3
Quantum Correlations

Quantum correlation is a key feature of quantum mechanics. Quantum correlation is crucial in quantum information, which involves transferring information through quantum systems. Over the years, it has been studied in many particle physics systems. Recently, interest in quantum correlation within neutrino systems has grown, driven by advances in neutrino experiments.

This chapter provides a thorough yet concise overview of quantum correlation and its different measures. It starts with the history of quantum correlation in section 3.1, then introduces the key terminology and mathematical foundations in section 3.2. Section 3.3 briefly covers the types of correlations and their measures. Finally, section 3.4 focuses on quantum correlation in neutrinos, with subsection 3.4.1 discussing mode entanglement and subsection 3.4.2 exploring quantum correlation measures through neutrino probability.

3.1 History of quantum correlations

The history of quantum correlations can be traced back to the early days of quantum mechanics. In 1927, Erwin Schrödinger proposed the idea of entanglement, which is a type of quantum correlation. Schrodinger's idea was that two particles could be linked together in such a way that they shared the same fate, even if they were separated by a large distance. Erwin Schrodinger [78] recognized the unusual qualities of what he dubbed entangled states in his renowned trilogy on "The Present Situation of Quantum Mechanics" in 1935. In the same year, Einstein, Podolsky, and Rosen (EPR) proposed a thought experiment involving two distant quantum particles to argue that quantum mechanics, as formulated, might be incomplete [79]. They published a paper in which they challenged the idea of entanglement. They argued that entanglement was a violation of locality, which is the principle that no information can travel faster than the speed of light. However, the EPR paradox argues that the measurement of particle A cannot have an instantaneous effect on particle B, because this would violate the principle of locality. According to EPR, any complete physical theory should adhere to the principle of local realism. The EPR paradox concludes that the Copenhagen interpretation of quantum mechanics must be incomplete and that there must be a more complete theory that can explain the behavior of quantum systems without violating the principle of locality.

Niels Bohr [278] promptly responded to the EPR report that quantum mechanics is complete. Schrödinger introduces the concept of "steering" in response to the EPR paradox, describing how one particle can nonlocally affect the state of another particle. W.H. Furry [279] also inspired by EPR and Schrodinger, discusses the discrepancies between quantum mechanics predictions of non-factorizable systems and models with spontaneous factorization in 1935. For over 30 years, the EPR-Bohr debate was considered more philosophical than practical, making it of little use to physicists. That is until John Stewart Bell [80] resurrected the topic in his groundbreaking 1964 article "On the Einstein-Podolsky-Rosen Paradox", which led to a radical shift in public opinion. Bell established what is now known as Bell's Theorem, i.e., "No local hidden variable theory can

reproduce all possible results of QM." Bell's theorem showed that correlations between the results of local measurements on two spatially separated subsystems can only have a local-realistic description if they satisfy specific constraints known as Bell-type inequalities. When Bell proposed an experiment to distinguish between quantum theory predictions and Einstein's arguments. This led to the formulation of the CHSH inequality by Clauser, Horne, Shimony, and Holt [86]. Freedman and Clauser experimentally tested the CHSH inequality soon after its discovery and observed violations, supporting quantum mechanics over Einstein's arguments [280].

Subsequent experiments by Aspect et al. [281] confirmed these findings. Over the years, numerous experiments have demonstrated the violation of the Clauser-Horne-Shimony-Holt (CHSH) inequality, although some loopholes remain [82]. According to Einstein, nature should respect the CHSH inequality. In contrast, quantum mechanics predicted violations of these inequalities; thus, rejecting local realism demonstrates the incompleteness of classical physics. The nonlocality resulting from quantum entanglement, the "spooky action at a distance," was determined as the fundamental property of quantum physics and stands in contrast to our understanding.

Quantum entanglement serves as the physical resource for generating nonlocal correlations. However, the relationship between entanglement and nonlocality is generally quite complex [82]. Werner demonstrated that the statistics generated from certain entangled states can indeed have a locally realistic description. Therefore, while entanglement is a necessary condition for the exhibition of nonlocal behavior, it is not sufficient on its own. From 1935 through the early 1990s, when Artur Ekert advocated using the correlations resulting from entangled states for cryptography, relatively few works were reported [282]. Entanglement is now seen as a resource for several quantum information applications. Bell's work [80] sparked new physics, such as quantum cryptography [282] and quantum teleportation [283], as well as quantum information and quantum communication.

The modern concept of entanglement dates back to 1989, when Werner extended it to all mixed quantum states [284]. This work initiated the theory of entanglement, which explores its properties and implications in fundamental tasks like quantum cryptography [282], quantum dense coding [285], and quantum teleportation [283]. The Horodecki family made significant contributions, including the discovery of bound entanglement [286], where states require entanglement to be created but cannot be used to extract pure entangled states [82]. Entanglement is crucial for the exponential speedup in quantum computing, as shown by Jozsa and Linden [287]. While Shor's algorithm [288] demonstrates this speedup for pure states, mixed-state quantum computation presents a more complex picture. Knill and Laflamme's algorithm [289] does not require entanglement but solves certain problems efficiently, indicating the existence of quantum correlations beyond entanglement [290].

Quantum discord, introduced by Zurek [83] and independently by Henderson and Vedral [291], captures quantum correlations present even in non-entangled states. Discord plays a role in the efficiency of Knill and Laflamme's algorithm [290]. Eastin showed that mixed-state quantum computation with zero discord can be efficiently simulated on a classical computer [292]. Zurek also provided a thermodynamic interpretation of quantum discord, demonstrating that a quantum Maxwell's demon can extract more work from a system than a classical one due to the ability to operate on the whole quantum state [293]. Quantum discord and related measures, such as information deficit [294], have been linked to various quantum tasks, including state merging [295, 296], entanglement generation, and entanglement distribution [297, 298]. Experiments have also demonstrated discord's role in remote state preparation [299] and information encoding [300]. Further details can be found in a comprehensive review by [301].
The quest to understand the diverse nature of quantum correlations has recently led to a renewed focus on the fundamental aspect of quantum mechanics, namely quantum coherence. Quantum

coherence, recognized as a valuable resource, has significant implications for technological applications as well as for studies on non-classicality under the influence of general open system effects. The study of quantum coherence as a resource theory began in the mid-2010s. A rigorous framework for the quantification of coherence was introduced [302]. Quantifying quantum coherence using information theory is crucial for the practical implementation of quantum resource theory.

3.2 Terminology and mathematical foundation

In this section, we introduce the terminology and mathematical framework essential for analyzing quantum correlations [303].

3.2.1 Quantum states

In the theory of quantum mechanics, a state vector $|\varphi\rangle$ in a Hilbert space $\mathcal{H}$ can fully represent any physical system. A qubit is a system with a two-dimensional Hilbert space. We assume, unless otherwise specified, that the dimension of the Hilbert space is arbitrary but finite. With Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ for two parties, Alice (A) and Bob (B) the Hilbert space as a whole is a tensor product of the subsystem spaces: $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. A pure state is defined as any system that can be characterized by a single state vector. However, the physical state of the considered system is not totally understood in a true experimental setup. The physical state of the system can be characterized using the density operator if the system is in the pure state $|\varphi_i\rangle$ with probability P_i

$$\rho = \sum_i P_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|, \tag{3.1}$$

such state is a mixed state.

The system's state is defined solely by two characteristics of the density operator are as follows:

- Trace of ρ equal to one i.e. $Tr[\rho] = 1$.
- For any vector $|\varphi\rangle$, ρ should be a positive operator, i.e., $\langle\varphi \mid \rho \mid \varphi\rangle \geq 0$ and this indicate that ρ is Hermitian.

Pure state: If and only if a one-dimensional projection operator functioning on $\mathcal{H}$ is present, the state ρ_S of a system is said to be pure. If $\varphi \in \mathcal{H}$ then $\rho_S = |\varphi\rangle \langle\varphi|$.
Mixed state: An ensemble of the pure state is called a mixed state, if the system is in a state $|\varphi\rangle$ with probability P_i and the sum of probabilities is one and i varies from 1 to n, then by a single vector, we can not define the system effective state. Then effective state can be defined as $\rho_S = \sum_i P_i \left|\varphi_i\right\rangle \left\langle\varphi_i\right|$. Two distinct ensembles can generate similar effective states.

3.2.2 Partial trace

In a system with two subsystems, A and B, one might be interested only in the state of subsystem A. This is described using the concept of the reduced density operator. If the entire system is represented by the density operatorρ_{AB}, then the state of the subsystem A is described by the reduced density operator. To obtain the reduced density operator, one can use operations such as partial trace. The partial trace is a mathematical operation that is used to reduce the dimension of a composite quantum system. If the total system is described by the density operator ρ_{AB}, then the subsystem A is described by the reduced density operator

$$\rho_A = Tr_B \left[\rho_{AB}\right], \tag{3.2}$$

where Tr_B denotes the partial trace over subsystem B. ρ is the density matrix of the composite system, and B is the subsystem that we want to trace out. The partial trace returns a density matrix for the subsystem A, which is the remaining subsystem after B has been traced out. This partial trace operation is defined for all density operators. Its physical significance lies in providing the correct measurement statistics for the subsystem A, making it a unique and essential operation. The detailed explanation of partial trace for the two-qubit and three-qubit scenarios is explained in the appendix A.1.

3.2.3 Partial transpose

The partial transpose is another operation used in the study of quantum systems. This operation is particularly useful in the study of quantum entanglement. For a composite system with two subsystems, A and B, described by the density matrix ρ_{AB}, the partial transpose operation involves transposing only the part of the density matrix that corresponds to one of the subsystems.

$$(\rho_{AB}^{T_B})_{mn,kl} = (\rho_{AB})_{mk,nl}. \tag{3.3}$$

Here, $\{|m\rangle\}, \{|k\rangle\}$ denote the basis elements of subsystem A, and $\{|n\rangle\}, \{|l\rangle\}$ denote those of subsystem B. The full bipartite density matrix ρ_{AB} is expressed in the product basis $\{|m\rangle \otimes |n\rangle\}$. In this representation, the matrix element $(\rho_{AB}^{T_B})_{mn,kl}$ on the left-hand side refers to the partially transposed density matrix with respect to subsystem B. On the right-hand side, the element $(\rho_{AB})_{mk,nl}$ corresponds to the original, untransposed density matrix. The partial transpose operation acts only on subsystem B, which results in the interchange of the indices n and l, while leaving the subsystem A indices m and k unchanged. The partial transpose is useful in criteria for entanglement, such as the Peres-Horodecki criterion, which states that if the partial transpose of ρ_{AB} has any negative eigenvalues, then the state is entangled. Partial transpose is also explained in detail in Appendix A.2.

3.3 Various measures of quantum correlations

Quantum correlations are a multifaceted concept within the domain of quantum mechanics, and they cannot be fully described by a single quantity or measure. Instead, there are various aspects used to characterize these correlations. Quantum correlation measures are organized into a hierarchical structure based on their strengths and properties in figure 3.2.

3.3.1 Nonlocality

Nonlocality is the strongest facet of quantum correction. The concept of nonlocality is important to quantum information theory. In 1964, Bell eliminated the notion that quantum mechanics would be incomplete due to the presence of hidden variables. When Bell proposed an experiment to differentiate between the predictions of quantum theory and Einstein's arguments, it led to the formulation of the CHSH inequality. However, quantum mechanics predicted violations of these inequalities, which challenged the concept of local realism. Nonlocality resulting from this is now recognized as a fundamental property of quantum physics, contrasting with classical intuitions. In this subsection, we briefly discuss some nonlocal measures of quantum correlations.

Bell's Inequality: It examines the quantum correlation existing between measurement outcomes performed on two spatially separated counterparts of a composite system. Violation of this inequality exhibits nonlocal correlations embedded in the system, which is one of the most crucial aspects of quantum theory. Maximal violation is realized by the largest eigenvalue of a Hermitian operator, known as the Bell operator [304]. A standard form of Bell-inequality is

CHSH inequality which is expressed as [86]

$$-2 \leq \mathcal{B}_{CHSH} \leq 2, \tag{3.4}$$

Here $\mathcal{B}_{CHSH}$ is known as the Bell operator. The general form of the Bell-CHSH inequality is expressed as [86]

$$E(a,b) - E(a,b') + E(a',b) + E(a',b') \leq 2, \tag{3.5}$$

Here, $E(a,b)$ represents the average product of the measurement outcomes of detectors A and B, respectively, when oriented in specific directions a and b. It is given by

$$E(a,b) = \int d\lambda\, \rho(\lambda)\, A(a,\lambda)\, B(b,\lambda), \tag{3.6}$$

where $A(a,\lambda)$ and $B(b,\lambda)$ denote the outcomes of measurements by detector A in direction a and detector B in direction b, respectively, under the setting of a particular hidden variable λ. Detectors A and B are performing measurements on two entangled subsystems. The parameters a, a', b, and b' represent different settings for these detectors. Alternatively, the CHSH inequality can be written in a simplified form

$$\langle AB - AB' + A'B + A'B' \rangle \leq 2. \tag{3.7}$$

For a system consisting of two spin-1/2 particles A and B, the combined state with Hilbert space defined as $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, is expressed in terms of the density matrix (ρ) as follows [305]

$$\rho = \frac{1}{4}\left[I \otimes I + (r.\sigma) \otimes I + I \otimes (s.\sigma) + \sum_{A,B=1}^{3} T_{AB}(\sigma_A \otimes \sigma_B) \right]. \tag{3.8}$$

Here I is the 2×2 identity matrix, r, s are the vectors in real space and σ stands for Pauli matrices. T is the correlation matrix, and its elements are given by $T_{AB} = Tr[\rho(\sigma_A \otimes \sigma_B)]$. A real matrix can be constructed as $T^\dagger T$ having eigenvalues u_i ($i = 1, 2, 3$) of which two largest positive eigenvalues are taken into account, denoted by u_i and u_j. Violation of Bell-CHSH inequality given in eq. 3.4 is possible *iff* $M(\rho) = u_i + u_j > 1$ [305].

Mermin and Svetlichny Inequalities: The violation of a Bell-type inequality, like the CHSH inequality, for a two-qubit state indicates nonlocality. Extending this concept to a three-party system is more complex, and the types of states are shown in figure 3.1. The Mermin inequality assumes that all three qubits are locally and realistically correlated, so its violation signals tripartite nonlocality among the qubits. However, it has been shown that even biseparable states can violate the Mermin inequality.

This led Svetlichny to develop a hybrid nonlocal-local realism-based inequality, known as the Svetlichny inequality. In a three-qubit system, nonlocality may be present if nonclassical correlations exist between two of the three qubits. Such a state would exhibit absolute nonlocality and violate the Mermin inequality. Consider a three-party system with the detector settings (A, B, C) and (A', B', C'). The two Mermin inequalities are [307]:

$$\begin{aligned} M_1 &\equiv ABC' + AB'C + A'BC - A'B'C' \leq 2, \\ M_2 &\equiv ABC - A'B'C - A'BC' - AB'C' \leq 2. \end{aligned} \tag{3.9}$$

and and Svetlichny inequality is [308]

$$\sigma \equiv M_1 + M_2 \leq 4. \tag{3.10}$$

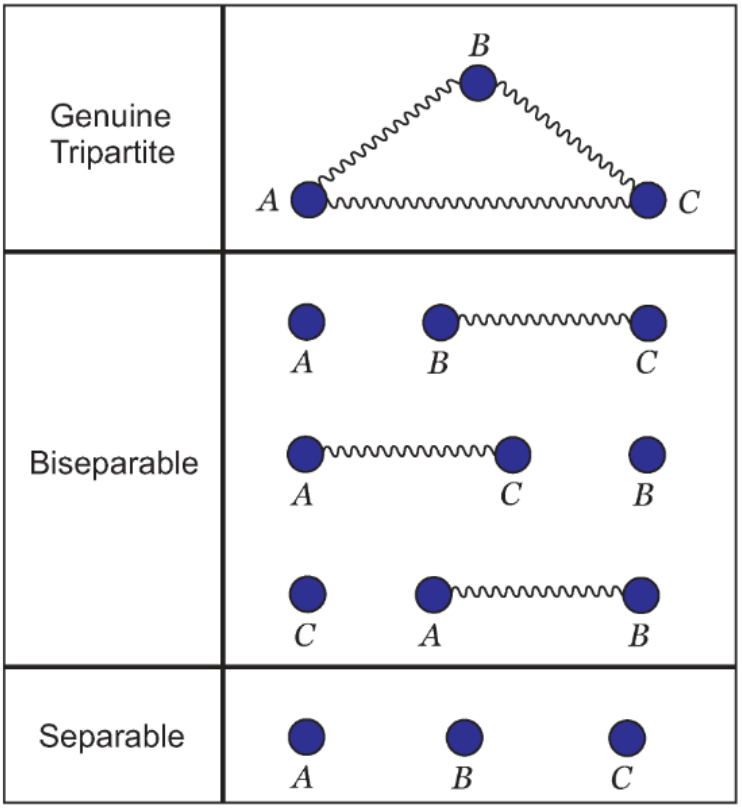

Figure 3.1.: Correlated states between three-qubit system [306]

In a three-party system, when each subsystem is locally correlated with the others, both Mermin and Svetlichny inequalities will be satisfied. When two subsystems are nonlocally correlated with each other while maintaining local correlations with the third subsystem, the Mermin inequalities will be violated, but the Svetlichny inequality will remain satisfied. When all subsystems are nonlocally correlated with one another, then both inequalities will be violated. This indicates that the Mermin parameters M_1 and M_2 can be viewed as absolute measures of tripartite nonlocality, while the Svetlichny parameter σ can be regarded as a measure of genuine tripartite nonlocality.

3.3.2 Steering

An intermediate feature between entanglement and nonlocality is known as steering. The concept of steering was introduced by Schrodinger in response to the EPR paradox, describing how one particle can nonlocally affect the state of another particle. Unlike nonlocality, steering implies an asymmetric nonlocal correlation between the subsystems of a given system. This means that one can manipulate or "steer" one subsystem by performing local operations on the other, but the reverse is not necessarily true. It refers to the ability of one subsystem (Alice) to nonlocally affect the state of another subsystem (Bob's) through local measurements. Steering demonstrates the nonlocal nature of quantum mechanics and is stronger than entanglement but weaker than Bell's nonlocality [310, 311].

In the context of steering, a task is set up with two parties. Alice prepares a bipartite quantum state and sends one part to Bob, repeating this process multiple times. Each time, they measure their respective parts and communicate their results classically. Alice's goal is to convince Bob that the state she prepares is entangled. Bob accepts that quantum mechanics accurately describes the measurements he makes. However, Bob does not trust Alice. If the correlations

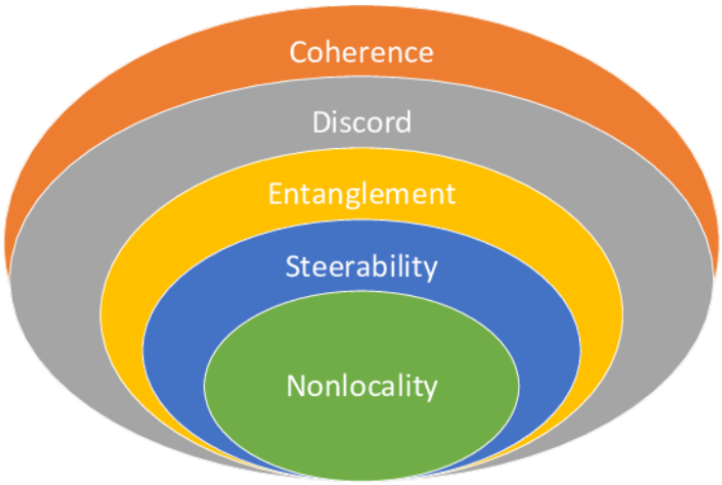

Figure 3.2.: Hierarchy breakdown of quantum correlation measures [309].

between Bob's measurements and Alice's reported results can be explained by a local hidden state model for Bob, he will not be convinced that the state is entangled. In such a scenario, Alice could have simply drawn a pure state at random from an ensemble, sent it to Bob, and then chosen her result based on her knowledge of this local hidden state. On the other hand, if the correlations cannot be explained by the local hidden state, the state must be entangled. Therefore, Alice will succeed in her task if and only if she can create genuinely different ensembles for Bob by steering his state [312].

3.3.3 Entanglement

A platform for testing and analyzing local hidden variable theories is provided by quantum correlations, which are founded on the principles of locality and realism. The study of correlations has an extensive literature in quantum systems. The most basic idea among them that can be accurately measured and described using the methods of quantum resource theories is entanglement. It is an excellent tool for performing numerous quantum tasks. If the state $|\psi_{AB}\rangle$ of the total system can be expressed as $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$, then the two subsystems are independent and separable. Quantum mechanics also permits superposition states such as

$$|\phi\rangle = \frac{1}{\sqrt{N}} \left(|\phi_1\rangle \otimes |\phi_2\rangle + |\phi_2\rangle \otimes |\phi_1\rangle \right), \tag{3.11}$$

where N is a normalization constant. This state is non-separable, indicating that it is entangled. An entanglement measure satisfies two key properties: it does not increase under local operations and classical communication (LOCC), and it vanishes for separable states. This property has led to a variety of intriguing applications. A pure state must not be factorizable if we assume that it is entangled. It has numerous potential uses in quantum information processing, such as entanglement swapping [313], quantum teleportation [283] and quantum cryptography [282]. Entanglement is not a single valued quantity. There are numerous ways to quantify entanglement. Different measures capture different amount of entanglement. These measures quantify the extent to which a quantum state ρ deviates from being separable.
For a pure state $|\psi_{AB}\rangle$, entanglement is typically quantified by the von Neumann entropy of the reduced density matrix ρ_j, obtained by tracing out all except subsystem j. This is given by

$$S = -\mathrm{Tr}(\rho_j \log_2 \rho_j) \equiv -\sum_j \lambda_j \log_2 \lambda_j, \tag{3.12}$$

where λ_j are the eigenvalues of density matrix ρ_j. This measure provides a non-zero value if the state exhibits entanglement in at least one bipartite division, making it a measure of absolute entanglement. For separable states, $S = 0$, while for a completely entangled state, $S = \log_2 d$, where d is the dimension of the system. A non-zero value of this measure indicates that all subsystems or parties are mutually entangled. Other entanglement measures are extensively studied in the literature for tripartite systems, i.e., entanglement of formation [132, 133], concurrence [134, 135] and negativity [136, 137]. Entanglement of formation is defined as the von Neumann entropy of the reduced density matrix as seen by party A or B or C. Simply, we can say it is the minimum amount of entanglement that three parties must share in order to produce the specified state. The good property of EOF is that it is zero only when the state is separable. Concurrence was defined as a simplified form of EOF and an extensively used measure. Negativity, a weaker measure, was defined to study the pure and mixed states.

3.3.4 Discord

Discord is a correlation measure that shows correlation even when entanglement is zero. For instance, discord is a feature that includes correlations caused by entangled but also by nonentangled states [83, 291], and it can be connected to quantum coherence as well. For pure states, quantum discord aligns with entanglement, but for mixed states, it can be non-zero even if the subsystems are not entangled. This implies that the separability of subsystems does not necessarily mean they are classically correlated. We have two quantum analogs of the classical mutual information: The original quantum mutual information (I), and the measurement-induced quantum mutual information (J). Quantum discord is given as the difference between the two mutual information. Therefore, for a state with density matrix ρ, the quantum discord is given in $QD = I(a) - J(a)$ [314].

Geometric discord was introduced to overcome the difficulty of calculating quantum discord, which quantifies the amount of non-classical correlation of an arbitrary quantum composite system in terms of its minimal distance from the set of classical states. For ρ, geometric discord can be shown to be [315]

$$GD(\rho) = \frac{1}{4}[\|\vec{y}\|^2 + \|T\|^2 - \lambda_{max}]\,, \tag{3.13}$$

where T is the correlation matrix defined above, $\vec{y}$ is the vector whose components are $y_m = \mathrm{Tr}(\rho(\sigma_m \otimes I))$, and λ_{max} is the maximum eigenvalue of the matrix $(\vec{y}\vec{y}^{\dagger} + TT^{\dagger})$.

3.3.5 Coherence

At the heart of quantum mechanics lies the superposition principle, which is fundamental to understanding quantum phenomena. Quantum coherence, a consequence of this principle, is essential for phenomena like interference to occur. Essentially, quantum coherence delineates a boundary between the classical and quantum worlds. In the realm of quantum information, coherence is a fundamental term that can be precisely defined within the framework of quantum resource theory.

In the realm of quantum information, coherence stands as a fundamental concept, serving as a quantifier for the quantum nature of a system. It is rigorously addressed within the framework of quantum resource theory and is intricately linked to various measures of quantum correlations, including entanglement and quantum discord [316]. Quantum coherence can be viewed as the foundational element encompassing all quantum correlations inherent in a given system.

In the context of quantum information theory, the quantification of coherence relies on

characterizing the set of incoherent states (I) and incoherent operations O^I. In a chosen reference basis $\{|i\rangle\}$, the states defined as $\rho_I = \sum_i d_i |i\rangle \langle i|$ where $d_i > 0$ and $\sum d_i = 1$, constitute a set of incoherent states. Incoherent operations are defined to map this set of incoherent states onto itself, i.e., $O^I(I) \in I$.

Within this framework of incoherent operations and states, the l_1-norm (sum of the absolute values of off-diagonal elements of the density matrix $\rho = |\psi\rangle \langle\psi|$) [302, 317] is introduced as a reliable measure or quantifier of coherence

$$\mathcal{C} = \sum_{i \neq j} |\rho_{ij}| \geq 0. \tag{3.14}$$

This measure possesses key properties of a coherence measure: (i) non$-$negativity ($\mathcal{C}(\rho) = 0$ iff $\rho \in I$), (ii) monotonicity under incoherent operations ($\mathcal{C}\left(O^I[\rho]\right) \leq \mathcal{C}(\rho)$) and (iii) convexity ($\mathcal{C}\left(\sum_k p_k \rho_k\right) \leq \sum_k p_k \mathcal{C}(\rho_k)$, where $\rho_k = \frac{Q_k \rho Q_k^\dagger}{p_k}$ (where Q_k is the Kraus operator) and $p_k = Tr\left(Q_k \rho Q_k^\dagger\right)$). The value of this measure for a $d-$dimensional maximally coherent state, defined as $\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle$, this measure attains the value $d-1$, which sets the maximum possible limit for the measure. The l_1- norm-based coherence measure is zero for a completely incoherent state, and, for three-flavor neutrino oscillation scenarios, it attains its maximum value $\mathcal{C}_{max} = 2$.

The framework of quantum correlations in quantum mechanics provides a way to understand and quantify the relationships between quantum systems. It is based on the idea that quantum systems can be correlated in a variety of ways, some of which are stronger than what is allowed by classical physics. As shown in figure 3.2, this hierarchy ranges from the strongest correlations to the weakest quantum correlations, with quantum coherence, quantum discord, entanglement, steering, and nonlocality. Each measure helps to identify and quantify different aspects of the quantum nature of correlations in composite systems. Quantum correlations have a number of potential applications in quantum information science. For example, quantum correlations can be used to develop new types of quantum computers that are much faster than traditional computers. Quantum correlations can also be used to develop new types of communication systems that are much more secure than traditional communication systems. In this section, we discussed various facets of quantum correlations. In the next section, we will explore these facets within the neutrino system.

3.4 Quantum correlations in neutrinos

The study of quantum correlation facets is an emerging topic, which is commonly studied in optical and electronic systems [318–320]. However, in recent years, the quest to probe such effects in high-energy physics systems, such as mesons and neutrinos, has emerged [89, 90, 92, 94–98, 103–108, 321–329]. Neutrino oscillations can only occur if neutrinos have a mass [22]. Flavor eigenstates (ν_e, ν_μ, ν_τ) are linear combinations of ν_1, ν_2, ν_3 mass eigenstates, with corresponding masses m_1, m_2 and m_3. In the scenario of neutrino oscillation, one can map the flavor state of the neutrino system as a single-particle mode-entangled state where distinct flavors of neutrino serve as different modes of oscillations and entanglement can be seen among these flavor modes. Measuring the quantumness embedded in a system is a complex task with many dimensions. Neutrino oscillation is a fascinating phenomenon that preserves and allows the measurement of the quantum nature of neutrinos over significant distances. Quantum correlation measure of neutrino oscillation can quantify the system's quantumness. This quantumness can be explored through various aspects such as nonlocality, entanglement, steering, and discord. Moreover, various nonclassical features can be expressed in terms of neutrino survival and oscillation probabilities that proclaim

these measures as legitimate quantities to be studied for neutrino systems.

This section offers a comprehensive overview of quantum correlations in neutrino oscillation. It starts with a detailed examination of mode entanglement in subsection 3.4.1, followed by subsection 3.4.2 explores quantum correlation measures in terms of neutrino oscillation probabilities.

3.4.1 Mode entanglement

The standard description of mode entanglement present in quantum information. Typically, mode entanglement refers to the entanglement between the various modes or degrees of freedom of a quantum system, such as the spatial modes of photons or the energy levels of particles. Mode entanglement is a type of quantum entanglement that occurs between two or more modes of a quantum system. A mode is a way of describing the different states that a quantum system can be in. In neutrino physics, neutrino oscillation is characterized by the weak interaction's mixing of neutrino flavors. This mixing results in the oscillation of neutrino flavors as they propagate through space due to the quantum superposition of different mass eigenstates. In neutrino physics, however, the term "mode entanglement" is used to describe the entanglement between the mass eigenstates in a specific neutrino flavor for a single particle, which is frequently referred to as different neutrino modes [89, 90, 103].

For two-flavors

It is a well-known fact that neutrino oscillation requires the flavor eigenstates ν_α to be represented as a linear combination of mass eigenstates ν_i as follows

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle , \tag{3.15}$$

where U represents the elements of the unitary mixing matrix. In the notion of two flavor neutrino oscillation, the 2×2 mixing matrix U is represented as,

$$U = \begin{pmatrix} U_{\alpha 1} & U_{\alpha 2} \\ U_{\beta 1} & U_{\beta 2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} , \tag{3.16}$$

where θ is the mixing angle. The time evolution of mass eigenstates is given by

$$|\nu_i(t)\rangle = e^{-\iota E_i t} |\nu_i\rangle , \tag{3.17}$$

where $|\nu_i\rangle$ are mass eigenstates at $t = 0$ given by eq. 3.15. Therefore the time-evolved flavor eigenstate is expressed as

$$\begin{aligned} |\nu_\alpha(t)\rangle &= \sum_i U_{\alpha i} e^{-\iota E_i t} |\nu_i\rangle \\ &= \sum_{i,\beta} U_{\alpha i} e^{-iE_i t} U^*_{\beta i} |\nu_\beta\rangle . \end{aligned} \tag{3.18}$$

$|\nu_\alpha\rangle$ and $|\nu_i\rangle$ denotes the flavor and mass eigen states respectively at $t = 0$.

$$|\nu_\alpha(t)\rangle = \bar{U}_{\alpha\alpha}(t) |\nu\rangle_\alpha + \bar{U}_{\alpha\beta}(t) |\nu\rangle_\beta , \tag{3.19}$$

where the coefficients in eq. 3.19, $\bar{U}_{\alpha\alpha}$ and $\bar{U}_{\alpha\beta}$ can be expressed as

$$\bar{U}_{\alpha\alpha} = \cos^2\theta\, e^{-iE_1 t} + \sin^2\theta\, e^{-iE_2 t}, \tag{3.20}$$

$$\bar{U}_{\alpha\beta} = -\sin\theta\cos\theta\, e^{-iE_1 t} + \sin\theta\cos\theta\, e^{-iE_2 t}. \tag{3.21}$$

Then, the density matrix corresponding to the state given in eq. 3.19 is expressed as

$$\rho_{\alpha\beta}(t) = \begin{pmatrix} \left|\bar{U}_{\alpha\alpha}(t)\right|^2 & \bar{U}_{\alpha\alpha}(t)\bar{U}^*_{\alpha\beta}(t) \\ \bar{U}^*_{\alpha\alpha}(t)\bar{U}_{\alpha\beta}(t) & \left|\bar{U}_{\alpha\beta}(t)\right|^2 \end{pmatrix}, \tag{3.22}$$

In the relativistic limit, neutrino flavor states are considered to be individual modes. In the two flavor neutrino system, it can be expressed as [89]

$$|\nu_\alpha\rangle \equiv |1\rangle_\alpha |0\rangle_\beta \equiv |10\rangle_{\alpha\beta}, \qquad |\nu\rangle_\beta \equiv |0\rangle_\alpha |1\rangle_\beta \equiv |01\rangle_{\alpha\beta}. \tag{3.23}$$

Using eq. 3.23 in eq. 3.19, we get

$$|\nu_\alpha(t)\rangle = \bar{U}_{\alpha\alpha}(t)\,|1\rangle_\alpha |0\rangle_\beta + \bar{U}_{\alpha\beta}(t)\,|0\rangle_\alpha |1\rangle_\beta, \tag{3.24}$$

where $\bar{U} = Ue^{-i\mathcal{H}_m t}U^\dagger$, $\mathcal{H}_m = diag(E_1, E_2)$ *i.e.,* the Hamiltonian in mass eigenbasis with eigen values E_i $(i = 1, 2)$. Eq. 3.24 indicates the entanglement between flavor modes in a single particle system at time t. Eq. 3.24 can also be written as

$$|\nu_\alpha(t)\rangle = \bar{U}_{\alpha\alpha}(t)\,|10\rangle_{\alpha\beta} + \bar{U}_{\alpha\beta}(t)\,|01\rangle_{\alpha\beta},$$

Similarly, we can write other eq.

$$\langle\nu_\alpha(t)| = \bar{U}^*_{\alpha\alpha}(t)\,\langle 10|_{\alpha\beta} + \bar{U}^*_{\alpha\beta}(t)\,\langle 01|_{\alpha\beta},$$

Then, the density matrix corresponding to the state given in eq. 3.24 is expressed as

$$\rho_{\alpha\beta}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \left|\bar{U}_{\alpha\beta}(t)\right|^2 & \bar{U}^*_{\alpha\alpha}(t)\bar{U}_{\alpha\beta}(t) & 0 \\ 0 & \bar{U}_{\alpha\alpha}(t)\bar{U}^*_{\alpha\beta}(t) & \left|\bar{U}_{\alpha\alpha}(t)\right|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.25}$$

For three-flavors

In the standard model of three-flavor neutrino oscillations, the three flavor states, ν_e, ν_μ and , ν_τ consist of a linear superposition of the mass eigenstates, ν_1, ν_2 and ν_3:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i}\,|\nu_i\rangle, \tag{3.26}$$

where $\alpha = e, \mu, \tau$ and $i = 1, 2, 3$. flavor state $|\nu_\alpha\rangle$ is the initial flavor state at time $t = 0$, which is associated with mass eigenstates $|\nu_i\rangle$ with 3×3 the PMNS unitary mixing matrix. There $U_{\alpha i}$ are elements of the PMNS matrix that consist of three mixing angles, i.e., θ_{12}, θ_{13}, θ_{23} and a charge conjugation and parity (CP) phase. The matrix U can be written as

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-\iota\delta} \\ -s_{12}c_{23} - c_{12}s_{13}s_{23}e^{\iota\delta} & c_{12}c_{23} - s_{12}s_{13}s_{23}e^{\iota\delta} & c_{13}s_{23} \\ s_{12}s_{23} - c_{12}s_{13}c_{23}e^{\iota\delta} & -c_{12}s_{23} - s_{12}s_{13}c_{23}e^{\iota\delta} & c_{13}c_{23} \end{pmatrix}, \tag{3.27}$$

where $s_{ij} = sin\theta_{ij}$ and $c_{ij} = cos\theta_{ij}$. For the sake of simplicity, we ignore the CP-violating phase until it is observed. Mass eigenstate time evolution is given as

$$|\nu_i(t)\rangle = e^{-\iota E_i t}\,|\nu_i\rangle, \tag{3.28}$$

where $|\nu_i\rangle$ are mass eigenstates at $t = 0$ given by eq. 3.26. Therefore the time evolved flavor eigenstate is expressed as

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} e^{-\iota E_i t} |\nu_i\rangle . \tag{3.29}$$

In the relativistic limit, neutrino flavor states are considered to be individual modes. In the three flavor neutrino system, it can be expressed as [89]

$$\begin{aligned} |\nu_e\rangle &\equiv |1\rangle_e |0\rangle_\mu |0\rangle_\tau \equiv |100\rangle_{e\mu\tau} , \\ |\nu_\mu\rangle &\equiv |0\rangle_e |1\rangle_\mu |0\rangle_\tau \equiv |010\rangle_{e\mu\tau} , \\ |\nu_\tau\rangle &\equiv |0\rangle_e |0\rangle_\mu |1\rangle_\tau \equiv |001\rangle_{e\mu\tau} . \end{aligned} \tag{3.30}$$

Using eq. 3.30 in eq. 3.28, we get time evolution for the tripartite quantum state

$$|\nu_\alpha(t)\rangle = \bar{U}_{\alpha e}(t) |100\rangle_{e\mu\tau} + \bar{U}_{\alpha\mu}(t) |010\rangle_{e\mu\tau} + \bar{U}_{\alpha\tau}(t) |001\rangle_{e\mu\tau} , \tag{3.31}$$

where α is a flavor state that generates at the initial time $t = 0$ and $\bar{U} = Ue^{-i\mathcal{H}_m t}U^\dagger$, $\mathcal{H}_m = diag(E_1, E_2, E_3)$ *i.e.,* the Hamiltonian in the mass eigenbasis with E_i $(i = 1, 2, 3)$. Eq. 3.31 determines the entanglement at time t between flavor-modes. As a result, we can use the framework of quantum resources to study the oscillations of neutrinos. We will focus on the density matrix to calculate the tripartite entanglement.

The density matrix corresponding to the state of eq. 3.31 is $\rho^\alpha_{ABC}(t) = |\nu_\alpha(t)\rangle \langle\nu_\alpha(t)|$, that is

$$\rho^\alpha_{ABC}(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho^\alpha_{22} & \rho^\alpha_{23} & 0 & \rho^\alpha_{25} & 0 & 0 & 0 \\ 0 & \rho^\alpha_{32} & \rho^\alpha_{33} & 0 & \rho^\alpha_{35} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \rho^\alpha_{52} & \rho^\alpha_{53} & 0 & \rho^\alpha_{55} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \tag{3.32}$$

where the elements of this matrix can be written as

$$\begin{aligned} &\rho^\alpha_{22} = \left|\bar{U}_{\alpha\tau}(t)\right|^2 ; \quad \rho^\alpha_{23} = \bar{U}_{\alpha\tau}(t)\bar{U}^*_{\alpha\mu}(t); \quad \rho^\alpha_{25} = \bar{U}_{\alpha\tau}(t)\bar{U}^*_{\alpha e}(t); \\ &\rho^\alpha_{32} = \bar{U}_{\alpha\mu}(t)\bar{U}^*_{\alpha\tau}(t); \quad \rho^\alpha_{33} = \left|\bar{U}_{\alpha\mu}(t)\right|^2 ; \quad \rho^\alpha_{35} = \bar{U}_{\alpha\mu}(t)\bar{U}^*_{\alpha e}(t); \\ &\rho^\alpha_{52} = \bar{U}_{\alpha e}(t)\bar{U}^*_{\alpha\tau}(t); \quad \rho^\alpha_{53} = \bar{U}_{\alpha e}(t)\bar{U}^*_{\alpha\mu}(t); \quad \rho^\alpha_{55} = \left|\bar{U}_{\alpha e}(t)\right|^2 . \end{aligned} \tag{3.33}$$

And the corresponding probabilities, $P_{\alpha e}(t) = \left|\bar{U}_{\alpha e}(t)\right|^2$, $P_{\alpha\mu}(t) = \left|\bar{U}_{\alpha\mu}(t)\right|^2$ and $P_{\alpha\tau}(t) = \left|\bar{U}_{\alpha\tau}(t)\right|^2$ and here α is the initial flavor state of neutrino where $\alpha = e, \mu, \tau$. If E is the energy of neutrinos and L is the distance traveled by neutrinos, then in the ultra-relativistic limit $L \equiv t$ and $E_i - E_j \approx \Delta m^2_{ij}/2E$.

3.4.2 Quantum correlation measures in terms of neutrino probability

The phenomenon of neutrino oscillation offers a natural framework for studying quantum correlations. These quantum correlation measures can be expressed and analyzed in terms of neutrino probabilities. Initially, Alok et al. [105] expressed various correlation measures in terms of

neutrino probabilities for two-flavor neutrino oscillations. The Bell inequality parameter, for example, is directly related to neutrino oscillation probabilities as $M(\rho) = 1 + 4P_{sur}P_{osc}$. A weaker yet popular and extensively studied feature of entanglement is concurrence, which can also be written in terms of probability as $C = 2\sqrt{P_{sur}P_{osc}}$. An even weaker measure of quantum correlations is quantum discord, which indicates that classicality and separability are not synonymous. The analytical formula for quantum discord is $D_\rho = \frac{8}{3}P_{sur}P_{osc}$. The maximum teleportation fidelity is also analyzed in terms of probability, given by $F_{max} = \frac{2}{3}(1 + P_{sur}P_{osc})$. A stronger measure of nonlocality, NAQC, and a measure of steering inequality is also studied in terms of probabilities [109, 330]. Subsequently, various measures have been analyzed in terms of neutrino oscillation probabilities for three-flavor neutrino oscillations. Some widely studied entanglement measures in terms of neutrino probabilities include the entanglement of formation, concurrence, negativity [94], tangle [114], genuinely multipartite concurrence (GMC), generalized geometric measure (GGM), three$-\pi$ and concurrence fill [120]. Coherence measures such as l_1-norm coherence [91], first-order coherence, and relative entropy of coherence [331] are also studied in terms of probabilities. Another measure, entropic uncertainty and quantum discord, was also studied in terms of oscillation probabilities [92]. Additionally, the temporal correlation measure LGtI is studied in terms of probabilities [98].

All these measures have been studied for neutrino oscillation in vacuum. Neutrino oscillation probabilities provide a rich framework for studying various quantum correlation measures. By expressing these measures in terms of probabilities, we can explore the fundamental quantum characteristics of neutrino systems.