

Robustness of Complex Networks Based on Minimum Sensitivity

4.1 INTRODUCTION

In the previous chapter, the relation between network structure and control indices has been presented. In this chapter, a pair of driver node (or control node) and output node will be analyzed based on sensitivity index. For a given pair of input-output nodes, transfer function enables to understand influence of input node on the selected output node. However, to understand robustness with respect to disturbances for a given input-output mapping, sensitivity needs to be analyzed. Sensitivity transfer function helps to understand robustness towards variations in transfer function. This chapter analyzes sensitivity between different node pairs of complex networks and associate it with network structure.

This chapter investigates the correlation between network structure and sensitivity transfer function of the complex networks. Here, the sensitivity transfer function between the pairs of the control and output nodes for random networked systems is analyzed. A relationship between the sensitivity and network properties such as distance between the control and output nodes in the networked systems is obtained. The sensitivity transfer function is related to the sum of product of the weights of all possible shortest path between the control and output nodes of a particular length in the networked system. In addition, the theoretical results are validated with the help of numerical examples. Finally, the proposed methods have been validated by using IEEE test systems such as IEEE-4 bus and IEEE-14 bus power networks.

The remaining sections of the chapter are organized as follows. Section 4.2 briefly reviews the preliminaries to describe the pole-zero-gain transfer function of complex networks and various classical control concepts in order to understand the sensitivity transfer function. Section 4.3 shows the effect of structural properties i.e. sum of product of the weights of all possible shortest path between the control and output nodes of a particular length on the sensitivity transfer function of the networked systems. Numerical examples of interconnected networked system are included in Section 4.4 followed by the standard IEEE test systems such as IEEE-4 bus and IEEE-14 bus power networks in Section 4.5. Finally, a summary of the chapter is presented in Section 4.6.

4.2 PRELIMINARIES

In this section, modeling and pole-zero-gain transfer function of the complex network are described.

The complex network dynamics is described as

$$\dot{x}(t) = Ax(t) + B_c u(t) \quad (4.1a)$$

$$y(t) = C_o x(t) \quad (4.1b)$$

where $A \in \mathbb{R}^{n \times n}$ is the Adjacency matrix of the networked system, $B_c \in \mathbb{R}^{n \times m}$ and $C_o \in \mathbb{R}^{r \times n}$ are input and output matrices respectively, $x \in \mathbb{R}^{n \times 1}$ and $y \in \mathbb{R}^{r \times 1}$ are state and output vectors respectively, and $u \in \mathbb{R}^{m \times 1}$ is control input vector in the networked system.

Here, the Adjacency matrix A of the networked system has eigenvalues $\lambda_i \in \mathbb{R}$. The control node c , output node o , diameter of network and peripheral nodes are used in the subsequent sections of this chapter have already been defined in Chapter 3. In this chapter, the complex network will be referred as networked system henceforth.

When the cardinality of input u_i is $m = 1$ then the generalized form of the transfer function for a given input-output nodes is

$$G_{co}(s) = \frac{N(s)}{D(s)} = k_{gf} \frac{s^{m_z} + b_{m_z-1}s^{m_z-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (4.2)$$

The pole-zero-gain form of Eq.(4.2) becomes

$$G_{co}(s) = \frac{Y(s)}{U(s)} = C_o(sI - A)^{-1}B_c = k_{gf} \frac{\prod_{i=0}^{m_z}(s - z_i)}{\prod_{i=0}^n(s - p_i)} \quad (4.3)$$

where m_z and n represent the number of network zeros and the number of poles respectively, and k_{gf} is the Gain Factor for the networked system.

Consider the standard block diagram of feedback networked system configuration as shown in Figure 4.1.

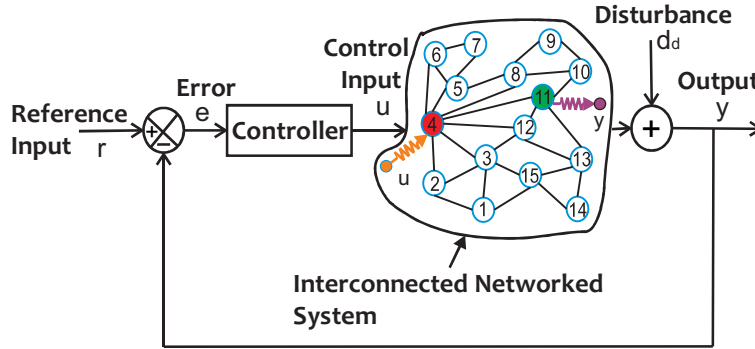


Figure 4.1 : Standard block diagram of feedback networked system configuration

It comprises of the interconnected networked system G and controller K forced by reference input r , and disturbances d_d . The system with transfer function matrices $G_{co}(s)$ (networked transfer function) and $K(s)$ (controller transfer function) is considered. Then the open-loop transfer function $G(s)$ can be denoted as $G(s) = G_{co}(s)K(s)$.

The input-output relationship can be written as

$$y = G(I + G)^{-1}r + (I + G)^{-1}d_d \quad (4.4a)$$

$$e = r - y = (I + G)^{-1}(r - d_d) \quad (4.4b)$$

The closed-loop transfer function with $d_d = 0$

$$M(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)} \quad (4.5)$$

The sensitivity transfer function of the networked system Eq.(4.1) is defined as follows.

Definition 4.1. The sensitivity transfer function (denoted by $NS(s)$) is the ratio of the percentage change in network output variable value $N(s)$ to the percentage change in parameter variation of element $D(s)$, then the sensitivity transfer function for a given input-output pair can be expressed as

$$NS(s)_{D(s)}^{N(s)} = \frac{\% \text{ change in } N(s)}{\% \text{ change in } D(s)} = \frac{\frac{\partial N(s)}{N(s)}}{\frac{\partial D(s)}{D(s)}} \quad (4.6)$$

where $NS(s)_{D(s)}^{N(s)}$ represents the sensitivity of variable $N(s)$ with respect to parameter $D(s)$.

In this chapter, the $NS(s)$ is represented a sensitivity for a given input-output pair in the network. Therefore, sensitivity transfer function $NS(s)_{G(s)}^{M(s)}$ of the overall transfer function $M(s)$ with respect to variation in open-loop transfer function $G(s)$ is defined as

$$NS(s)_{G(s)}^{M(s)} = \frac{\frac{\partial M(s)}{M(s)}}{\frac{\partial G(s)}{G(s)}} = \frac{G(s)}{M(s)} \frac{\partial M(s)}{\partial G(s)} = \frac{1}{1 + G(s)} = (1 + G(s))^{-1} \quad (4.7)$$

In the next section, the effect of Gain Factor k_{gf} and DC-Gain g on the sensitivity transfer function of the networked system defined in Eq.(4.1) is discussed.

4.3 EFFECT OF NETWORK STRUCTURE ON SENSITIVITY

The relationship between the Gain Factor k_{gf} , DC-Gain g and sensitivity transfer function of the complex network is discussed. The sensitivity transfer function $NS(s)$ of the networked system defined in Eq.(4.1) is obtained in terms of the Gain Factor k_{gf} in the next subsection.

4.3.1 Effect of Gain Factor on Sensitivity

At low frequency ($s = 0$), the pole-zero-gain transfer function Eq.(4.2) becomes

$$G_{co}(0) = \frac{N(0)}{D(0)} = k_{gf} \frac{b_0}{a_0} \quad (4.8)$$

where k_{gf} is the Gain Factor for a given input-output nodes. Assume $k_g := |\frac{b_0}{a_0}|$ is constant and $k_g \in \mathbb{R}^+ \cup \{0\}$.

Therefore, the sensitivity transfer function of Eq.(4.1) with Gain Factor k_{gf} can be written as

$$NS(0)_{G(0)}^{M(0)} = NS(0)_{k_{gf}}^{M(0)} = \frac{1}{1 + G_{co}(0)K(o)} = \frac{1}{1 + k_{gf}k_gK(0)} = (1 + k_{gf}k_gK(0))^{-1} \quad (4.9)$$

where k_{gf} is the Gain Factor which is defined in Eq.(4.2) and $K(0) \in \mathbb{R}^+ \cup \{0\}$.

The relation between the network structure i.e. distance between the control node c and output node o of the networked system, the Gain Factor k_{gf} and sensitivity transfer function of the networked systems is presented in Lemma 4.1.

Lemma 4.1. For the complex network defined in Eq.(4.1), for a given pair of input and output nodes the sensitivity at low frequency is defined as

$$NS(0)_{k_{gf}}^{M(0)} = (1 + K(0)k_g \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij})^{-1} \quad (4.10)$$

where p_q is the total number of the paths of the shortest distance d between the control node c and output node o and w_{ij} is weight of the edge j of path i .

Proof: Substituting the value of Gain Factor k_{gf} from Eq.(3.6) (in Chapter 3) into Eq.(4.9), the sensitivity transfer function of the networked system is

$$NS(0)_{k_{gf}}^{M(0)} = \frac{1}{1 + (K(0)k_g \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij})} = (1 + (K(0)k_g \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij}))^{-1} \quad (4.11)$$

Here, $k_{gf} = w = \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij}$, where, p_q is the total number of the paths of the shortest distance d and w_{ij} is weight of the path i of particular edge j of the path. It is clear that the sensitivity transfer function of the networked system depends on the $(o, c)^{th}$ element of A^d , where d is the distance between the control node c and output node o , because the Gain Factor k_{gf} has value of the $(o, c)^{th}$ element of A^d from Theorem 3.1 given in Chapter 3. For better performance and thus minimum sensitivity, the value of the Gain Factor k_{gf} should be high. From Eq.(4.11), it can be inferred that the sensitivity of the networked system reduces on addition of new path between respective nodes of the particular length. Further, it can be inferred that increase in link weights also results in reduction in sensitivity. This completes the proof of Lemma 4.1.

It is a well known fact that large DC-Gain brings better tracking accuracy and disturbance rejection. When the open-loop DC-Gain is large, then the closed-loop DC-Gain is nearly equal to 1. This implies that the sensitivity becomes small. Therefore, the networked system properties such as distance and diameter of the networks with respect to the DC-Gain (denoted by g) and the sensitivity is presented in the next subsection.

4.3.2 Effect of Distance on Sensitivity

The sensitivity transfer function of Eq.(4.1) with DC-Gain g can be written as

$$NS(0)_{G(0)}^{M(0)} = NS(0)_g^{M(0)} = \frac{1}{1 + G_{co}(0)K(0)} = \frac{1}{1 + K(0)|g|} = (1 + K(0)|g|)^{-1} \quad (4.12)$$

where g is the DC-Gain which is defined in Eq.(3.14) (in Chapter 3) and $K(0) \in \mathbb{R}^+ \cup \{0\}$.

From Eq.(4.12), the sensitivity transfer function of the closed-loop networked system depends on the DC-Gain g of the open-loop networked system. It can be observed that the sensitivity transfer function of the closed-loop networked system is reduced by a factor $(1 + K(0)|g|)$ as compared to the open-loop networked system.

Now, the relationship between the network properties i.e. distance between the control node c and output node o of the networked system, the DC-Gain g and sensitivity transfer function for peripheral nodes is obtained.

Lemma 4.2. For the complex network defined in Eq.(4.1), the sensitivity at the low frequency for the peripheral nodes i.e. $d = n - 1$ becomes

$$NS(0)_g^{M(0)} = \left(1 + K(0) \frac{\text{SOP of weights of paths of length } (n-1) \text{ between } o \text{ \& } c}{\det(A)} \right)^{-1} \quad (4.13)$$

where SOP denotes the sum of product.

Proof: From Eq.(4.12), the sensitivity transfer function with DC-Gain g of the networked system is denoted as

$$NS(0)_{G(0)}^{M(0)} = NS(0)_g^{M(0)} = \frac{1}{1 + G_{co}(0)K(0)} = \frac{1}{1 + K(0)|g|} = (1 + K(0)|g|)^{-1} \quad (4.14)$$

where g is the DC-Gain in the networked system. Substituting the value of DC-Gain g from Eq.(3.15) (in Chapter 3) in Eq.(4.14),

$$NS(0)_g^{M(0)} = \frac{1}{1 + K(0) \frac{SOP \text{ of weights of paths of length } (n-1) \text{ between } o \text{ \& } c}{\det(A)}} \quad (4.15)$$

$$= \left(1 + K(0) \frac{SOP \text{ of weights of paths of length } (n-1) \text{ between } o \text{ \& } c}{\det(A)} \right)^{-1}$$

It is clear that sensitivity transfer function of the networked system depends only on the $(o, c)^{th}$ element of $\frac{A^d}{\det(A)}$ when $d = n - 1$ because the DC-Gain g has value of the $(o, c)^{th}$ element of $\frac{A^d}{\det(A)}$ from Theorem 3.1 given in Chapter 3. For better performance and thus minimum sensitivity the absolute value of the DC-Gain g should be high. From Eq.(4.15), it could be inferred that the sensitivity of the networked system reduces on addition of new path between peripheral nodes of $(n - 1)$ length. Further, it could be inferred that increase in link weights also results in reduction of sensitivity. This completes the proof of Lemma 4.2.

In what follows next, numerical examples are analyzed based on sensitivity for a given pair of input-output nodes.

4.4 SIMULATION RESULTS AND DISCUSSION

In this section, four numerical examples are presented. The sensitivity for a given input-output nodes is analyzed based on Gain Factor as well as DC-Gain. In particular, the robustness of the networked system is determined based on sensitivity through these examples.

To understand influence of network topology on sensitivity for a given pair of input-output, the following illustrative examples are used as below.

Example 4.1. Consider two weighted networks of $n = 16$ nodes as shown in Figure 4.2.

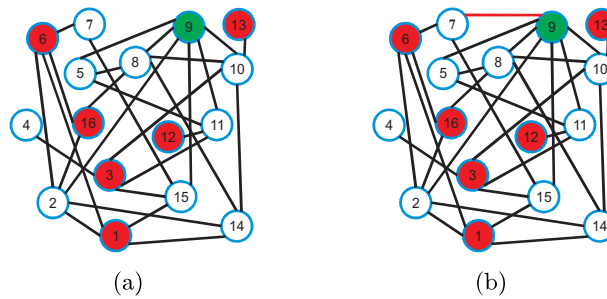


Figure 4.2 : Two weighted networks of $n = 16$ nodes have 6 individual driver nodes which are $\{1, 3, 6, 12, 13, 16\}$ (shown in red color) and one output node which is $\{9\}$ (shown in green color) with the same distance $d = 2$ between a pair of control and output nodes but topology with a single change in topology (a) The original network with 28 edges (b) One edge (shown in red color) between node 7 and node 9 added in original network in Figure 4.2(a)

The original weighted networked system shown in Figure 4.2(a) represents an undirected topology, where, the Adjacency matrix $A \in \mathbb{R}^{16 \times 16}$ is represented as

$$A = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.3 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0 & 1.3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2.2 & 0 & 0 & 0 & 0 & 1.6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 1.5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.2 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 1.4 & 1.3 & 0 & 0 & 0 & 0 & 2.7 & 0 & 3.2 \\ 0 & 0.6 & 0 & 0 & 1.5 & 0 & 0 & 1.4 & 0 & 0.5 & 0.7 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1.3 & 0.5 & 0 & 0 & 0 & 1.9 & 2.8 & 0 & 0 & 0 \\ 0 & 0 & 2.2 & 0 & 3 & 0 & 0 & 0 & 0.7 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 1.2 & 0 & 0 & 0 & 0 & 0 & 2.7 & 0 & 2.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 1.6 & 0 & 0 & 0 & 3.2 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 & 0 & 2.5 & 0 & 3.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the weighted Adjacency matrix $A \in \mathbb{R}^{16 \times 16}$ of modified networked system shown in Figure 4.2(b) is represented as

$$A = \begin{bmatrix} 0 & 0.1 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0.3 & 0 \\ 0.1 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 1.2 & 0 & 1.3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 2.2 & 0 & 0 & 0 & 0 & 1.6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 1.5 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.5 \\ 0 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 3.2 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 1.4 & 1.3 & 0 & 0 & 0 & 0 & 2.7 & 0 & 3.2 \\ 0 & 0.6 & 0 & 0 & 1.5 & 0 & 0.7 & 1.4 & 0 & 0.5 & 0.7 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1.3 & 0.5 & 0 & 0 & 0 & 1.9 & 2.8 & 0 & 0 & 0 \\ 0 & 0 & 2.2 & 0 & 3 & 0 & 0 & 0 & 0.7 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 1.2 & 0 & 0 & 0 & 0 & 0 & 2.7 & 0 & 2.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 1.6 & 0 & 0 & 0 & 3.2 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.3 & 0 & 0 & 0 & 2.5 & 0 & 3.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The input matrix and output matrix of the networked system can be denoted as $B_i \in \mathbb{R}^{16 \times 1}$ and $C_i \in \mathbb{R}^{1 \times 16}$, where the i^{th} element corresponding to the driver node (control node) and output node is 1 and the other entries are 0. This networked system has 6 individual driver nodes which

are $\{1, 3, 6, 12, 13, 16\}$ and one output node which is $\{9\}$. The input matrix B_i , $i = 1, 2, \dots, 6$ can be represented as $B_1, B_2, \dots, B_6$ corresponding to these driver nodes. The input can be applied to these driver nodes and the output can be taken from the node 9.

In these networks, consider the same distance ($d = 2$) between the pair of control node and output node. The topology of both are almost identical except a link of weight 0.7 (shown in red color in Figure 4.2(b)) between nodes 7 and 9 (in Figure 4.2(b) added compared to original Figure 4.2(a)). Now, calculate the sensitivity for different sets of control and output nodes for both the networks. For network shown in Figure 4.2(a), the sensitivity for different sets of control and output nodes is given in Table 4.1. For network shown in Figure 4.2(b), the sensitivity for different sets of control and output nodes is given in Table 4.2. It can be observed that with small change in structure, the sensitivity is different for both the networks. The comparison between the sensitivity calculated for the networks as shown in Figure 4.2 is summarized in Table 4.1 and Table 4.2. From the Table 4.1 and Table 4.2, it can be observed that the sensitivity is minimum corresponding to the pair of control and output nodes $\{12, 9\}$ as compared to other pairs in the networks. It can also be observed that the sensitivity for network shown in Figure 4.2(b) is reduced as compared to the network shown in Figure 4.2(a). This is because addition of a link between node 7 and 9 increases sum of product of weights of path of length 2. Hence, small modification in topology has substantial influence on sensitivity.

Table 4.1 : Sensitivity for different sets of control and output nodes of the networked system shown in Figure 4.2(a)

Set of Control and Output Nodes	Sensitivity NS	$\frac{NS_i - NS_{min}}{NS_{min}}$	Remark
$\{12, 9\}$	0.3586	0	Minimum Sensitivity
$\{1, 9\}$	0.8660	1.414	Increased by 141%
$\{3, 9\}$	1.0	1.788	Increased by 178%
$\{6, 9\}$	0.9174	1.558	Increased by 155%
$\{13, 9\}$	0.7488	1.088	Increased by 108%
$\{16, 9\}$	0.5882	0.640	Increased by 64%

Table 4.2 : Sensitivity for different sets of control and output nodes of the modified networked system shown in Figure 4.2(b)

Set of Control and Output Nodes	Sensitivity NS	$\frac{NS_i - NS_{min}}{NS_{min}}$	Remark
$\{12, 9\}$	0.3386	0	Minimum Sensitivity
$\{1, 9\}$	0.8510	1.513	Increased by 151%
$\{3, 9\}$	1.0	1.953	Increased by 195%
$\{6, 9\}$	0.7690	1.271	Increased by 127%
$\{13, 9\}$	0.7360	1.173	Increased by 117%
$\{16, 9\}$	0.5390	0.591	Increased by 59%

Example 4.2. Consider two weighted networks of $n = 15$ nodes as shown in Figure 4.3.

The original weighted networked system shown in Figure 4.3(a) represents an undirected topology, where, the Adjacency matrix $A \in \mathbb{R}^{15 \times 15}$ is represented as

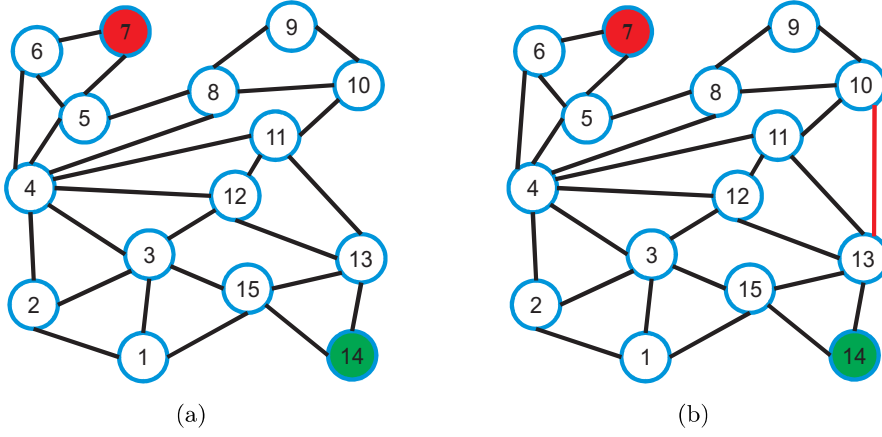


Figure 4.3 : Two weighted networks of $n = 15$ nodes has a pair of input-output nodes which is $\{7, 14\}$ (shown in red and green color respectively) with the same distance $d = 5$ (or $diam = 5$ of the network) between this pair but topology with a single change in topology (a) The original network with 27 edges (b) One edge (shown in red color) between node 10 and node 13 added in original network in Figure 4.3(a)

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 1 & 0 & 2.1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2.1 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0.5 \\ 0 & 3 & 0.1 & 0 & 5 & 2 & 0 & 0.5 & 0 & 0 & 0.6 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 3 & 2 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 6 & 0 & 0 & 0 & 2 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 3 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 2 & 5 & 0 & 0 \\ 0 & 0 & 0.7 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 3 & 0 & 0.2 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 2 \\ 4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 2 & 0 \end{bmatrix}$$

Now, an edge with different weights is added in original network shown in Figure 4.3(a) and modified network is shown in Figure 4.3(b). Therefore, the weighted Adjacency matrix $A \in \mathbb{R}^{15 \times 15}$ of modified networked system shown in Figure 4.3(b) is also represented as above weighted Adjacency matrix with adding the elements $a_{10,13} = a_{13,10} = 5$ for one case and the elements $a_{10,13} = a_{13,10} = 0.4$ for another case.

The input and output matrices of the networked system can be denoted as $B \in \mathbb{R}^{15 \times 1}$ and $C \in \mathbb{R}^{1 \times 15}$, respectively, where the i^{th} element corresponding to the driver node (control node) and output node is 1 and the other entries are 0. This networked system has 1 individual driver node

which is node 7. The input can be applied to this driver node and the output can be taken from node 14.

In these networks, consider the same distance $d = 5$ (or $diam = 5$ of the network) between the pair of control node and output node. The topology of both are almost identical except a link (shown in red color in Figure 4.3(b)) between nodes 10 and 13 (in Figure 4.3(b) added compared to original Figure 4.3(a)). Now, calculate the sensitivity for a set of control and output nodes $\{7, 14\}$ is 0.034. For network shown in Figure 4.3(a), the sensitivity for a set of control and output nodes $\{7, 14\}$ is 0.007 for edge weight 5 and is 0.026 for edge weight 0.4. It can be observed that with small change in structure, the sensitivity is different for both the networks. The comparison between the sensitivity calculated for the networks as shown in Figure 4.3 is summarized as: it can be observed that the sensitivity for network shown in Figure 4.3(b) is reduced as compared to network shown in Figure 4.3(a). This is because the sum of product of weights of path of length 5 is higher in case when link weight is 5 than the case when link weight is 0.4.

Example 4.3. Consider two weighted networks of $n = 21$ nodes as shown in Figure 4.4.

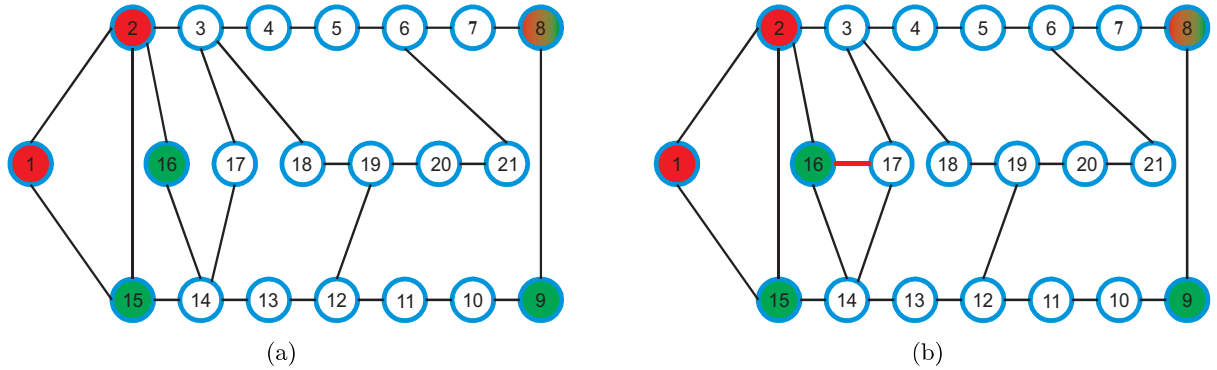


Figure 4.4 : Two weighted networks of $n = 21$ nodes have pairs of input-output nodes which are $\{(1, 8), (1, 9), (2, 9), (8, 15), (8, 16)\}$ (shown in red and green color respectively) with the same distance $d = 7$ (or $diam = 7$ of the network) between these pair but topology with a single change in topology (a) The original network with 26 edges (b) One edge (shown in red color) between node 16 and node 17 added in original network in Figure 4.4(a)

The original weighted networked system shown in Figure 4.4(a) represents an undirected topology, where, the Adjacency matrix $A \in \mathbb{R}^{21 \times 21}$ is represented as

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 5 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0.1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 & 0 & 2 & 4 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 3.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.5 & 0 & 4.5 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 \end{bmatrix}$$

Now, the weighted Adjacency matrix $A \in \mathbb{R}^{21 \times 21}$ of modified networked system shown in Figure 4.4(b) is also represented as above weighted Adjacency matrix with adding the elements $a_{16,17} = a_{17,16} = 7$.

The input and output matrices are defined as $B \in \mathbb{R}^{21 \times 1}$ and $C \in \mathbb{R}^{1 \times 21}$, respectively, where, the i^{th} element corresponding to the driver node (control node) and output node is 1 and the other entries are 0. This networked system has pairs of input-output nodes which are $\{(1,8), (1,9), (2,9), (8,15), (8,16)\}$. The input can be applied to these driver nodes and the output can be taken from the corresponding node, respectively.

In these networks, consider the same distance $d = 7$ (or $diam = 7$ of the network) between the pair of control node and output node. The topology of both are almost identical except a link (shown in red color in Figure 4.4(b)) between nodes 16 and 17 (in Figure 4.4(b) added compared to original Figure 4.4(a)). Now, calculate the sensitivity for different sets of control and output nodes for both the networks. For networks shown in Figure 4.4, the sensitivity for different sets of control and output nodes is given in Table 4.3. It can be observed that with small change in structure, the sensitivity is different for both the networks. The sensitivity may increase or decrease depending on the sum of product of weights of path of length 7 decreases or increases respectively. The comparison between the sensitivity calculated for the networks as shown in Figure 4.4 is summarized in Table 4.3. From the Table 4.3, it can be observed that the sensitivity for network shown in Figure 4.4(b) is different as compared to network shown in Figure 4.4(a). This is because the small modification in topology has substantial influence on sensitivity.

Table 4.3 : Comparison between the sensitivity calculated for different sets of control and output nodes of the networked system shown in Figure 4.4

Set of Control and Output Nodes	Sensitivity for Original Network Shown in Figure 4.4(a)	Sensitivity for Modified Network Shown in Figure 4.4(b)
{1,8}	0.9964	0.9976
{1,9}	0.9453	0.9801
{2,9}	0.8218	0.2887
{8,15}	0.9955	0.9881
{8,16}	0.9970	0.9942

Now the theoretical result is validated with the help of real world Freeman Organizational networked system [Freeman, 1979] with $n = 34$ nodes as given below.

Example 4.4. Consider a Freeman Organizational networked system with $n = 34$ nodes in this example.

The Freeman Organizational networked system represents an undirected topology, where, the Adjacency matrix as $A \in \mathbb{R}^{34 \times 34}$, here, the elements $a_{ij} = 1$ if node i and j are adjacent or otherwise 0 with $i \neq j = 1, 2, \dots, 34$. The input and output matrices are defined as $B \in \mathbb{R}^{34 \times 1}$ and $C \in \mathbb{R}^{1 \times 34}$, respectively, where the i^{th} element corresponding to the driver node (control node) and output node is 1 and the other entries are 0. Here, consider the control node $c = 1$ and output nodes j , where $j = 1, 2, \dots, 34$, one output node at a time is considered to obtain sensitivity. The sensitivity transfer function between the single control node 1 and all nodes considered as the output nodes 1 – 34 with respect to DC-Gain g are plotted in Figure 4.5.

From the Figure 4.5, the sensitivity transfer function of the control and output nodes pair {1,1} is minimum as compared to other pairs of the control 1 and output 1 – 34 nodes in the Freeman Organizational networked system.

4.5 APPLICATION TO POWER NETWORKS

Theoretical results are validated with the help of some real-world IEEE bus test systems such as IEEE-4 bus and IEEE-14 bus power networked systems.

4.5.1 Modeling of Power Networks

Consider a power networked system with l generators and $h > l$ buses which are indexed by $gp_1, gp_2, \dots, gp_l$ and $bp_1, bp_2, \dots, bp_h$, respectively. The generator terminal buses which are indexed $bp_1, bp_2, \dots, bp_l$, where, each bus is connected to exactly one generator and the remaining buses are called load buses numbered from $bp_{l+1}, bp_{l+2}, \dots, bp_h$. Here, the total number of nodes in the power networked system is $n = l + h$.

A power networked system with n nodes can be represented by a graph $G := (V, E)$, where, $V := \{1, 2, \dots, n\}$ and $E \subseteq V \times V$ are the set of nodes and set of edges respectively. The weighted Adjacency matrix of the power networked system G is defined as A , where, the elements $a_{ij} \in \mathbb{R}^+ \cup \{0\}$ are nonzero if node i and node j are adjacent, or are 0 otherwise. It is assumed that the power networked system G is independently controllable by q sets of driver nodes from the superset $N_D = \{N_1, N_2, \dots, N_q\}$, where, each set i.e. N_i , $i = 1, 2, \dots, q$ contains $m \leq n$ nodes and it can control the entire power network. The cardinality of a set of driver nodes is fixed for the power network G . The superset of control inputs corresponding to the superset of driver nodes N_D is defined as

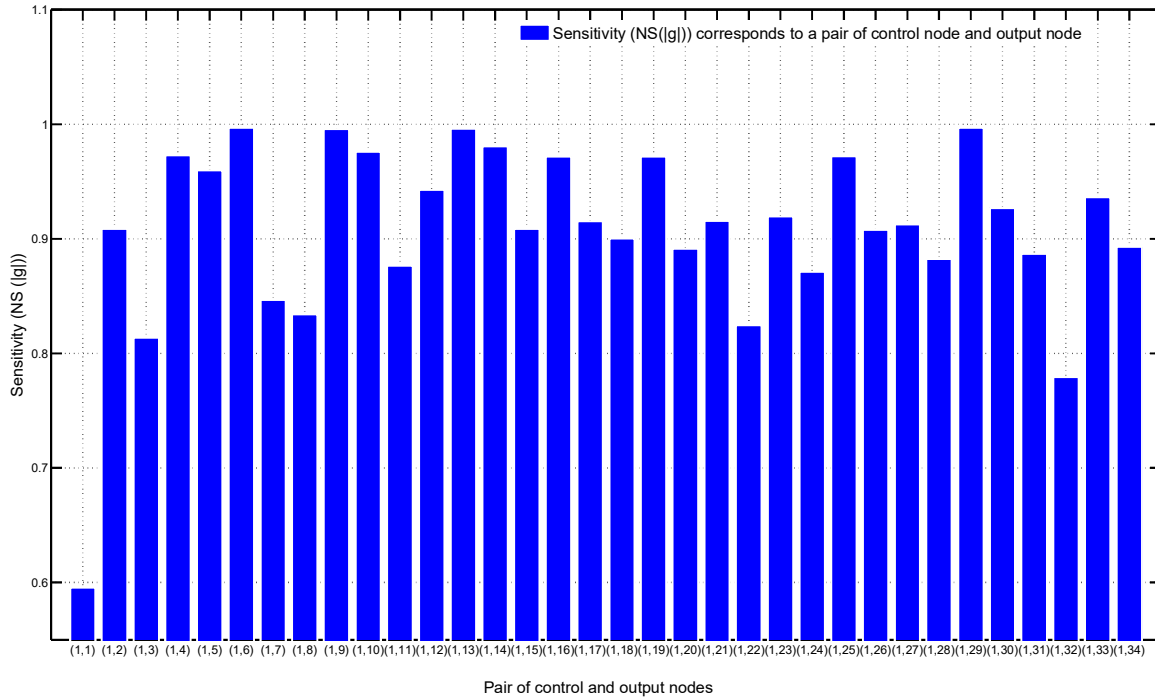


Figure 4.5 : Sensitivity $NS(g)$ of Freeman Organizational network of 34 nodes with respect to DC-Gain g verses all possible combinations of the single control node 1 and all output nodes 1 – 34 considered in Example 4.4.

$\bar{U} := \{U_1, U_2, \dots, U_q\}$, where, $U_i := \{u_{i1}, u_{i2}, \dots, u_{im}\}$, $i = 1, 2, \dots, q$. The cardinality of U_i is m . The set of control inputs U_i injected into the network corresponding to particular set of driver nodes N_i . An input injected node is described by driver node or control node.

Now, the descriptor power networked system dynamics can be described as

$$E\dot{x}(t) = Ax(t) + B_c U \quad (4.16a)$$

$$y(t) = C_o x(t) \quad (4.16b)$$

where $E \in \mathbb{R}^{n \times n}$ is a singular matrix, $A \in \mathbb{R}^{n \times n}$ is the Adjacency matrix of the power networked system, $B_c \in \mathbb{R}^{n \times m}$ and $C_o \in \mathbb{R}^{r \times n}$ are the input and output matrices, respectively, $x \in \mathbb{R}^{n \times 1}$ and $y \in \mathbb{R}^{r \times 1}$ are the state and output vectors, respectively, and $U \in \mathbb{R}^{m \times 1}$ is the input vector in the power networked system. The Eq.(4.16) is the general descriptor power networked system dynamics with control input U .

When the cardinality of input U_i is $m = 1$ then the generalized form of the pole-zero-gain transfer function for a given input-output nodes becomes

$$G_{co}(s) = \frac{Y_o(s)}{U_c(s)} = C_o(sE - A)^{-1}B_c = k_{gf} \frac{\prod_{i=0}^{m_z}(s - z_i)}{\prod_{i=0}^n(s - p_i)} \quad (4.17)$$

where $G_{co}(s)$ is transfer function between the control node c and output node o , m_z and n represent the number of zeros and the number of poles of the transfer function in the power networked system, respectively, and k_{gf} is the open-loop Gain Factor in the power networked system.

In what follows next IEEE bus test systems such as IEEE-4 bus and IEEE-14 bus power networked systems are analyzed for a given input-output nodes. To understand influence of network

topology on sensitivity for a given pair of input-output, the following illustrative Example 4.5 is used as shown in Figure 4.6.

Example 4.5. Consider two weighted IEEE-14 bus power networked system with $n = 14$ nodes shown in Figure 4.6.

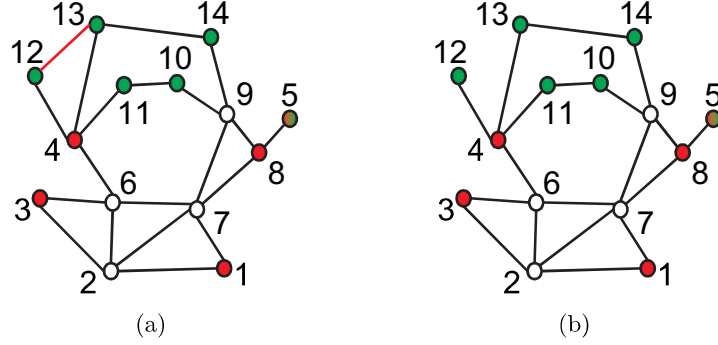


Figure 4.6 : Two weighted IEEE-14 bus power networked systems of $n = 14$ nodes have a pair of input-output nodes with the same distance $d = 4$ between these pair but topology with a single change in topology (a) The original network with 20 edges (b) One edge (shown in red color) between nodes 12 and 13 deleted from original network in Figure 4.6(a)

The original weighted IEEE-14 bus power networked system shown in Figure 4.6(a) represents an undirected topology, where, the weighted Adjacency matrix $A \in \mathbb{R}^{14 \times 14}$ is represented as

$$A = \begin{bmatrix} 0 & 0.54 & 0 & 0 & 0 & 0 & 0.67 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.54 & 0 & 0.14 & 0 & 0 & 0.88 & 1.22 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.14 & 0 & 0 & 0 & 2.55 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3.65 & 0 & 0 & 0 & 0 & 1.33 & 2.41 & 0.66 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.88 & 2.55 & 3.65 & 0 & 0 & 4.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.67 & 1.22 & 0 & 0 & 0 & 4.22 & 0 & 1.77 & 0.55 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 1.77 & 0 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.55 & 5 & 0 & 4.1 & 0 & 0 & 0 & 2.11 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4.1 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.33 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.41 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.99 & 0 \\ 0 & 0 & 0 & 0.66 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.99 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.11 & 0 & 0 & 0 & 7 & 0 \end{bmatrix}$$

Now, the weighted Adjacency matrix $A \in \mathbb{R}^{14 \times 14}$ of modified IEEE-14 bus power networked system shown in Figure 4.6(b) is also represented as above weighted Adjacency matrix with adding the elements $a_{12,13} = a_{13,12} = 0$, because the edge between these nodes is deleted from the original network as shown in Figure 4.6(a).

The input and output matrices of the IEEE-14 bus power networked system can be defined as $B \in \mathbb{R}^{14 \times 1}$, and $C \in \mathbb{R}^{1 \times 14}$, respectively, where, the i^{th} element is 1 and the other entries are 0. Here,

the matrix $E \in \mathbb{R}^{14 \times 14}$ is a singular matrix. This IEEE-14 bus power networked system has pairs of input-output nodes which are $\{(1, 11), (1, 12), (1, 13), (3, 5), (3, 10), (3, 14), (4, 5), (5, 11), (5, 13), (8, 12)\}$. The input can be applied to these driver nodes and the output can be taken from the corresponding node, respectively.

In these networks, consider the same distance ($d = 4$) between the pair of control node and output node. The topology of both are almost identical except a link (shown in red color in Figure 4.6(a)) between nodes 12 and 13 (in Figure 4.6(a) deleted as compared to modified Figure 4.6(b)). Now, calculate the sensitivity for different sets of control and output nodes for both the networks is given in Table 4.4. It can be observed that with small change in structure, the sensitivity is different for both the networks. The sensitivity may increase or decrease depending on the sum of product of weights of path of length 4 decreases or increases respectively. The comparison between the sensitivity calculated for the networks as shown in Figure 4.6 is summarized in Table 4.4. From the Table 4.4, it can be observed that the sensitivity for network shown in Figure 4.6(b) is different as compared to network shown in Figure 4.6(a). This is because the small modification in topology has substantial influence on sensitivity.

Table 4.4 : Comparison between the sensitivity calculated for different sets of control and output nodes of the IEEE-14 bus power networked system shown in Figure 4.6

Set of Control and Output Nodes	Sensitivity for Original Network Shown in Figure 4.6(a)	Sensitivity for Modified Network Shown in Figure 4.6(b)
$\{1, 11\}$	0.6044	0.2815
$\{1, 12\}$	0.6713	0.3901
$\{1, 13\}$	0.7760	0.4704
$\{3, 5\}$	0.2426	0.2130
$\{3, 10\}$	0.9199	1.0
$\{3, 14\}$	0.9527	1.0
$\{4, 5\}$	0.4756	1.0
$\{5, 11\}$	0.2483	0.0866
$\{5, 13\}$	0.4282	0.1770
$\{8, 12\}$	1.0	1.0

Example 4.6. Consider a weighted interconnected IEEE-4 bus power networked system with $n = 4$ nodes shown in Figure 4.7.



Figure 4.7 : An IEEE-4 bus power networked system (a) An IEEE-4 bus test feeder system, and (b) Network representation of the IEEE-4 bus network of $n = 4$ nodes, which has 4 individual driver nodes i.e. $\{1, 2, 3, 4\}$ and the input is applied to the control node (shown in red color) and the output is taken from the output node (shown in green color)

An IEEE-4 bus power networked system shown in Figure 4.7 represents a undirected topology, where, the weighted Adjacency matrix $A \in \mathbb{R}^{4 \times 4}$ and the singular matrix $E \in \mathbb{R}^{4 \times 4}$ are defined

as

$$A = \begin{bmatrix} 0 & 0.733 & 0 & 0 \\ 0.733 & 0 & 0.860 & 0 \\ 0 & 0.860 & 0 & 0.801 \\ 0 & 0 & 0.801 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The input and output matrices are defined as $\mathbf{B} \in \mathbb{R}^{4 \times 1}$, and $\mathbf{C} \in \mathbb{R}^{1 \times 4}$, respectively, where, the i^{th} element is equal to 1 with other entries are 0. This IEEE-4 bus networked system has 4 individual driver nodes which are 1, 2, 3, and 4. The input matrix $\mathbf{B}_i$, $i = 1, 2, \dots, 4$, where, the i^{th} element is 1 and the other entries are 0, can be represented as $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_4$ corresponding to these driver nodes and the output nodes are 1, 2, 3, and 4. The input can be applied to these driver nodes and the output can be taken from the same nodes as well as others. Therefore, there are ten (${}^4C_2 + 4$) possibilities of pairs of the control node and output node, which are $\{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$.

The sensitivity $NS(|g|)$ with respect to DC-Gain g verses all possible combination of the control and output nodes is plotted in Figure 4.8.

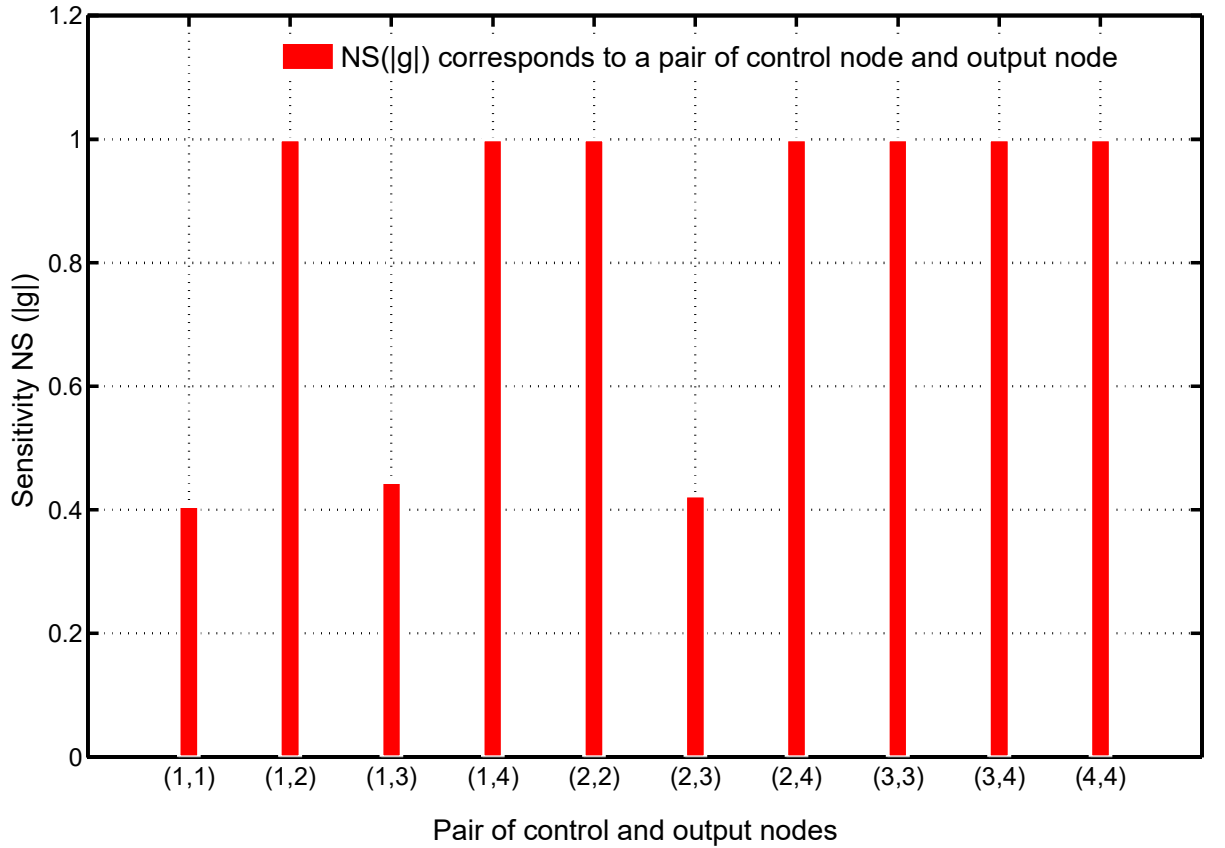


Figure 4.8 : Sensitivity $NS(|g|)$ with respect to DC-Gain g verses all possible pairs of the control and output nodes considered in Example 4.6

The minimum sensitivity is obtained of a pair $\{1,1\}$ corresponding to maximum DC-Gain g as compared to the other pair of the control and output nodes in the IEEE-4 bus power networked system which means the disturbance rejection of this value is better as compared to other pairs.

4.6 SUMMARY

The chapter has investigated the correlation between the network structure and sensitivity of the complex networks. It has been established that the sensitivity is related to the sum of product of weights of all possible shortest paths between the pair of control and output nodes in the networked system. In particular, the effect of the Gain Factor and DC-Gain on the sensitivity transfer function of the networked system has been studied. In addition, the theoretical results have been validated with the help of the numerical examples. Finally, the proposed theory has been applied successfully to analyze the robust properties of the real-world standard IEEE test systems such as IEEE-4 bus and IEEE-14 bus power networks. As a conclusion, addition of links affects the sensitivity of the system. Sensitivity gets reduced by addition of new link. Higher the value of link weight, lower would be the sensitivity and vice-versa. The next chapters will investigate the algorithms to obtain optimal driver nodes (or control nodes). As it is noted about multiple possibilities of driver nodes (control nodes), it is necessary to rank different driver nodes. The next two chapters will discuss this issue.

...