Declaration

I hereby declare that the work presented in this Thesis titled *Graph Theoretic Aspects of Quantum Information Processing* submitted to the Indian Institute of Technology Jodhpur in partial fulfillment of the requirements for the award of the degree of Doctor of Philosophy, is a bonafide record of the research work carried out under the supervision of Dr. Subhashish Banerjee. The contents of this thesis in full or in parts, have not been submitted to, and will not be submitted by me to, any other Institute or University in India or abroad for the award of any degree or diploma.

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Certificate

This is to certify that the thesis titled *Graph Theoretic Aspects of Quantum Information Processing*, submitted by *Supriyo Dutta (P14MT001)* to the Indian Institute of Technology Jodhpur for the award of the degree of *Doctor of Philosophy*, is a bonafide record of the research work done by him under my supervision. To the best of my knowledge, the contents of this report, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

Subhahsish Benerjee
Ph.D.Thesis Supervisor

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List of Symbols

Symbol	Description
$\otimes$	Kronecker product or tensor product
^{t}	Transpose of a matrix or a vector.
†	Conjugate transpose of a matrix or a vector.
$\mathbb{R}$	Set of all real numbers.
$\mathbb{C}$	Set of all complex numbers.
$\mathbb{C}^n$	Complex Hilbert space of dimension n.
G	Combinatorial graph.
$V(G)$	Vertex set of the graph G.
$\|V(G)\|$	Number of vertices in the graph G.
$E(G)$	Edge set of the graph G.
$\|E(G)\|$	Number of edges in the graph G.
$\hat{e}$	Directed edge.
e	Undirected edge.
w_{uv}	Edge weight of the edge (u, v).
$A(G)$	Adjacency matrix of the graph G.
d_v	Degree of the vertex v.
$D(G)$	Degree matrix of the graph G.
$L(G)$	Laplacian matrix of the graph G
$Q(G)$	Signless Laplacian matrix of the graph G
$K(G)$	Laplacian matrix of the graph G in general. We use the symbol when the Laplacian or signless Laplacian matrix is not explicitly mentioned.
$\langle C_\mu \rangle_G$	Induced subgraph of G generated by the vertex set C_μ.
$\langle C_\mu, C_\nu \rangle_G$	Subgraph of G generated by the vertex set $C_\mu \cup C_\nu$ and edges joining vertices in C_μ and C_ν.
$\rho_l(G)$	Density matrix corresponding to the graph G with respect to $L(G)$.
$\rho_q(G)$	Density matrix corresponding to the graph G with respect to $Q(G)$.
$\rho(G)$	Density matrix of the graph G when the Laplacian or signless Laplacian matrix is not explicitly mentioned.
$\mathcal{H}$	Hilbert space.
$\mathcal{H}^{(n)}$	Hilbert space with dimension n.
$\mathcal{H}^{(n)} \otimes \mathcal{H}^{(m)}$	Product Hilbert space of $\mathcal{H}^{(n)}$ and $\mathcal{H}^{(m)}$.
$\mathcal{H}_A$	Hilbert space corresponding to the party A.
$\mathcal{H}_A^{(n)}$	Hilbert space corresponding to the party A with dimension n.
$\|\psi\rangle$	Quantum state vector ψ.
$\langle\psi\|$	Conjugate transpose of $\|\psi\rangle$.
$\langle\phi\|\psi\rangle$	Inner product between $\langle\phi\|$ and $\|\psi\rangle$.
$\|\phi\rangle \langle\psi\|$	Outer product between $\|\phi\rangle$ and $\langle\psi\|$.
$A = (a_{ij})_{m\times n}$	Matrix A of order $m \times n$ whose entries are a_{ij}.
$\Lambda(A)$	Spectra of a matrix A.
$\Lambda(G)$	Spectra of a graph G.