Declaration

I hereby declare that the work presented in this thesis titled *Corona Product in Signed Graph Theory: Modelling, Structural Balance, and Control* submitted to Indian Institute of Technology Jodhpur in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of Dr. Sandeep Kumar Yadav. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to any other Institute or University in India or abroad for the award of any degree or diploma.



Amrik Singh
P14VSS002

Certificate

This is to certify that the thesis titled *Corona Product in Signed Graph Theory: Modelling, Structural Balance, and Control*, submitted by *Amrik Singh (P14VSS002)* to Indian Institute of Technology Jodhpur for the award of the degree of *Doctor of Philosophy*, is a bonafide record of the research work done by him under my supervision. To the best of my knowledge, the contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.



Dr. Sandeep Kuamr Yadav
Ph.D. Thesis Supervisor

Abstract

Signed networks help in studying complex relationships and interactions in various domains, offering insights into the dynamics of positive and negative influences within interconnected systems.

I extend the concept of the corona product of two graphs to signed graphs, building on the marked graph framework introduced by Beineke and Harary in 1978. This approach provides a visual method for understanding the process of network generation, aiding in the analytical determination of the resulting signed graph properties. I explore both the structural and spectral properties of the corona product of signed graphs. Additionally, I introduce the concept of signed corona graphs, which are formed by iteratively applying the corona product to a fixed small signed graph, known as the seed graph. These signed corona graphs serve as a model for generating large, evolving signed networks. This study includes an examination of the structural properties of corona graphs, such as the statistics of signed edges, various types of signed triangles, and degree distribution. I also investigate the algebraic conflict within signed corona graphs produced by specially structured seed graphs. By carefully selecting a seed graph, it is possible to generate corona graphs that mirror the properties of real-world signed networks.

I have expanded the corona product of signed graphs into a generalized corona product of signed graphs motivated by the generalized corona product of unsigned graphs. I study the structural balance and spectral properties of these graphs. By utilizing the notion of coronal of a graph, I determine the formulas of characteristic, Laplacian, and signless Laplacian polynomials of generalized corona product of signed graphs. I have also identified sufficient conditions for the generalized corona product of some special collections of signed graphs to be co-spectral.

Following this, I analyze the controllability of the corona product of signed graphs and derive sufficient conditions based on the spectra of corona product of signed graphs.

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List of Abbreviations

Abbreviation	*Full form*
ER	Erdős Rényi
LTI	Linear Time Invariant
WS	Watts-Strogatz Model