bell states and Figures 2.10 and 2.12 for Werner states, respectively. The results obtained using the modified operator and discord are consistent and explain the usefulness of states for which fidelity exceeds the classical bound but the Bell inequality is not violated. The applications of weak measurements to restore correlations and efficiency also return similar results for the modified operator and discord in both cases.

2.6 SUMMARY

In this chapter, a modification to the existing Bell inequality using cluster coefficients is presented. The existing inequality fails to identify nonlocal correlations in certain classes of mixed states for a given range of state parameters. However, the violation of modified inequality indicates the existence of quantum correlations between qubits for pure and mixed entangled systems for the whole range of state parameters. Therefore, the modified operator succeeds in capturing nonclassicality even in those mixed entangled and separable states which do not violate the original Bell inequality but are identified by discord as useful resources exhibiting nonlocal correlations. The analytical relation between the optimal expectation value of modified Bell operator and concurrence of the input state is also demonstrated. The analytical results presented in this chapter are in complete agreement with numerical results. In addition, nonlocal correlations of two-qubit pure and mixed states are also analyzed in the noisy environment. Our results show that the increase in noise reduces the nonlocality of quantum systems under consideration. Although there exist several mechanisms to reduce effects of the noisy environments, in this chapter, we focused on the scheme of weak measurement and its reversal operations. For weak measurement parameter close to unity, we found that nonlocal correlations in the underlying state can be restored to its original value, similar to an ideal scenario. Table 2.1 shows the summary of analytical results for the

Table 2.1 : Summary of analytically calculated expressions of the Bell cumulant operator for various two qubit states

S.No.	state	ideal condition	with AD	with ADWM
1)	generalized-*Bell* state	$2\sqrt{2}C(\rho_{Bell})$	$2\sqrt{2}(1-d)C(\rho_{Bell})$	$\frac{2\sqrt{2}C(\rho_{Bell})}{1+\eta^2+(1-\eta^2\sqrt{1-C(\rho_{Bell})})}$
2)	*Werner* state	$2\sqrt{2}\left(\frac{2C(\rho_w)+1}{3}\right)$	$2\sqrt{2}(1-d)\left(\frac{2C(\rho_w)+1}{3}\right)$	$\frac{8\sqrt{2}\left(\frac{2C(\rho_w)+1}{3}\right)}{4\eta+\left(1+\frac{2C(\rho_w)+1}{3}\right)d^2\bar{w}^2}$
3)	*Horodecki* state	$2\sqrt{2}C(\rho_H)$	$2\sqrt{2}(1-d)C(\rho_H)$	$\frac{2\sqrt{2}C(\rho_H)}{1+C(\rho_H)d\bar{w}}$
4)	*MEMS* state (for $\gamma<\frac{2}{3}$)	$2\sqrt{2}C(\rho_{MEMS})$	$2\sqrt{2}(1-d)C(\rho_{MEMS})$	$\frac{6\sqrt{2}C(\rho_{MEMS})}{3+3d\bar{w}+d^2\bar{w}^2}$
5)	*MEMS* state (for $\gamma\geq\frac{2}{3}$)	$2\sqrt{2}C(\rho_{MEMS})$	$2\sqrt{2}(1-d)C(\rho_{MEMS})$	$\frac{4\sqrt{2}C(\rho_{MEMS})}{2+2d\bar{w}+d^2\bar{w}^2C(\rho_{MEMS})}$

here $\eta = 1+d(1-w)$.

Bell cumulant operator for different two-qubit states in ideal scenario, in the presence of amplitude damping noise and with the weak measurement operators. Furthermore, we also analyzed an operational interpretation of the proposed operator for quantum information processing by establishing a relation between the maximum expectation value of the proposed operator and teleportation fidelity.

The identification of nonlocal correlations in pure and mixed two-qubit entangled states further motivates us to address nonlocality in multiqubit systems. The cases of three and four qubit entangled states are much more complicated due to the increased number of entangled classes. The increased complexity certainly makes the analysis highly involved and interesting. For this purpose, we analyze nonlocality and efficiencies of multiqubit systems in next few chapters of this Thesis.

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