CHAPTER 3
Bent Waveguide eigenvalue problem

"*To me, the transfer from the self-adjoint paradise to non-self-adjoint torments was like opening Pandora's box...*"

David Krejcirik

The problem of optical wave propagation in dielectric straight waveguides has been systematically studied as a self-adjoint eigenvalue problem with real eigenvalues. In terms of the underlying physics, such real eigenvalues meant no losses during the wave propagation. However, when the waveguides were bent, experiments showed that the wave propagation became lossy [Marcuse, 1971].

In this chapter, optical wave propagation in dielectric bent waveguides is mathematically analyzed. It is shown that the corresponding eigenvalue problem is a non-self-adjoint eigenvalue problem and has non-real-valued eigenvalues. The imaginary part of the eigenvalues is a measure of loss. For large-bent radii, the eigenvalue problem for bent waveguides behaves as an eigenvalue problem for straight waveguides, and the non-real-valued eigenvalues approach the real-valued eigenvalues of the straight waveguide problem. By expressing the bent waveguide eigenvalue operator as a sum of the self-adjoint operator and the non- self-adjoint operator, asymptotic behavior of guided modes, and their lossy nature are investigated.

Brief orientation about the content of this chapter: In Sec. 3.1, the eigenvalue problem for the bent optical waveguide is defined on the space $\mathcal{L}^2(0,\infty)$. By posing the eigenvalue problem in the operator theoretic setting with the radial inner product, in Sec. 3.2, we prove mathematically that this eigenvalue problem (EVP) is a non-self-adjoint EVP. After that, it is established that eigenvalues corresponding to the eigenvalue problem are non-real by showing that the imaginary part of the eigenvalue is non-zero. In Sec. 3.3, with the help of a suitable transformation, we obtain an equivalent eigenvalue problem for the bent waveguide on $(-\infty,\infty)$ and use it to explain the behavior of the eigenvalue problem for large bent radius R. It shows the bent waveguides behave like straight waveguides when the parameter $R \to \infty$. In Sec. 3.4, we express the non-self-adjoint operator corresponding to the bent waveguides as the sum of the self-adjoint operator and a non-self-adjoint operator. Then, using this setup, we will discuss other properties of the bent waveguide eigenvalue problem for a large bending radius.

3.1 MATHEMATICAL MODEL OF BENT WAVEGUIDE

This section describes an eigenvalue problem corresponding to bent waveguides and the space setting on which it is defined. As discussed in Sec. 1.4.1, on inserting the functional form for the electric field and the magnetic field Eq. (1.24) in Maxwell's equations, one gets two sets of equations: One for the transverse electric (TE) polarization and the other set for the transverse magnetic (TM) polarization. From these sets of equations, we obtain a governing equations Eq. (1.28) and Eq. (1.30) for the principal component ψ ($\psi = \tilde{E}_y$ for the TE mode, $\psi = \tilde{H}_y$ for the TM

mode) for 1-D bent waveguide [Hiremath *et al.*, 2005] as

$$r^2\psi_{rr}(r) + r\psi_r(r) + (n^2(r)k^2r^2 - \gamma^2R^2)\psi(r) = 0, \quad 0 < r < \infty. \tag{3.1}$$

Here, we use the notation that the subscript r in ψ_r represents the derivative of ψ with respect to r. Associated boundary conditions for the guided modes are given by

$$\begin{aligned} &|\psi(r)| \to 0 \quad \text{as} \quad r \to 0, \\ &|\psi(r)| \to 0 \quad \text{as} \quad r \to \infty \text{ and } \psi \text{ is an outgoing wave.} \end{aligned}$$

Here $\psi(r)$ is the mode profile, $k = 2\pi/\lambda$ is the wavenumber corresponding to free space wavelength λ and $n(r)$ is the piece-wise constant refractive index profile given by

$$n(r) = \begin{cases} n_s, & 0 < r < R - d, \\ n_f, & R - d \le r \le R, \\ n_c, & R < r < \infty, \end{cases}$$

where d is the width of the guiding core and n_s, n_f, and n_c are the constant refractive index of the substrate, the film and the cladding medium as shown in Fig. 1.6. For the uniqueness of the solution, one needs to add the outgoing wave condition to the exterior boundary condition at ∞. Note that ψ is continuous at the interfaces. Non-guided modes also exist, for which boundary conditions are different.

3.1.1 Eigenvalue Problem Formulation

Now, we will set up the eigenvalue problem for the guided modes of the bent waveguide. Express the Eq. (3.1) as

$$\left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi.$$

Define an operator L_R as

$$L_R\psi := \left(\frac{1}{k^2}\frac{r}{R}\psi_r\right)_r + n^2(r)\frac{r}{R}\psi,$$

then we have

$$L_R\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi, \qquad 0 < r < \infty. \tag{3.2}$$

This is the (weighted) eigenvalue problem for bent optical waveguides with the piece-wise constant refractive index profile. The eigenvalues are given by $\dfrac{\gamma^2}{k^2}$ and the weight function is $\dfrac{R}{r}$. This EVP involves discontinuous coefficient $n(r)$; it is defined on the semi-infinite interval $(0, \infty)$, and it contains a parameter R (=bent radius).

We are looking for the non-trivial bounded solutions, (i.e. guided modes) of eigenvalue problem (3.2) in the inner product space

$$\mathbb{V}_{r,R} = \{\psi|\ \psi, \frac{r}{R}\psi_r \in \mathcal{L}^2(0,\infty)\}, \tag{3.3}$$

where $\mathcal{L}^2(0,\infty)$ is the space of square-integrable functions with the radial inner product defined as

$$\langle\psi,\phi\rangle_r = \langle\psi(r),\phi(r)\rangle_r = \int_0^\infty \psi(r)\phi^*(r)\frac{R}{r}dr. \tag{3.4}$$

And the corresponding norm is defined as

$$\|\psi\|_r{}^2 = \langle \psi(r), \psi(r) \rangle_r = \int_0^\infty \psi(r)\psi^*(r)\frac{R}{r}dr. \tag{3.5}$$

Here asterisk * represents the complex conjugate. Later in Sec. 3.3, we need to define this inner product and the norm in radial coordinate r in the transformed coordinate x. We emphasize the choice of the working coordinate by using the appropriate subscripts, e.g., $\langle \cdot, \cdot \rangle_r$, $\langle \cdot, \cdot \rangle_x$, etc. Without loss of the generality, unless it is explicitly stated, all the mathematical analysis in this work is for the functions in $\mathbb{V}_{r,R}$. Rather than enforcing a strict equality between functional spaces, we make the natural assumption that the domain of the operator L_R is continuously embedded in the space $V_{r,R}$. This ensures that the space all functions acted upon by L_R (and later, by its adjoint operator) possess the necessary regularity and boundary behavior prescribed by $V_{r,R}$, thereby allowing the operator L_R to be well-defined.

The first question about an eigenvalue problem is whether or not the operator involved is self-adjoint. To verify the adjointness behavior of the operator L_R, we will first find the adjoint of the operator in the next section and do further analysis to get new insight into the model.

3.2 ADJOINT OF OPERATOR

The first result of this section shows that operator L_R is non-self-adjoint because it is not symmetric. Later in Sec. (3.3), we will see how the non-self-adjointness of the operator L_R transforms into a self-adjointness, denoted as L_∞, as the bent radius tends to infinity.

Theorem 3.2.1 (Non-self-adjointness of L_R)**.** *The operator L_R defined in Eq.* (3.2) *corresponding to the bent waveguide eigenvalue problem is non-self-adjoint on the inner product space defined in* (3.3) *with the inner product* (3.4)*.*

Proof. According to the definition of the non-self-adjoint operator, one can prove this theorem by showing that at least one condition mentioned in the definition does not hold. Therefore, one needs to find the adjoint of the operator L_R.

Let $\psi(r) \in \mathbb{V}_{r,R}$ and $\phi(r) \in \mathbb{V}_{r,R}$ be modal solutions of a given bent waveguide with corresponding propagation constants γ_1 and γ_2 respectively. Consider

$$\langle L_R\psi, \phi \rangle_r \;=\; \int_0^\infty L_R\psi\phi^*\frac{R}{r}dr = \int_0^\infty \Big(\frac{r}{k^2}\psi_r\Big)_r \phi^*\frac{1}{r}dr + \int_0^\infty n^2(r)r\psi\phi^*\frac{1}{r}dr. \tag{3.6}$$

The first integral term on the RHS can be simplified as

$$\begin{aligned}\int_0^\infty \Big(\frac{r}{k^2}\psi_r\Big)_r \phi^*\frac{1}{r}dr \;=\; &\Big[\Big(\frac{r}{k^2}\psi_r\Big)\phi^*\frac{1}{r} - \frac{r}{k^2}\psi\Big(\phi^*\frac{1}{r}\Big)_r\Big]_0^\infty \\ &+ \int_0^\infty \psi\Big(\frac{r}{Rk^2}\phi^*_{rr} - \frac{1}{Rk^2}\phi^*_r + \phi^*\frac{1}{rRk^2}\Big)\frac{R}{r}dr.\end{aligned} \tag{3.7}$$

On the right-hand side of the Eq. (3.7), the first bracketed term is for the boundary contribution, and the second term (integral term) contributes to finding the adjoint of the operator L_R. Let

us first evaluate the boundary contribution. The boundary contribution terms $\left[\left(\frac{r}{k^2}\psi_r\right)\phi^*\frac{1}{r} - \frac{r}{k^2}\psi\left(\phi^*\frac{1}{r}\right)_r\right]_0^\infty$ in Eq. (3.7) can be simplified as $\left[\frac{1}{k^2}\left(\psi_r\phi^* - \psi\phi_r^* + \psi\phi^*\frac{1}{r}\right)\right]_0^\infty$.

At $r = 0$, the modal solutions $\psi(r)$ and $\phi(r)$ are represented by the Bessel functions of the first kind [Hiremath *et al.*, 2005]. The Bessel functions of the first kind $J_{\gamma R}(\cdot)$ are zero at $r = 0$ for $\gamma R \neq 0$. In the present context, γR are non-zero (for finite R, β and α are non-zero). This gives

$$\left[\psi_r\phi^* - \psi\phi_r^* + \psi\phi^*\frac{1}{r}\right]_{r=0} = 0.$$

For $r \to \infty$, the modal solutions $\psi(r)$ and $\phi(r)$ are represented by the Hankel functions of the second kind [Hiremath *et al.*, 2005]. The asymptotic expressions for the Hankel functions of the second kind for $r \to \infty$ are

$$\begin{aligned}\psi &\sim A_{c_1}\sqrt{\frac{2}{\pi n_c k r}}\exp -\imath\left(n_c k r - \gamma_1 R\frac{\pi}{2} - \frac{\pi}{4}\right),\\ \psi_r &\sim \left(-\frac{1}{2r} - \imath n_c k\right)\psi.\end{aligned} \tag{3.8}$$

Using these expressions, one gets

$$\begin{aligned}\psi_r\phi^* - \psi\phi_r^* &\sim -2\imath n_c k\psi\phi^*\\ \therefore \left[\psi_r\phi^* - \psi\phi_r^* + \psi\phi^*\frac{1}{r}\right]_{r\to\infty} &\sim \left(-2\imath n_c k + \frac{1}{r}\right)\psi\phi^*.\end{aligned}$$

Also

$$\psi\phi^* \sim A_{c_1}A_{c_2}^*\frac{2}{\pi n_c k r}e^{-\imath\left(-(\gamma_1-\gamma_2^*)R\frac{\pi}{2}\right)},$$

where A_{c_1} and $A_{c_2}^*$ are constants. Using the above two equations, one gets

$$\left[\psi_r\phi^* - \psi\phi_r^* + \psi\phi^*\frac{1}{r}\right]_{r\to\infty} \sim \left(-2\imath n_c k + \frac{1}{r}\right)A_{c_1}A_{c_2}^*\frac{2}{\pi n_c k r}e^{-\imath\left(-(\gamma_1-\gamma_2^*)R\frac{\pi}{2}\right)}.$$

As $r \to \infty$, the term on the RHS tends to 0.

$$\therefore \left[\psi_r\phi^* - \psi\phi_r^* + \psi\phi^*\frac{1}{r}\right]_{r\to\infty} \sim 0.$$

Thus, one gets the contribution of the boundary term in the Eq. (3.7) as

$$\left[\left(\frac{r}{k^2}\psi_r\right)\phi^*\frac{1}{r} - \frac{r}{k^2}\psi\left(\phi^*\frac{1}{r}\right)_r\right]_0^\infty = 0. \tag{3.9}$$

Therefore, the adjointness of the eigenvalue problem for the bent waveguide depends on the operator L_R itself, not on its boundary conditions. Using the results from Eq. (3.7), and Eq. (3.9) in Eq. (3.6),

one gets

$$\begin{aligned}\langle L_R\psi,\phi\rangle_r &= \int_0^\infty \psi\left[\frac{r}{Rk^2}\phi^*_{rr}-\frac{1}{Rk^2}\phi^*_r+\left(n^2(r)\frac{r}{R}+\frac{1}{rRk^2}\right)\phi^*\right]\frac{R}{r}dr,\\ &= \int_0^\infty \psi L^*_R\phi^*\frac{R}{r}dr=\langle\psi,L^*_R\phi\rangle_r,\end{aligned}$$

where L^*_R is the formal adjoint of the operator L_R and is defined as

$$L^*_R\phi := \frac{r}{Rk^2}\phi_{rr}-\frac{1}{Rk^2}\phi_r+\left(n^2(r)\frac{r}{R}+\frac{1}{rRk^2}\right)\phi.$$

Rearrange L^*_R as

$$L^*_R\phi := \left(\frac{1}{k^2}\frac{r}{R}\phi_r\right)_r+n^2(r)\frac{r}{R}\phi+\left(\frac{1}{rRk^2}\phi-\frac{2}{Rk^2}\phi_r\right).$$

For the eigenvalue problem under discussion, from Eq. (3.2) we have

$$L_R=\frac{d}{dr}\left(\frac{1}{k^2}\frac{r}{R}\frac{d}{dr}\right)+n^2(r)\frac{r}{R}, \tag{3.10}$$

and its adjoint operator is

$$L^*_R = \frac{d}{dr}\left(\frac{1}{k^2}\frac{r}{R}\frac{d}{dr}\right)+n^2(r)\frac{r}{R}+\left(\frac{1}{rRk^2}-\frac{2}{Rk^2}\frac{d}{dr}\right). \tag{3.11}$$

As $L^*_R\neq L_R$, L_R is not self-adjoint. So, the operator L_R corresponding to the eigenvalue problem for the bent waveguide is a non-self-adjoint operator.

□

From the boundary terms in Eq. (3.7), it follows that the adjoint operator L^*_R satisfies the same boundary conditions as the operator L_R. Note that, according to the semi-analytic study [Hiremath *et al.*, 2005], in the limiting case (i.e., $R\to\infty$), the operator L_R, and its adjoint operator L^*_R coincide, indicates that there domains should not differ significantly. Here, we assume that the domain of operator L^*_R is also contained in the space $\mathbb{V}_{r,R}$ defined in (3.3).

After proving the non-self-adjointness of the operator L_R, a next natural question to ask is about the nature of its eigenvalues. In the case of the non-self-adjoint operator defined on the infinite domain, their eigenvalues can be real or non-real, such as in the case of the convection-diffusion operator (2.1) [Davies, 2002]. Motivated by the phenomenological arguments, an earlier reported semi-analytic work Ref. [Hiremath *et al.*, 2005] has assumed that these eigenvalues are non-real and investigated the modal properties of the bent optical waveguides. In the following theorem, mathematical analysis establishes this fact about the eigenvalues.

Theorem 3.2.2 (Non-real eigenvalues)**.** *Eigenvalues of the bent waveguide eigenvalue problem for the guided modes defined in Eq.* (3.2) *with the given boundary conditions are non-real.*

Proof. Let ψ be an eigenfunction with the eigenvalue $\frac{\gamma^2}{k^2}$. Using the Eq. (3.2), consider

$$\langle L_R\psi,\psi\rangle_r=\left\langle\frac{\gamma^2}{k^2}\frac{R}{r}\psi,\psi\right\rangle_r.$$

Using the adjoint of L_R on the L.H.S. and conjugate the linearity of the inner product on the RHS, one gets

$$\langle \psi, L_R^* \psi \rangle_r = \left\langle \psi, \frac{\gamma^{*2}}{k^2} \frac{R}{r} \psi \right\rangle_r ,$$

where ψ is an eigenfunction of Eq. (3.2). From this, one gets

$$\left\langle \psi, L_R^* \psi - \frac{\gamma^{*2}}{k^2} \frac{R}{r} \psi \right\rangle_r = 0.$$

Thus

$$L_R^* \psi = \frac{\gamma^{*2}}{k^2} \frac{R}{r} \psi, \ (\text{as } \psi \neq 0), \tag{3.12}$$

for any eigenfunction ψ of Eq. (3.2) (with corresponding eigenvalue γ^2/k^2). Since $L_R \neq L_R^*$, for eigenfunction ψ we have

$$\begin{aligned} \langle L_R \psi, \psi \rangle_r - \langle L_R^* \psi, \psi \rangle_r &\neq 0, \\ \left\langle \frac{\gamma^2}{k^2} \frac{R}{r} \psi, \psi \right\rangle_r - \left\langle \frac{\gamma^{*2}}{k^2} \frac{R}{r} \psi, \psi \right\rangle_r &\neq 0, \\ \left(\frac{\gamma^2}{k^2}_r - \frac{\gamma^{*2}}{k^2} \right) \left\langle \frac{R}{r} \psi, \psi \right\rangle_r &\neq 0. \end{aligned}$$

From Eq. (1.25), one has

$$-\frac{4\imath\alpha\beta}{k^2} \left\langle \frac{R}{r} \psi, \psi \right\rangle_r \neq 0.$$

This is true if and only if

$$\alpha \neq 0, \quad \beta \neq 0, \quad \left\langle \frac{R}{r} \psi, \psi \right\rangle_r = \int_0^\infty \frac{R}{r} \psi \psi^* \frac{R}{r} dr \neq 0.$$

Thus α can not be zero, which means the eigenvalues $\gamma^2/k^2 = (\beta - \imath\alpha)^2/k^2$ of the operator L_R are non-real. Thus, we proved mathematically that eigenvalues of the bent waveguide EVP are non-real. □

To get further insight into the mathematical properties of the bent waveguide model, in the next section we study the behavior of the governing Eq. (3.1) and the properties of its solutions for the large radii R.

3.3 BEHAVIOR FOR LARGE R

In the previous section, the non-real characteristics of the eigenvalues of the eigenvalue problem (3.2) were proved to correspond to the fixed bent radius R. What happened if the bent radius is large, i.e., when R tends to infinity, will behavior of the eigenvalue problem and its eigenvalues still be the same? The answer is no. This and the next section verified this answer mathematically. This study will be facilitated by the use of a suitable transformation.

Lemma 3.3.1 (Bent waveguide EVP in $x-$coordinate)**.** *Under the transformation defined by [Heiblum and Harris, 1975]*

$$r = R\exp\left(\frac{x}{R}\right), \tag{3.13}$$

the bent waveguide EVP (3.2) *becomes*

$$\frac{1}{k^2}\left(\psi(x)\right)_{xx} + \left(N_R^2(x) - \frac{\gamma^2}{k^2}\right)\psi(x) = 0, \quad -\infty < x < \infty, \tag{3.14}$$

and the boundary conditions get transformed into

$$\begin{aligned} &|\psi(x)| \to 0 \quad as \quad x \to -\infty, \\ &|\psi(x)| \to 0 \quad as \quad x \to \infty \text{ and } \psi \text{ is an outgoing wave.} \end{aligned}$$

The term $N_R(x)$ in Eq. (3.14) *denotes the transformed refractive index profile given by*

$$N_R(x) = n\left(R\exp\left(\frac{x}{R}\right)\right)\exp\left(\frac{x}{R}\right), \tag{3.15}$$

where

$$n\left(R\exp\left(\frac{x}{R}\right)\right) = \begin{cases} n_s, & -\infty < x < R\ln\left(\frac{R-d}{R}\right), \\ n_f, & R\ln\left(\frac{R-d}{R}\right) \le x \le 0, \\ n_c, & 0 < x < \infty. \end{cases}$$

Thus the transformed refractive index is given by Eq. (3.15) *becomes*

$$N_R(x) = \begin{cases} n_s\exp\left(\frac{x}{R}\right), & -\infty < x < R\ln\left(\frac{R-d}{R}\right), \\ n_f\exp\left(\frac{x}{R}\right), & R\ln\left(\frac{R-d}{R}\right) \le x \le 0, \\ n_c\exp\left(\frac{x}{R}\right), & 0 < x < \infty. \end{cases} \tag{3.16}$$

Proof. Using $r = R\exp\left(\frac{x}{R}\right), dr = \exp\left(\frac{x}{R}\right)dx$. We get $\frac{r}{R}\psi_r = \psi_x$. Then the Eq. (3.2) transformed into

$$\frac{1}{k^2}\psi_{xx} + n^2\left(R\exp\left(\frac{x}{R}\right)\right)\exp\left(\frac{2x}{R}\right)\psi = \frac{\gamma^2}{k^2}\psi, \quad -\infty < x < \infty. \tag{3.17}$$

The refractive index profile $n(r)$ becomes

$$n\left(R\exp\left(\frac{x}{R}\right)\right) = \begin{cases} n_s, & 0 < R\exp\left(\frac{x}{R}\right) < R-d, \\ n_f, & R-d \le R\exp\left(\frac{x}{R}\right) \le R, \\ n_c, & R < R\exp\left(\frac{x}{R}\right) < \infty, \end{cases} = \begin{cases} n_s, & -\infty < x < R\ln\left(\frac{R-d}{R}\right), \\ n_f, & R\ln\left(\frac{R-d}{R}\right) \le x \le 0, \\ n_c, & 0 < x < \infty. \end{cases}$$

From the above, we will get the transformed index profile in Eq. (3.16) and the EVP Eq. (3.14). In the coordinate x, the inner product space $\mathbb{V}_{r,R}$ becomes

$$\mathbb{V}_{x,R} = \left\{\psi | \psi, \psi_x \in \mathcal{L}^2(-\infty, \infty)\right\}, \tag{3.18}$$

with the inner product and the norm defined as

$$\langle\psi, \phi\rangle_x = \int_{-\infty}^{\infty}\psi\phi^* dx, \qquad \|\psi\|_x^2 = \int_{-\infty}^{\infty}|\psi|^2 dx. \tag{3.19}$$

The transformed refractive index profile is geometrically depicted in Fig. 3.1. Similarly, using the transformation, we will get the boundary conditions. □

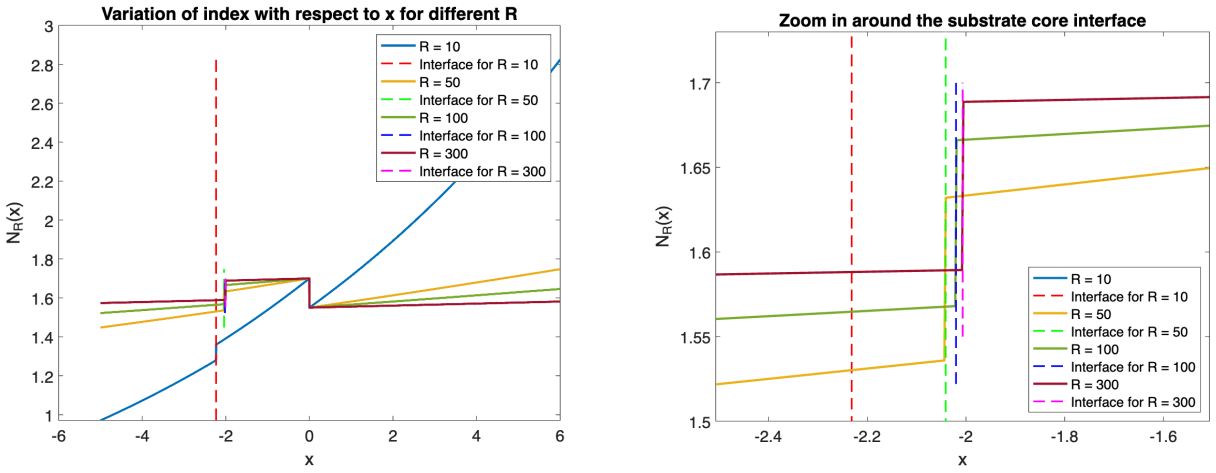


Figure 3.1. : Refractive index profile $N_R(x)$ in x co-ordinate for a fixed bent radius $R, -R\ln\left(1-\frac{d}{R}\right)$ is width of the guiding core, and for $(n_s, n_f, n_c) = (1.55, 1.7, 1.6)$, and $d = 2\,\mu$m width of core of bent waveguide. Dotted vertical lines represent the interface between the core and substrate region depending on the bent radius R. The interface between core and cladding is fixed at $x = 0$ in this setup.

One can view this transformation as 'a kind of change of co-ordinates' from polar r to cartesian x. With Eq. (3.14) and Eq. (3.16), one can study the bent waveguide model in an equivalent form, i.e., an equivalent model of the bent waveguide model [Heiblum and Harris, 1975; Kumar and Hiremath, 2022].

Theorem 3.3.2 (EVP for the large bent radii)**.** *As $R \to \infty$, the non-self-adjoint EVP of bent waveguides changes into self-adjoint EVP.*

Proof. This statement has a nice physical interpretation, which we shall comment on later. To study the behavior of the governing Eq. (3.14) for large R, first, we will study the form of the transformed refractive index profile in Eq. (3.16). Taking limit as $R \to \infty$ of the Eq. (3.16), one gets

$$\lim_{R\to\infty} N_R(x) = N_\infty(x) = \begin{cases} n_s, & -\infty < x < -d, \\ n_f, & -d \le x \le 0, \\ n_c, & 0 < x < \infty. \end{cases}$$

Note that this is nothing but the refractive index profile of the $1-$D straight waveguide with a core width d as shown in Fig. 3.2. We call this limiting case of the bent waveguide for $R \to \infty$ as the *corresponding* straight waveguide [Hiremath *et al.*, 2005].

For $R \to \infty$, Eq. (3.14) becomes

$$\frac{1}{k^2}(\psi_\infty)_{xx} + \left(N_\infty^2(x) - \frac{\gamma_\infty^2}{k^2}\right)\psi_\infty = 0, \quad -\infty < x < \infty,$$

where

$$\psi_\infty(x) = \lim_{R\to\infty} \psi(x) \tag{3.20}$$

is a mode of the corresponding straight waveguide with refractive index profile $N_\infty(x)$. γ_∞ is the propagation constant of the mode of the corresponding straight waveguide. From the theory of straight waveguides [Hunsperger, 1995], we know that γ_∞ is a real number. Denote

$$\gamma_\infty = \beta_s, \text{ where } \beta_s \in \mathbb{R} \setminus \{0\}. \tag{3.21}$$

Here, β_s represents the phase constant of a mode of the corresponding straight waveguide.

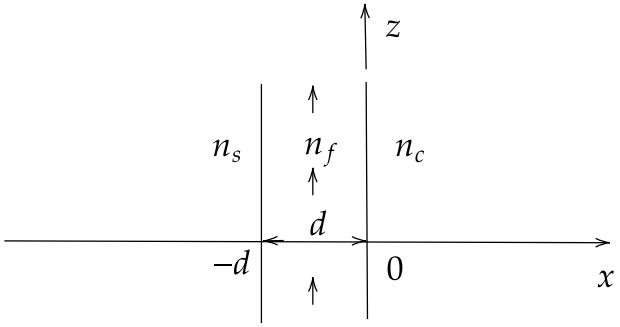


Figure 3.2. : An equivalent straight waveguide setting corresponding to $R \to \infty$.

For the limiting case $R \to \infty$, the eigenvalue problem becomes

$$L_\infty \psi_\infty = \frac{\gamma_\infty^2}{k^2}\psi_\infty = \frac{\beta_s^2}{k^2}\psi_\infty, \quad -\infty < x < \infty,$$

where

$$L_\infty \psi_\infty := \frac{1}{k^2}(\psi_\infty)_{xx} + N_\infty^2(x)\psi_\infty. \tag{3.22}$$

It has been shown in Ref. [Joly and Poirier, 1995] that this is a self-adjoint eigenvalue problem with real eigenvalues $\frac{\beta_s^2}{k^2}$ with $\beta_s \in \mathbb{R} \setminus \{0\}$. Thus, as $R \to \infty$, the non-self-adjoint EVP of bent waveguides changes into self-adjoint EVP. □

The physical interpretation of this result is that for large bent radii R, the modes of bent waveguides behave like the modes of straight waveguides. This observation is consistent with the results of the semi-analytic study [Hiremath *et al.*, 2005]. The transformation in transverse electric and transverse magnetic modes w.r.t. the bent radius is shown in Fig. 3.3 and Fig. 3.4. This also validates that bent modes transform into straight modes w.r.t. bent radius R, and the curves shift in the outward direction.

3.4 SPLITTING OF THE OPERATOR

Since we have access to the operator L_R and its adjoint operator L_R^*, we can split L_R as

$$L_R = \frac{L_R + L_R^*}{2} + \frac{L_R - L_R^*}{2}.$$

This splitting of L_R helps us understand the dependence of the bent waveguide's eigenvalue problem (i.e., the guided modes) on the bent radius. Let

$$P = \frac{L_R + L_R^*}{2} \quad \text{and} \quad Q = \frac{L_R - L_R^*}{2}.$$

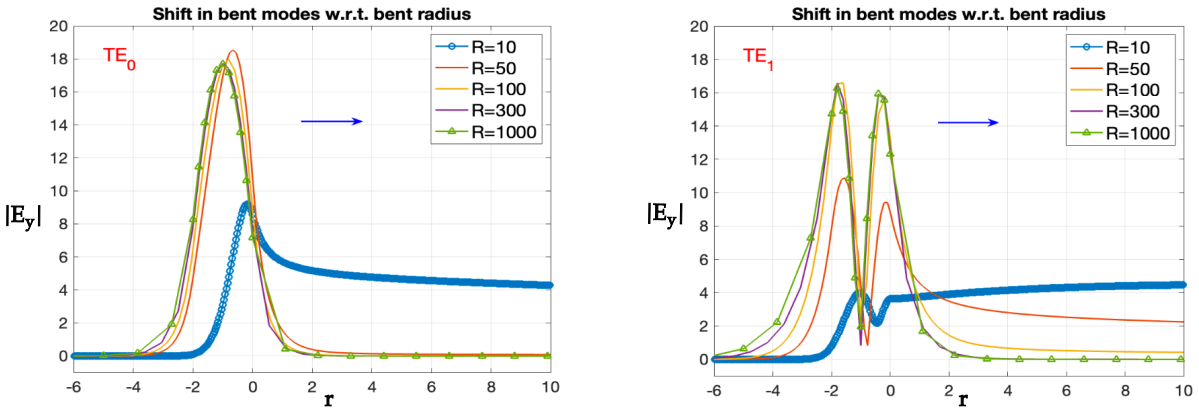


Figure 3.3. : These figures show how TE_0 and TE_1 modes transform w.r.t. distinct bent radius R. Here $(n_s, n_f, n_c) = (1.55, 1.7, 1.6)$, vacuum wavelength is $1.55\,\mu$m, and $d = 2\ \mu$m. In the case of TE_0 mode, for $R = 10\ \mu$m, $\gamma = 6.4410 - \imath 6.4959 \times 10^{-2}$, $R = 50\ \mu$m, $\gamma = 6.6773 - \imath 3.9200 \times 10^{-6}$, $R = 100\,\mu$m, $\gamma = 6.7336 - \imath 4.4601 \times 10^{-12}$, $R = 300\,\mu$m, $\gamma = 6.5258 - \imath 8.6828 \times 10^{-14}$, and for $R = 1000\,\mu$m, $\gamma = 6.7929 - \imath 0$. In the case of TE_1 mode with the corresponding radii, we have $\gamma = 5.8295 - \imath 1.3898 \times 10^{-1}$, $\gamma = 6.4121 - \imath 6.1262 \times 10^{-3}$, $\gamma = 6.4772 - \imath 7.1368 \times 10^{-5}$, $\gamma = 6.5258 - \imath 1.1345 \times 10^{-13}$, and for $\gamma = 6.5441 - \imath 0$.

Therefore

$$L_R = P + Q.$$

Note that $P^* = P$ and $Q^* = -Q$. Therefore

$$L_R^* = (P+Q)^* = P^* + Q^* = P - Q.$$

Using Eq. (3.10) and Eq. (3.11), one gets

$$P = \frac{r}{Rk^2}\frac{d^2}{dr^2} + n^2(r)\frac{r}{R} + \frac{1}{2rRk^2},$$

and

$$Q = \frac{1}{Rk^2}\frac{d}{dr} - \frac{1}{2rRk^2}.$$

From Eq. (3.2) and Eq. (3.12), we have

$$(P+Q)\psi = \frac{\gamma^2}{k^2}\frac{R}{r}\psi = \frac{(\beta^2 - \alpha^2 - 2\imath\alpha\beta)}{k^2}\frac{R}{r}\psi, \tag{3.23}$$

$$(P-Q)\psi = \frac{\gamma^{*2}}{k^2}\frac{R}{r}\psi = \frac{(\beta^2 - \alpha^2 + 2\imath\alpha\beta)}{k^2}\frac{R}{r}\psi. \tag{3.24}$$

Adding Eq. (3.23) and Eq. (3.24), one gets

$$P\psi(r) = \frac{(\beta^2 - \alpha^2)}{k^2}\frac{R}{r}\psi(r),$$

i.e.,

$$\left(\frac{r}{Rk^2}\frac{d^2}{dr^2} + n^2\frac{r}{R} + \frac{1}{2rRk^2}\right)\psi(r) = \frac{(\beta^2 - \alpha^2)}{k^2}\frac{R}{r}\psi(r). \tag{3.25}$$

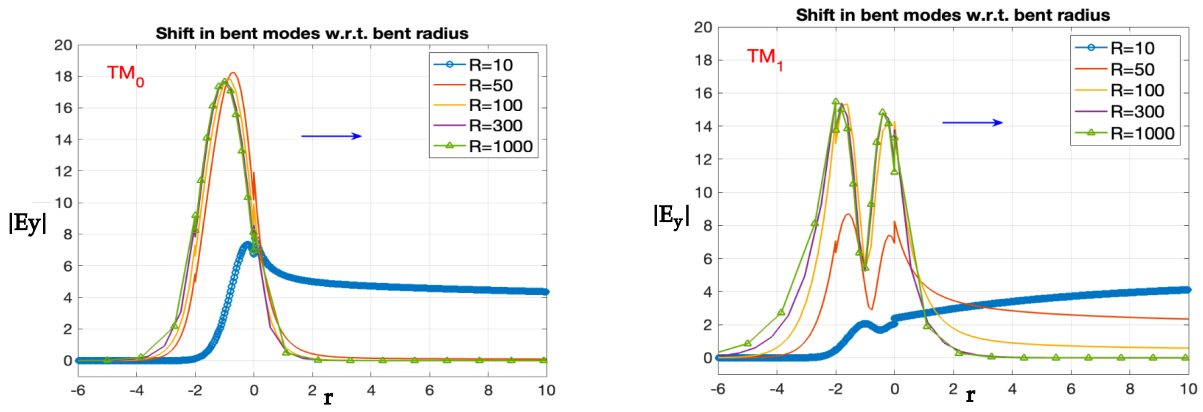


Figure 3.4. : These figures show how TM_0 and TM_1 modes transform w.r.t. distinct bent radius R. Here $(n_s, n_f, n_c) = (1.55, 1.7, 1.6)$, vacuum wavelength is $1.55\,\mu$m, and $d = 2\,\mu$m. In the case of TM_0 mode, for $R = 10\,\mu$m, $\gamma = 6.4271 - \imath 7.5809 \times 10^{-2}$, $R = 50\,\mu$m, $\gamma = 6.6686 - \imath 5.1351 \times 10^{-6}$, $R = 100\,\mu$m, $\gamma = 6.7261 - \imath 7.4738 \times 10^{-12}$, $R = 300\,\mu$m, $\gamma = 6.7698 - \imath 3.2181 \times 10^{-15}$, and for $R = 1000\,\mu$m, $\gamma = 6.7861 - \imath 0$. In the case of TM_1 mode, for the corresponding radii, we have $\gamma = 5.8337 - \imath 1.8683 \times 10^{-1}$, $\gamma = 6.4010 - \imath 8.2563 \times 10^{-3}$, $\gamma = 6.4638 - \imath 1.4063 \times 10^{-4}$, $\gamma = 6.5126 - \imath 8.0146 \times 10^{-13}$, and for $\gamma = 6.5316 - \imath 8.0358 \times 10^{-15}$.

As $P = P^*$, this is a self-adjoint eigenvalue problem with pure real eigenvalues.

Subtracting Eq. (3.23) from Eq. (3.24), one gets

$$Q\psi(r) = -\frac{2\imath\alpha\beta}{k^2}\frac{R}{r}\psi(r),$$

i.e.,

$$\left(\frac{1}{Rk^2}\frac{d}{dr} - \frac{1}{2rRk^2}\right)\psi(r) = -2\imath\frac{\alpha\beta}{k^2}\frac{R}{r}\psi(r). \tag{3.26}$$

As $Q^* = -Q$, this is a skew-Hermitian eigenvalue problem with purely imaginary and possibly zero eigenvalues.

Thus, the non-self-adjoint operator L_R can be expressed as a sum of the self-adjoint differential operator of the second order and the non-self-adjoint differential operator of the first order. Now we will study how these self-adjoint and non-self-adjoint operators behave as $R \to \infty$.

Theorem 3.4.1 (Asymptotic behavior of eigenvalues of operator Q)**.** *As $R \to \infty$, zero is the only eigenvalue for the EVP* (3.26) *of the skew-Hermitian operator Q.*

Proof. For the operator Q, with the transformation defined in Eq. (3.13), the EVP (3.26) becomes

$$Q\psi(x) = -\frac{2\imath\alpha\beta}{k^2}e^{-\frac{x}{R}}\psi(x),$$

where $Q\psi(x) := \frac{1}{Rk^2}e^{-\frac{x}{R}}\frac{d\psi}{dx} - \frac{1}{2R^2k^2}e^{-\frac{x}{R}}\psi$. Taking limit $R \to \infty$, we get the EVP:

$$Q_\infty\psi_\infty = -\frac{2\imath\alpha_\infty\beta_\infty}{k^2}\psi_\infty, \qquad -\infty < x < \infty, \tag{3.27}$$

where

$$\begin{aligned} Q_\infty\psi_\infty &= \lim_{R\to\infty}\left[\frac{1}{Rk^2}e^{-\frac{x}{R}}\frac{d\psi}{dx} - \frac{1}{2R^2k^2}e^{-\frac{x}{R}}\psi\right], \\ \alpha_\infty\beta_\infty\psi_\infty &= \lim_{R\to\infty}\alpha\beta\psi(x). \end{aligned}$$

and ψ_∞ is as defined in Eq. (3.20), $\beta_\infty = \lim_{R\to\infty}\beta$, and $\alpha_\infty = \lim_{R\to\infty}\alpha$.

Note that

$$\begin{aligned} \|Q\psi\|_x &= \left\|\frac{1}{Rk^2}e^{-\frac{x}{R}}\frac{d\psi}{dx} - \frac{1}{2R^2k^2}e^{-\frac{x}{R}}\psi\right\|_x, \\ \lim_{R\to\infty}\|Q\psi\|_x &= \lim_{R\to\infty}\left\|\frac{1}{Rk^2}e^{-\frac{x}{R}}\frac{d\psi}{dx} - \frac{1}{2R^2k^2}e^{-\frac{x}{R}}\psi\right\|_x, \\ &\leq \frac{1}{k^2}\left\|\left(\frac{d\psi}{dx}\right)_\infty\right\|_x \lim_{R\to\infty}\frac{1}{R} + \frac{1}{2k^2}\|\psi_\infty\|_x \lim_{R\to\infty}\frac{1}{R^2}. \end{aligned}$$

Here $\|\cdot\|_x$ is the norm defined by Eq. (3.19) in the transformed co-ordinate x. Also, $\|\psi_\infty\|_x < \infty$ and $\left\|\left(\frac{d\psi}{dx}\right)_\infty\right\|_x < \infty$ [Joly and Poirier, 1995]. Therefore, as $R\to\infty$, $\lim_{R\to\infty}\|Q\psi\|_x$ will be zero, i.e., the LHS of Eq. (3.27) tends to zero. This implies

$$\lim_{R\to\infty}\alpha\beta\psi(x) = 0, \qquad \forall\ x\in(-\infty,\infty).$$

As $\psi(x)$ is not identically equal to zero for all $x\in(-\infty,\infty)$,

$$\Rightarrow \lim_{R\to\infty}\alpha\beta = 0.$$

The semi-analytic study of the bent waveguides shows that $\alpha = \alpha(R)$ and $\beta = \beta(R)$ are continuous functions of R [Hiremath *et al.*, 2005].

$$\begin{aligned} \therefore \left(\lim_{R\to\infty}\alpha\right)\left(\lim_{R\to\infty}\beta\right) &= 0, \\ \alpha_\infty\beta_\infty &= 0. \end{aligned}$$

Thus zero is the only eigenvalue of the EVP (3.27). i.e. As $R\to\infty$, zero is the only eigenvalue for the EVP (3.26). □

Taking limit as $R\to\infty$ of Eq. (1.25), one gets

$$\gamma_\infty = \lim_{R\to\infty}\gamma = \lim_{R\to\infty}(\beta - \imath\alpha) = \beta_\infty - \imath\alpha_\infty$$

As a corollary of the Theorem (3.4.1), we get the following result for the asymptotic behavior of the real part and the imaginary part of the bent mode propagation constants.

Corollary 3.4.2 (Asymptotic behavior of propagation constant)**.** $\alpha_\infty = \lim_{R\to\infty}\alpha = 0$ *and* $\beta_\infty = \beta_s$.

Proof. From Eq. (3.21), $\gamma_\infty = \beta_s$, where $\beta_s\in\mathbb{R}\setminus\{0\}$.

$$\therefore\ \gamma_\infty = \beta_\infty - \imath\alpha_\infty = \beta_s.$$

Comparing the real and the imaginary parts, one gets

$$\alpha_\infty = \lim_{R\to\infty} \alpha = 0, \qquad \text{and} \qquad \beta_\infty = \beta_s.$$

□

This result has a nice physical interpretation. It shows that as the bending radius R tends to ∞, the attenuation constant of the bent modes tends to zero, and the phase constant β (= the real part of the propagation constant) tends to the (phase) propagation constant of the corresponding equivalent straight waveguide. This mathematical result of the asymptotic behavior of eigenvalues is consistent with the reported semi-analytic results [Hiremath *et al.*, 2005]. Next, we will show that such asymptotic behavior is also seen at the level of the operators involved in the corresponding eigenvalue problems.

Theorem 3.4.3 (Asymptotic behavior of L_R)**.** *When the bent radius is large, the asymptotic behavior of self-adjoint operator P and non-self-adjoint operator L_R is the same as that of the operator L_∞, i.e., both operators P and L_R approach to L_∞.*

Proof. For the operator P from Eq. (3.25), with the transformation (3.13), and using Eq. (3.15), one gets

$$P\psi(x) = \frac{\beta^2 - \alpha^2}{k^2} e^{-\frac{x}{R}} \psi(x),$$

where $P\psi(x) := \frac{1}{k^2} e^{-\frac{x}{R}} \psi_{xx} - \frac{1}{Rk^2} e^{-\frac{x}{R}} \psi_x + \left(N_R^2(x) e^{-\frac{x}{R}} + \frac{1}{2R^2k^2} e^{-\frac{x}{R}} \right) \psi(x)$. Taking limit as $R \to \infty$, one gets

$$\frac{1}{k^2} (\psi_\infty)_{xx} + \left(N_\infty^2(x) - \frac{\beta_\infty^2}{k^2} \right) (\psi_\infty) = 0.$$

Thus as $R \to \infty$, eigenvalue problem for the operator P is

$$P_\infty \psi_\infty = \frac{\beta_\infty^2}{k^2} \psi_\infty = \frac{\beta_s^2}{k^2} \psi_\infty, \qquad -\infty < x < \infty,$$

where

$$P_\infty \psi_\infty = \frac{1}{k^2} (\psi_\infty)_{xx} + N_\infty^2(x)(\psi_\infty),$$

which is the same as the straight waveguide operator obtained in Eq. (3.22).

This means that for $R \to \infty$, the self-adjoint operator P and the non-self-adjoint operator L_R behave the same as the operator L_∞. In other words, with $R \to \infty$ behavior of the bent waveguide is similar to that of the straight waveguide, which agrees with the already done studies [Hiremath *et al.*, 2005].

□

3.5 CONCLUDING REMARKS

This chapter investigated the EVP arising in modeling guided modes of one-dimensional optical bent waveguides. A study of the adjoint operator in the function theoretic setting showed

that this is a non-self-adjoint EVP. Using appropriate transformation, we rigorously proved that when the bent radius approaches infinity, the bent waveguide EVP behaves like a straight waveguide EVP. We further expressed the bent waveguide EVP as the sum of a self-adjoint EVP and a non-self-adjoint EVP. With these split problems, we mathematically proved that for the limiting case of bent radius $R \to \infty$, the non-real-valued eigenvalues of the bent waveguide problem tend to real-valued eigenvalues of the corresponding straight waveguide problem, i.e. the real part of the eigenvalues of the bent waveguide problem tend to eigenvalues of the corresponding straight waveguide problem, whereas the imaginary part of the eigenvalues of the bent waveguide problem tend to zero. In terms of physics, this is interpreted as the propagation constants for the bent waveguide modes approaching the propagation constants of the straight waveguide modes, and the bent mode attenuation constants approaching zero. To the best of our knowledge, these are the first results that mathematically formulate the outcomes reported in semi-analytic and numerical works on the nature of bent mode propagation constants.