

Basics of quantum correlation

In this chapter, a brief overview of quantum correlation is described. Quantum correlation is a fundamental feature of quantum mechanics and plays an inherent role in the quantum information sector which deals with the transfer of information using various quantum mechanical systems. The quantum correlations have been studied over the years in the several systems of particle physics. In recent years the study of the quantum correlation measures has gained popularity in the neutrino system because of the experimental advancements in the neutrino sector. This chapter starts with the description of the origin of the quantum correlations in Section 4.1, followed by the introduction of its different measures in Section 3.2, with a brief introduction of the temporal correlation in Subsection 3.2.1. In Section 3.3 the quantum correlation in the system of neutrinos is described.

3.1 Quantum correlations

The foundation of quantum information started with the development of the modern theory of quantum mechanics in the 1920s. The theory of quantum mechanics has been tested in several microscopic systems [236–238] and is found to be astonishingly successful. Although quantum mechanics has been remarkably successful in explaining the physical phenomena at a small scale, its apparent absurdity and capricious nature make it very difficult to understand clearly. Quantum information helps us to access the quantum world more easily. Unlike the classical concepts, the notion of quantum mechanics about the physical realm seems to be apparently absurd, since according to the quantum theory, a physical entity cannot possess an exact well defined value before its measurement. Such eccentric feature is comprehensible, as in quantum theory a system is allowed to exist as a superposition of different states which leads to the probabilistic determination of the dynamical entities in a quantum system. In addition to this, in the quantum picture, the physical observables are only allowed to possess discrete values. The concept of wave-particle duality also makes quantum mechanics exceptionally distinct from the laws of classical physics. The modern quantum theory is based on the *uncertainty principle*, proposed by Werner Heisenberg in 1926 [239], according to which it is not possible to predict the physical properties of any two non-commuting dynamical observables simultaneously.

Albert Einstein was one of the biggest critics of the quantum theory who along with his colleagues, Boris Podolsky and Nathan Rosen argued that the quantum theory cannot be the complete theory of nature [240]. In a thought experiment presented in their paper, it was concluded that two non-commuting observables, say position and momentum, can be measured simultaneously and allowed to possess definite values even before performing a measurement on the system. This was in direct contradiction with the uncertainty principle. This led Einstein to draw the conclusion that the quantum theory was incomplete as it could not predict the definite value of a particular observable without performing any kind of measurement on the system. He introduced the concept of some *hidden variables* (λ) which, if included in the quantum theory, would be able to determine the exact properties of the dynamic observables even before performing the measurement. The classical view on which EPR is based, are established on two principle perceptions, known as *locality*

and *realism*. The outlines of the terms are interpreted as follows.

- *Locality*: The measurement performed on one system does not influence the properties of the observable determined by taking a measurement on the second system, provided that the two systems are produced from the same source and then separated from each other after a while.
- *Realism*: Any dynamical observable possesses a definite value prior to any measurement performed on the system. The measurement only reveals the value of the observable

A long time after the proposal of the EPR paradox along with the idea of hidden variables, in 1952 the EPR thought experiment was put forward in a much simpler way by David Bohm who considered a system of spin-0 particle decaying into two spin-1/2 fermions [241]. The composite spin-0 state is represented as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle),$$

where $|\uparrow\rangle, |\downarrow\rangle$ denotes the set of orthonormal basis vectors in the spin space. If the spin of one particle is measured along a specific direction (e.g. z -axis) and is found to be in, say spin-up state $|\uparrow\rangle$, then according to the quantum theory, the measurement performed on the other particle is automatically forced to be in the spin-down state $|\downarrow\rangle$, to conserve the total spin of the combined system. Such outcomes of the measurement are independent of the spatial distance between the particles. This was understood as if the information of the measurement of one particle reached the other particle instantaneously *i.e.* exceeding the speed of light in free space. This exhibits the violation of the law of the special theory of relativity. Such non-local behaviour was named as "spooky action at a distance" by Einstein. The negative correlation between the outcomes of the spin measurements in Bohm's experiment is not possible if the concept of locality is taken into consideration. Therefore, according to the local theories, such anti-correlation between the spin measurements along any direction can only be possible, if the particles already acquire pre-existing values.

The set of outcomes of the spin measurement on either of the particles is distinct for different trials of the experiment. If $P(S_a^j)$ is the probability of the predetermined outcome from the measurement done on j -th particle ($j = 1, 2$) along a specific direction a ($a = x, y, z$), one can write the following expression based on the measurement conducted on a single particle along three different directions

$$P(S_x^j = S_y^j) + P(S_y^j = S_z^j) + P(S_z^j = S_x^j) \geq 1, \quad (3.1)$$

provided that S_a^j acquires only two values, +1 (spin-up) and -1 (spin-down). $P(S_a^1 = S_{a'}^1)$ ($a \neq a'$) represents the equality of probability of the spin measurement along two different directions a and a' for the j -th particle. The existence of anti-correlation between the spin values along any specific direction a between the two spin-1/2 particles implies $S_a^1 = -S_a^2$ which further results in $P(S_x^1 = S_y^1) = P(S_x^1 \neq S_y^2)$. From this, the following inequality, known as Bell's theorem, can be written by the following expression

$$P(S_x^i \neq S_y^{j'}) + P(S_y^i \neq S_z^{j'}) + P(S_z^i \neq S_x^{j'}) \geq 1 \quad (3.2)$$

The above inequality given by Eqn. (3.2) is based on the assumption of the locally realistic theory. If one tries to analyze this inequality in terms of quantum mechanics, the local realistic theory seems to be inadequate. In the above mentioned spin-1/2 particle system, the probability of obtaining anti-correlated spin values along different directions is expressed as $(1 + \cos\theta)/2$, θ ($\theta \in [0, \pi]$) being the angle between the two axes along the two different directions for performing

measurements existing in the same plane. This gives the value of $1/4$, if the axes are aligned with each other at an angle $2\pi/3$. The value is smaller than unity and thus it clearly violates the above inequality. Therefore, the quantum mechanics is evidently inexplicable by the notion of the pre-determinant values of the observables. In his original work, Bell defined the correlation function $C(x, y)$ which is equal to the expected value of the product of the outcomes $S_x^1 S_y^1$ in such a way that $C(x, y) = P(S_x^1 S_y^2 = 1) - P(S_x^1 S_y^2 = -1) = 1 - 2P(S_x^1 S_y^2 \neq 1)$. The correlation functions are defined in such a way that the measurement of one system immediately foresees the outcome obtained from the other system. Using this in Eqn. (3.2), Bell's original inequality was obtained, given by [242]

$$C(x, y) + C(y, z) + C(z, x) \leq 1. \quad (3.3)$$

If the measurements executed on any system violate the inequality of Eqn. (3.3), it implies that the system cannot be described within the extent of any local realistic theory. Instead, it requires a quantum mechanical description.

To understand the experimental test of Bell inequality, let us consider two systems 1 and 2 which are spatially separated after being interacted previously. This can result in the codependence of the measurement outcomes obtained from the systems A_1 and A_2 along the directions α_1 and α_2 respectively. However, if the concept of locality is assumed, then it must be the case that any additional randomness that might affect System 1 after it separates from System 2 must be independent of any additional randomness that might affect System 2 after it separates from System 1. Thus, the locality requires the existence of some extra variable λ which can account for the correlation existing between A_1 and A_2 . In other words, λ behaves like an external source, which is common to both the systems and is accountable for the codependency of A_1 and A_2 . This additional quantity λ is interpreted as the *local hidden variable*. The joint probability distribution of the composite system can be defined as

$$P_{\alpha_1, \alpha_2}(A_1, A_2) = \int_{\Lambda} P_{\alpha_1}(A_1|\lambda) P_{\alpha_2}(A_2|\lambda) \rho(\lambda) d\lambda. \quad (3.4)$$

Here the integration takes place over the entire hidden variable space (Λ). The term $P_{\alpha_1(\alpha_2)}(A_1(A_2)|\lambda)$ denotes the probability of obtaining $A_1(A_2)$ as the outcome from the System 1(2) if the axis of detection is chosen along the direction $\alpha_1(\alpha_2)$, $\rho(\lambda)d\lambda$ represents the probability density of finding the correlated results A_1 and A_2 . Based on the concept of the local hidden variables, the CHSH inequality was developed in 1969 by J. Clauser, M. Horn, A. Shimony and R. Holt [243], which is expressed as

$$E(\alpha_1 \alpha_2) - E(\alpha_1 \alpha'_2) + E(\alpha'_1 \alpha_2) + E(\alpha'_1 \alpha'_2) \leq 2. \quad (3.5)$$

Here $E(\alpha_1 \alpha_2) = \int E(\alpha_1|\lambda) E(\alpha_2|\lambda) \rho(\lambda) d\lambda$ represents the expectation value of the product of the outcomes of the experiment A_1 and A_2 along the directions α_1 and α_2 respectively. The violation of CHSH inequality exhibits the non-local behaviour of the system.

3.2 Measures of quantum correlation

In general, the term correlation implies the dependence of certain properties measured in different parts of a composite system. It is essential to understand the nature of correlation present among the subsystems of the main concerned system which play the crucial role in the study of quantum information processing. The concept of non-local correlation represents the most radical contrast to the classical descriptions. However, there are several other measures of quantum correlations that are capable of capturing the quantum effects embedded in the system. The hierarchy of these measures is illustrated in Fig. 3.1. It is clearly shown in the figure that

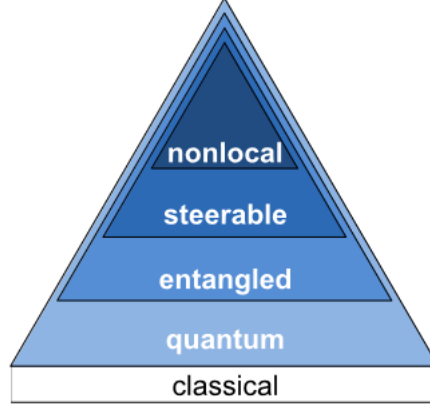


Figure 3.1: Hierarchy in strength of differet kinds of quantum correlation

the non-locality is the strongest measure of non-classicality *i.e.* any composite system that exhibits non-local features, will also exhibit the weaker measures of quantum correlation, known as steering and entanglement. Different aspects of quantum correlations are introduced below.

- **Entanglement:** Entanglement is a form of quantum correlation that was introduced first by Schrodinger in 1935. The idea of entanglement arises due to the superposition of the measurements connected by some quantum correlation so that the measurement performed on one particle is associated with a definite measurement result obtained for the other particle. To understand this, let us consider a system composed of two subsystems, A and B corresponding to two separate Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The resultant Hilbert space available to the composite system is $\mathcal{H}_A \otimes \mathcal{H}_B$. If the composite system can be represented as $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$, it implies that the subsystems A and B are completely independent *i.e.* any measurement performed on A does not affect the result of the measurement done on B . Otherwise, if $|\psi\rangle_{AB}$ cannot be factorised as the direct product, the $|\psi\rangle_A$ and $|\psi\rangle_B$ are said to be entangled. If A and B are entangled, the measurement carried out on A collapses its state $|\psi\rangle_A$ which also alters the original superposition. The example of an entangled state can be given as follows

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A |0\rangle_B). \quad (3.6)$$

Here $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, |1\rangle_B\}$ are the respective set of basis vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$. The measurement result on the whole system is available, but the subsystem measurements are completely random and their separate descriptions do not make any physical meaning. For example, if measurement is performed in the eigenbasis of A and the outcome is $|0\rangle$ ($|1\rangle$), the state $|\phi\rangle_{AB}$ collapses to $|0\rangle_A \otimes |1\rangle_B$ ($|1\rangle_A |0\rangle_B$).

The concept of entanglement is the bedrock of the EPR paradox discussed previously. The entanglement does not disturb the law of causality, as it is a form of the non-local quantum correlation *i.e.* the distant events, which are not causally connected, are correlated. There are several measures that quantify the entanglement embedded in a system. One such measure for describing quantum entanglement is the von-Neumann entropy. Unlike statistical entropy which indicates the presence of randomness in a system, the entropy in information theory represents the uncertainty present in the information *i.e.* the amount of information that can be extracted from the state of a composite system. The entropy of a quantum system is

represented as von Neumann entropy, given by

$$S = -\text{Tr}(\rho \log_2 \rho) = -\sum_i \eta_i \log_2 \eta_i. \quad (3.7)$$

Here ρ is the density matrix and η_i 's are the eigenvalues of ρ . For a two level system, $\rho = |\psi_{AB}\rangle \langle \psi_{AB}|$. The von Neumann entropy gives non-zero values if the two subsystems A and B are entangled.

- **Steering:** Quantum steering is a stronger measure of non-local quantum correlation in comparison with entanglement. The concept of quantum steering is based on the idea that the outcome of a measurement carried out on one subsystem can be manipulated by the measurement on the other subsystem *i.e.* in the composite system $|\psi_{AB}\rangle$ if the measurement of A is influenced by B , then the reverse scenario does not occur. This implies an inherent asymmetric correlation between the subsystems. The influence imposed by one subsystem on the other is understood in terms of a *local hidden state model*. A subsystem is said to be steerable if such a model cannot be constructed [244, 245].
- **Discord:** Quantum discord is a measure of non-classical correlations between two subsystems of a quantum system [246, 247]. It is a weaker measure of quantum correlation which does not impose the necessity of quantum entanglement among the subsystems. For pure states, quantum discord can be put in one-to-one correspondence with entanglement, while in the case of mixed states, it gives a non-zero value, even if the subsystems are not entangled. In other words, it is not essential that if the subsystems are separable, they are classically correlated. It is very difficult to obtain an analytical expression for quantum discord which requires an optimization procedure. To resolve this issue, another measure named geometric discord is introduced [248].

3.2.1 Temporal correlation

The quantum correlation measures discussed earlier are based on the measurement outcomes of spatially separated sections of a system. The non-classical feature of a system can also be realized by taking measurements on a single system at different times, which is termed as the temporal correlation. Such sequential measurements performed on a system reveal whether the dynamics of the system follow a classical or quantum description. In 1985, such a temporal correlation, known as *Leggett-Garg inequality (LGI)* was developed based on the measurements at four different instances [249]. The construction of LGI is based on the assumptions of the classical standpoints: (1) *macrorealism (MR)* *i.e.* the measurement performed on the system only unveils the pre-determined values and (2) *non-invasive measurement (NIM)*, which implies the possibility of measurement without disturbing the system. Thus, the violation of this inequality discloses the non-classicality of the system and excludes the possibility of the existence of any hidden variable theory. Regardless, the imposition of the NIM condition makes it difficult to test the inequality in experiments. Therefore, the NIM assumption is replaced by *stationarity* which implies the dependence of the temporal correlation only on the time difference between two measurements. The modified inequality is known as Leggett-Garg type Inequality (LGtI), which is discussed in detail further in Chapter 4.

3.3 Quantum correlation in neutrino

The measures of quantum correlation are tested in several particle physics systems over the years *e.g.* neutral kaon [250–252] and B meson systems [253, 254]. The neutrino is another interesting candidate to test the fundamental features of quantum mechanics. Several studies on the quantum correlations in the neutrino system are available [255–259]. The neutrinos are

available in three flavours in nature (ν_e , ν_μ and ν_τ) which oscillate among each other due to the presence of their non-zero mass. As discussed in Chapter 1, neutrino oscillation is understood by considering the idea of the flavour and mass eigenstate connected by a unitary transformation. The mass eigenstates interfere due to the presence of quantum coherence among them while the neutrino propagates through the medium which results in the oscillation phenomena. The presence of quantum coherence in the neutrino system over macroscopic distances indicates the existence of *single particle mode entanglement* among the mass eigenstates in a particular neutrino flavour. This insinuates the potential of the neutrino system for the study of quantum correlation. Further, the current and upcoming experimental facilities in the neutrino sector are encouraging to test various correlation measures.

The concept of mode entanglement in a single particle system is elucidated in several works considering different physical systems. Although the unitary equivalence cannot be established between the Fock spaces associated with the flavour and mass eigenstates according to quantum field theory (QFT), as the vacuum for flavour eigenstates cannot be annihilated by the annihilation operators corresponding to the vacuum of the mass eigenstates, in the relativistic limit (> 1 MeV.), the unitarity relation can be restored and the QFT flavour states can be represented as the usual active neutrino flavours [260]. To understand the mode entanglement in the neutrino system, each flavour mode is considered to be associated with the occupation number representation using unitary equivalence defined on the flavour basis, as follows [255, 256, 261]

$$\begin{aligned} |\nu_e\rangle &\equiv |1\rangle_e \otimes |0\rangle_\mu \otimes |0\rangle_\tau = |100\rangle \\ |\nu_\mu\rangle &\equiv |0\rangle_e \otimes |1\rangle_\mu \otimes |0\rangle_\tau = |010\rangle \\ |\nu_\tau\rangle &\equiv |0\rangle_e \otimes |0\rangle_\mu \otimes |1\rangle_\tau = |001\rangle. \end{aligned} \quad (3.8)$$

Here $|0\rangle_\alpha$ and $|1\rangle_\alpha$ represent the absence and presence of a neutrino in a particular mode α respectively. In this representation, a tripartition of the Hilbert space of the quantum states is available, in which the entanglement can be considered. Correspondingly, a neutrino state which is entangled as a two-qubit state is said to be mode entangled.

Similar representations can also be understood in the case of the mass eigenstates as, $|\nu_1\rangle \equiv |1\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$, $|\nu_2\rangle \equiv |0\rangle_1 \otimes |1\rangle_2 \otimes |0\rangle_3$, $|\nu_3\rangle \equiv |0\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3$.

Following this representation formalism, the time evolution of the flavour state, given by Eqn. (1.14), can be expressed as

$$|\nu_\alpha(t)\rangle = \tilde{U}_{\alpha e}(t) |100\rangle + \tilde{U}_{\alpha \mu}(t) |010\rangle + \tilde{U}_{\alpha \tau}(t) |001\rangle, \quad (3.9)$$

where, $\tilde{U} = U e^{-iH_m t} U^\dagger$, U being the PMNS matrix. It is clearly evident from Eqn. (3.9) that the time evolved neutrino flavour state is the entangled superposition of the flavour eigenstates *i.e.* the existence of the entangled flavour modes within a single particle. The corresponding density matrix $\rho_\alpha(t)$ is written as

$$\begin{aligned} \rho_\alpha(t) &= |\tilde{U}_{\alpha e}|^2 |100\rangle \langle 100| + |\tilde{U}_{\alpha \mu}|^2 |010\rangle \langle 010| + |\tilde{U}_{\alpha \tau}|^2 |001\rangle \langle 001| \\ &\quad + \tilde{U}_{\alpha e} \tilde{U}_{\alpha \mu}^* |100\rangle \langle 010| + \tilde{U}_{\alpha e} \tilde{U}_{\alpha \tau}^* |100\rangle \langle 001| + \tilde{U}_{\alpha \mu} \tilde{U}_{\alpha e}^* |010\rangle \langle 100| \\ &\quad + \tilde{U}_{\alpha \mu} \tilde{U}_{\alpha \tau}^* |010\rangle \langle 001| + \tilde{U}_{\alpha \tau} \tilde{U}_{\alpha e}^* |001\rangle \langle 100| + \tilde{U}_{\alpha \tau} \tilde{U}_{\alpha \mu}^* |001\rangle \langle 010| \end{aligned} \quad (3.10)$$

The above state can be understood as an 8×8 matrix and is expressed as [262]

$$|\rho_\alpha(t)\rangle = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\tilde{U}_{\alpha e}|^2 & 0 & \tilde{U}_{\alpha e}\tilde{U}_{\alpha\mu}^* & \tilde{U}_{\alpha e}\tilde{U}_{\alpha\tau}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{U}_{\alpha\mu}\tilde{U}_{\alpha e}^* & 0 & |\tilde{U}_{\alpha\mu}|^2 & \tilde{U}_{\alpha\mu}\tilde{U}_{\alpha\tau}^* & 0 \\ 0 & 0 & 0 & \tilde{U}_{\alpha\tau}\tilde{U}_{\alpha e}^* & 0 & \tilde{U}_{\alpha\tau}\tilde{U}_{\alpha\mu}^* & |\tilde{U}_{\alpha\tau}|^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (3.11)$$

The mode entanglement, and consequently the measures of quantum correlations can be represented in terms of neutrino flavour oscillation probabilities ($|\tilde{U}_{\alpha\beta}|^2$) [257, 259]. Several measures of mode entanglement exhibit sensitivity to the larger mass squared difference taking the matter effect into account according to the SM framework [259]. In the presence of model independent NC NSI the behaviour of the correlation measures is expected to be different, which is discussed in detail in Chapter 4.