1
Introduction

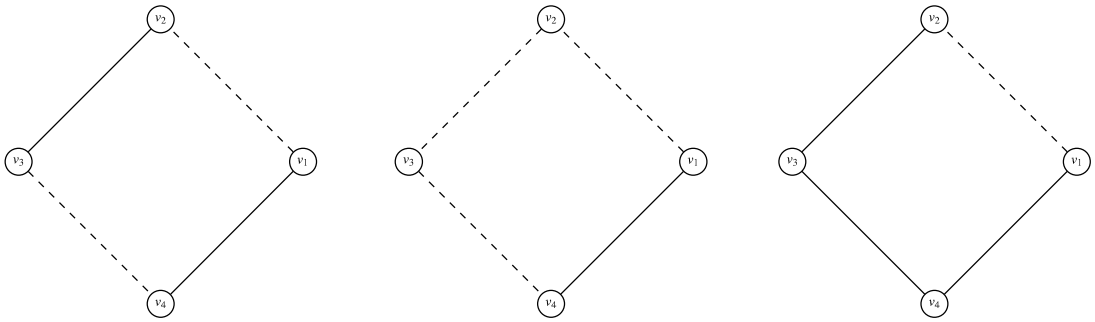
In 1735, the Swiss mathematician Leonhard Euler laid the foundation of graph theory by introducing the famous Konigsberg Bridge problem. In particular, the term 'graph' was first introduced by an English mathematician James Joseph Sylvester in 1878. There is a significant advancement in Graph theory as it emerged as a core branch of mathematics. Interestingly, few mathematicians like Vandermonde, Leibniz, Sylvester who contributed to graph theory also contributed significantly to matrix determinants. In fact matrix theory and linear algebra play an indispensable role in graph theory [Godsil and Royle, 2013; Biggs, 1993; Bapat, 2010]. But, most of these studies are on simple graphs, that is, on graphs whose edges are undirected having weight equal to 1. A positive edge (an edge having positive weight) between two vertices depicts the affinity of these vertices towards each other. In order to represent a system in which affinity and aversion co-exist we need signed graphs. The pioneer work in this direction is due to the Austrian psychologist Fritz Heider in 1946, when he distinguished between two major types of relations, likes and dislikes among human beings. In his seminal work on analysis of balanced cognitive units in the article 'Attitudes and cognitive organization' in 1946 [Heider, 1946], he introduced the concept of the balanced state of a relationship among three social entities. Later, in 1956, the American mathematician Frank Harary modeled the cognitive structure of balance, which is consistent with Heider's concept of balance, by introducing the concept of signed graphs [Cartwright and Harary, 1956; Harary et al., 1953]. Also, every square matrix can be represented in the form of weighted signed graph, also called as Coates digraph. Thus, there is an interplay between matrix theory and these graphs [Brualdi and Cvetkovic, 2008]. The determinant and the permanent are basic concepts in matrix having wide uses, and significant signed graph theoretic interpretation [Harary, 1969; Farrell et al., 2000; Wanless, 2005; Minc, 1984]. Although, mathematically weighted graphs and weighted signed graphs can be considered same but due to the virtue of signs of edges, these graphs display interesting phenomena, hence in the thesis, we use the word weighted signed graph rather than the just weighted graph. Also, the terms graph and network are used interchangeably in the thesis. In all the figures in the thesis, the dark line shows positive edge (weight +1), dotted or dashed line shows negative edge (weight -1).

1.1 MOTIVATION

Strong balance is one of the important aspects of signed networks which is determined by the balance of cycles in a signed graph. A cycle in a signed graph is called strongly balanced when it has an even number of negative edges, otherwise, it is strongly unbalanced [Cartwright and Harary, 1956; Harary et al., 1953], see Figure 1.1. When all the cycles in the signed graphs are balanced then the graph is considered to be strongly balanced, otherwise, it is strongly unbalanced. This leads to the following theorem.

Theorem 1.1. *[Harary et al., 1953] A signed graph is strongly balanced if and only if either all its edges are positive or the vertices can be partitioned into two subsets (or clusters) such that each positive edge joins vertices in the same subset and each negative edge joins vertices in different subsets.*

An example of strongly balanced graph is given in Figure 1.2 (left), where we can check

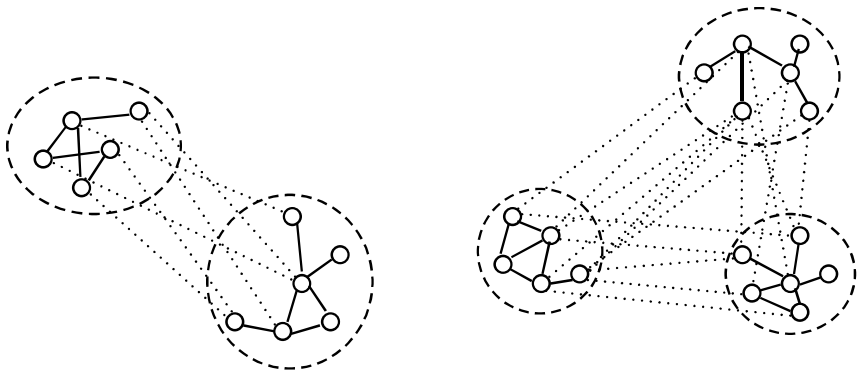


(a) Strongly balanced and also weakly balanced

(b) Strongly unbalanced but weakly balanced

(c) Strongly unbalanced and also weakly unbalanced

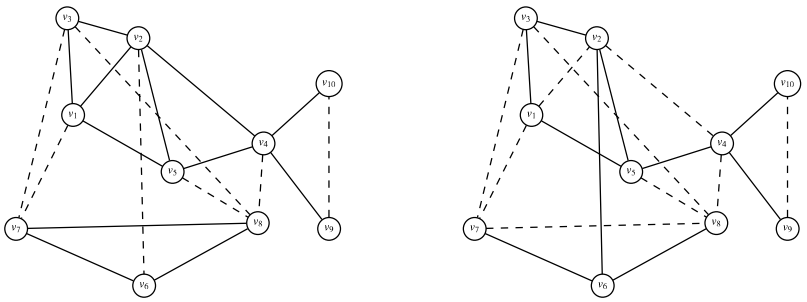
Figure 1.1 : Signed cycles.



Strongly Balanced

Weakly Balanced

Figure 1.2 : Signed graphs. The dotted circles encompass the clusters of vertices. Each edge inside a cluster is positive, while each edge between the clusters is negative. For strongly balanced graph there are exactly two clusters, while for weakly balanced graph there can be more than two clusters.



(a) G_1 (b) G_2

Figure 1.3 : Signed graphs G_1 and G_2 both are strongly unbalanced. The only strongly unbalanced cycle in G_1 is the cycle on vertices v_4, v_9, v_{10}, while G_2 has many strongly unbalanced cycles.

that all the cycles are strongly balanced. But counting all the cycles in a signed graph is an "NP-Complete problem", that is, it can not be solved in polynomial time [Arora and Barak, 2009]. Thus examining all the cycles to determine whether a signed graph is strongly balanced or not, is a challenging task. However, the spectra, that is, the set of the eigenvalues (including multiplicities) of signed graph can be helpful. There is a spectral criteria which can determine whether a given signed graph is strongly balanced or not [Acharya, 1980]. According to the spectral criteria, a signed graph is strongly balanced if and only if it is cospectral with its underlying unsigned graph (graph in which all the negative edges are replaced by positive edges).

An obvious question that arises is, what if only very few of the cycles are not strongly balanced? Indeed consider two signed graphs having the same set of vertex labeled cycles, only differing in signs on the edges. Let us assume that in one, a very small fraction of the cycles is strongly unbalanced while in other a very large fraction of the cycles is strongly unbalanced. Will it be fair to say that both are equally strongly unbalanced? For example, consider two strongly unbalanced signed graphs G_1 and G_2 having the same set of vertex labeled cycles in Figure 1.3. Signed graph G_1 has only one strongly unbalanced cycle (on the vertices v_4, v_9, v_{10}) while G_2 has several. Intuitively, G_1 is less unbalanced as compared to G_2. To outline this, the concept of degree of unbalance came into existence [Cartwright and Harary, 1956]. The measures of the degree of unbalance can be helpful in (i) distinguishing two signed graphs based upon their degree of unbalance, (ii) sign prediction of their edges [Chiang et al., 2014]. The eigenvalues of a signed graph play an important role in developing a measure of the degree of unbalance. In literature, there are such measures with conflicting results [Estrada and Benzi, 2014; Cartwright and Harary, 1956; Pelino and Maimone, 2012]. According to some measures, the real signed networks have a high degree of unbalance [Estrada and Benzi, 2014] while others say that they have low degree of unbalance [Cartwright and Harary, 1956; Pelino and Maimone, 2012]. Thus, distinguishing real signed networks based on the degree of unbalance becomes a problem. This calls for developing a coherent measure for the degree of unbalance which can distinguish signed networks and also give a good sign prediction accuracy.

In 1967, Davis gave the concept of weak balance [Davis, 1967]. All the cycles are considered to be weakly balanced except those which have only one negative edge, see Figure 1.1. And, if all the cycles in signed graph are weakly balanced then the graph is called weakly balanced otherwise, weakly unbalanced, see Figure 1.2. After witnessing the importance of the spectra of strongly balanced graphs it is imperative to analyze the spectra of weakly balanced graphs. Unfortunately, like Acharya's spectral criteria for the strong balance, there is no spectral criteria which can determine whether a signed graph is weakly balanced or not, thus it is an open problem. However, similar to strongly balanced graph, weakly balanced graphs exhibit an interesting phenomenon of graph partitioning as given in following theorem.

Theorem 1.2. *[Davis, 1967] A signed graph is weakly balanced if and only if either all its edges are positive or the vertices can be partitioned into two or more subsets (or clusters) such that each positive edge joins vertices in the same subset and each negative edge joins vertices in different subsets.*

The next task is to exploit this phenomenon of partitioning to find the eigenvalues of some weakly balanced graphs which are induced from a complete graph.

Eigenvalues, determinant, permanent and weighted signed digraphs are closely related. Note that the eigenvalues of a square matrix A of order n are the roots of the characteristic polynomial $\det(A - \lambda I)$, where, det represents the determinant, I is the identity matrix of order n, and values of λ give the eigenvalues. Interestingly, the determinant, and its counterpart, permanent of a square matrix A can be calculated using the linear subdigraphs in the weighted signed digraph G corresponding to A [Brualdi and Cvetkovic, 2008]. These linear subdigraphs

are the union of vertex disjoint directed which partition G and each vertex having indegree and outdegree equal to 1. But as mentioned before, counting the cycles takes non polynomial time, thus calculating the determinant and permanent using linear subdigraph is not efficient. This motivates us to find the subdigraphs (other than linear subdigraphs) in G which can be exploited to calculate the determinant and the permanent in a way to reduce their complexities.

1.2 HIGHLIGHTS OF THE THESIS

In an attempt towards developing a reasonable measure of the degree of unbalance, we first give a measure of the degree of unbalance using the eigenvalues of signed graphs. Later we use this measure for sign prediction in signed graphs. The developed method can be used to distinguish two signed networks based on their degree of unbalance. Also, the sign prediction accuracy of the developed method is better than the state of art methods. We also find that cycles of shorter lengths can predict the sign of an edge in these signed networks better than the longer cycles. Considering weak balance into account, we find the spectrum of some weakly balanced signed graphs induced from complete graphs. As an application of signed digraphs towards matrix determinant and permanent, we exploit the blocks of the digraph corresponding to the matrix. We show that subdigraphs of the blocks which partition the digraph are the handy tools to calculate the characteristic and permanent polynomials of the matrix and hence its determinant and permanent, respectively. This partition is termed as $\mathscr{B}$-partition. We use $\mathscr{B}$-partitions to calculate the determinant and permanent of various graphs without loops. Finally, we analyzed the parameterized complexity of this developed method using the $\mathscr{B}$-partitions for the determinant and permanent. For a class of matrices, the parametrized complexities beat the state of art complexities. Based on the above discussion, we divide the work and present different chapters of this thesis as follows.

1.2.1 Literature review

Chapter 2 presents notation used in thesis and a literature review of the previous work in the context of this study.

1.2.2 Measuring balance of signed networks and its applications to sign prediction.

In Chapter 3, we propose a parameterized walk based measure for lack of balance in signed networks inspired by the Katz measure of similarity of two vertices in a network. We show that the performance of the proposed measure is better than a recently proposed walk based measure of lack of balance for an undirected version of the real world signed networks Epinions, Slashdot and Wiki-election. The proposed measure can be used to distinguish signed social networks on the basis of their degree of lack of balance. We also establish that cycles of shorter lengths can predict the sign of an edge in these signed networks better than the longer cycles by using the Katz prediction rule.

1.2.3 Eigenvalues of some signed graphs with negative cliques.

In a signed graph G, a negative clique is a complete subgraph having negative edges only. Negating the signs of edges in a complete graph with negatives cliques gives a weakly balanced graph. In Chapter 4, we give characteristic polynomial expressions and eigenvalues of some weakly balanced signed graphs. This includes signed cycle graph, signed path graph, a complete graph with disjoint negative cliques, a complete-cycle weakly balanced graph, and complete-path weakly balanced graph.

1.2.4 Characteristic and Permanent polynomials of Matrix.

There is a signed digraph corresponding to every square matrix over $\mathbb{C}$. In Chapter 5 we generate a recurrence relation using the Laplace expansion to calculate the characteristic and the permanent polynomials of a square matrix. Solving this recurrence relation, we found that the characteristic and the permanent polynomials can be calculated in terms of the characteristic and the permanent polynomials of some specific induced subdigraphs of blocks in the digraph, respectively. Interestingly, these induced subdigraphs are vertex-disjoint and they partition the digraph. Similar to the characteristic and the permanent polynomials; the determinant and the permanent can also be calculated. Therefore, this chapter provides a combinatorial meaning to these useful quantities of the matrix theory.

1.2.5 B-partitions and applications to determinant and permanent of graphs.

Let G be a graph (directed or undirected) having k number of blocks. A $\mathscr{B}$-partition of G is a partition into k vertex-disjoint subgraph $(\hat{B}_1, \hat{B}_1, \ldots, \hat{B}_k)$ such that $\hat{B}_i$ is induced subgraph of B_i for $i = 1, 2, \ldots, k$. The terms $\prod_{i=1}^{k} \det(\hat{B}_i)$, $\prod_{i=1}^{k} \mathrm{per}(\hat{B}_i)$ are the det-summands and the per-summands, respectively, corresponding to the $\mathscr{B}$-partition. The determinant (permanent) of a graph having no loops on its cut-vertices is equal to the summation of the det-summands (per-summands), respectively, corresponding to all the possible $\mathscr{B}$-partitions. Thus, in Chapter 6 we calculate the determinant and the permanent of some graphs, which include block graph, block graph with negatives cliques, signed unicyclic graph, mix complete graph, negative mix complete graph, and star mix block graphs.

1.2.6 Parametrized complexity of matrix determinant and permanent.

In Chapter 7, first, we develop an algorithm to find out $\mathscr{B}$-partitions and then found parameterized complexities of matrix determinant and permanent. For a class of matrices, the parameterized complexities beat the complexities of state of art algorithms.

1.2.7 Conclusion and future work

Finally in Chapter 8 we conclude the chapter and outline the future works.