Chapter 6

A general class of shock models with dependent inter-arrival times

We introduce and study a general class of shock models with dependent inter-arrival times of shocks that occur according to the HPGGP. A lifetime of a system affected by a shock process from this class is represented by the convolution of inter-arrival times of shocks. This class contains many popular shock models, namely, the extreme shock model, the generalized extreme shock model, the run shock model, the generalized run shock model, specific mixed shock models, etc. For systems operating under shocks, we derive and discuss the main reliability characteristics (namely, the survival function, the failure rate function, the mean residual lifetime function and the mean lifetime) and study relevant stochastic comparisons. Finally, we provide some numerical examples and illustrate our findings by the application that considers an optimal mission duration policy.

6.1 Introduction

A new point process called the PGGP, has been introduced by Cha and Mercier [2021]. It has a complex mathematical structure, which makes its implementation in applications limited. However, its *special case*, namely, the HPGGP, not only is mathematically tractable but also has many nice statistical properties. The HPGGP possesses the dependent increments property with a general dependence structure and hence, it may be applied to a wider class of problems. Further, the inter-arrival times of this process are also dependent. Moreover, it contains many well-known processes (namely, the HPP, the Pólya process, etc.) as the particular cases. Thus, the HPGGP can be viewed as a more general and flexible process that do not have limitations of the Poisson and the renewal processes.

On the other hand, there exists many popular shock models, which have a great potential in different applications, namely, the run shock model, the generalized run shock model, the generalized extreme shock model, some special mixed shock models. To the best of our knowledge, they have not yet been studied with sufficient generality taking into account the dependent inter-arrival times. Thus, in this chapter, our goal is to develop and study a *unified class* of shock models with dependent inter-arrival times of shocks that occur

according to the HPGGP. Moreover, we assume that the distribution of the fatal shock that causes the system's failure is a DPH distribution, which describes a rather general model of failure. The survival function and other statistical measures (e.g., mean, variance, etc.) of the DPH can be written in matrix forms that are convenient in computations by using the relevant mathematical packages. Consequently, the results to be developed for the proposed class of shock models are also mathematically tractable.

The contribution and novelty of this chapter can be summarized as follows:

(a) We propose the unified class of shock models with the specified dependence structure and a general discrete failure model. This class contains many popular shock models, namely, the run shock model, the generalized run shock model, the generalized extreme shock model, some special mixed shock models, etc;

(b) We assume that shocks occur in accordance with the HPGGP, which is a rather general process with dependent increments and inter-arrival times, whereas the failure model is defined by the DPH distribution;

(c) For the proposed class of shock models, we study some important reliability indices, namely, the survival function, the failure rate function, the mean residual life function and the mean lifetime. Moreover, the derived results are in matrix forms that can easily be calculated using the relevant mathematical packages.

The rest of the chapter is organized as follows. In Section 6.2, we define the class of shock models to be considered. In section 6.3, we derive some important reliability indices and study some stochastic comparisons for the proposed class of shock models. In Section 6.4, we give some numerical examples to illustrate the developed results. Lastly, in Section 6.5, we illustrate our findings by considering the relevant optimal mission duration policy.

6.2 Model Formulation and Specific Cases

Let L be a random variable defining a lifetime of a system incepted into operation at time $t = 0$ and subject to external random shocks. We assume that shocks is the only cause of its failure. Let X_i be a random variable representing the time between the $(i-1)$-th and the i-th shocks, $i \geq 1$. Further, let Y_i be a random variable defining the magnitude (or, the damage caused by) by the i-th shock, $i = 1, 2, 3, \ldots$. Furthermore, let M be a random variable representing the fatal shock, i.e., the number of shocks arrived till the failure of the system. We define now a general class of shock models for which the lifetime of a system can be represented as a random sum of the inter-arrival times of shocks, i.e., $L = \sum_{i=1}^{M} X_i$. We study this class of shock models under the following assumptions.

Assumptions:

- Shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$;
- $M \sim DPH(\boldsymbol{a}, A)$ (see Definition 1.2.19);

- $\{Y_i : i \in \mathbb{N}\}$ is a sequence of iid random variables. Moreover, $\{Y_i : i \in \mathbb{N}\}$ and $\{X_i : i \in \mathbb{N}\}$ are independent;
- M and $\{X_i : i \in \mathbb{N}\}$ are independent.

Below we list some well known shock models that belong to this class.

- **Extreme shock model [Gut and Husler, 1999, 2005]:** In this model,

$$P(M = m) \quad = \quad P(Y_1 \leq \gamma, Y_2 \leq \gamma, \ldots, Y_{m-1} \leq \gamma, Y_m > \gamma) = p(1-p)^{m-1},$$

  where $p = P(Y_1 > \gamma)$. Clearly, $M \sim DPH(1, 1-p)$. Further, note that M and $\{X_i : i \in \mathbb{N}\}$ are independent. Thus, the extreme shock model is a member of the defined class of shock models.

- **Generalized extreme shock models-I and II [Bozbulut and Eryilmaz, 2020]:** In these models, it is assumed that shocks may arrive from more than one possible sources (say, n sources). Let θ_i be the probability that shocks come from source i, and let p_i be the probability that the magnitude of a shock from source i exceeds the critical level γ; $i = 1, 2, \ldots, n$. In both models, a system fails upon occurrence of a shock of size, at least γ. Thus, shocks that come from source i are harmless with probability $1-p_i$; $i = 1, 2, \ldots, n$. In Model-1, it is assumed that shocks may arrive from different sources over time, i.e., two consecutive shocks may come from two different sources. Note that, in this model, $M \sim DPH(1, 1-p)$, where $p = \sum_{i=1}^{n} \theta_i p_i$ and $\sum_{i=1}^{n} \theta_i = 1$ [see Bozbulut and Eryilmaz, 2020]. In Model-2, it is assumed that shocks initially come from a specified source, say i, and after the i-th source completes its task, shocks continue to arrive from another source. In this case, $M \sim DPH(\boldsymbol{a_{gem}}, A_{gem})$, where $\boldsymbol{a_{gem}} = (\theta_1, \theta_2, \ldots, \theta_n)$, $A_{gem} = diag(1-p_1, 1-p_2, \ldots, 1-p_n)$ and $\sum_{i=1}^{n} \theta_i = 1$. If $n = 1$, then both models reduce to the classical extreme shock model. Clearly, in both models, M follows the discrete PH distribution, and M and $\{X_i : i \in \mathbb{N}\}$ are independent. Therefore, these two models belong to the defined class of shock models.

- **Run shock model [Mallor and Omey, 2001]:** In this model, $M \sim DPH(\boldsymbol{a_{rm}}, A_{rm})$, where $\boldsymbol{a_{rm}} = (1, 0, \ldots, 0)_{1\times r}$ and

$$A_{rm} = \begin{pmatrix} 1-p & p & 0 & \cdots & 0 \\ 1-p & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p & 0 & 0 & \cdots & p \\ 1-p & 0 & 0 & \cdots & 0 \end{pmatrix}_{r\times r}$$

  (see Subsection 1.2.3 for mathematical details of this model). Consequently, the run shock model belongs to the defined class of shock models.

- **Mixed shock model [Eryilmaz and Tekin, 2019]:** In this model, the classical run and extreme shock models are combined. Let γ_1 and γ_2 be two fixed threshold values such that $\gamma_1 < \gamma_2$. Here, the failure of a system takes place when either r

consecutive shocks of size, at least γ_1 or a single shock of size, at least γ_2 occur(s). In this case, $M \sim DPH(\boldsymbol{a_{mm}}, A_{mm})$, where $\boldsymbol{a_{mm}} = (1, 0, \ldots, 0)_{1\times r}$ and

$$A_{mm} = \begin{pmatrix} 1-p_1-p_2 & p_1 & 0 & \cdots & 0 \\ 1-p_1-p_2 & 0 & p_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-p_1-p_2 & 0 & 0 & \cdots & p_1 \\ 1-p_1-p_2 & 0 & 0 & \cdots & 0 \end{pmatrix}_{r\times r},$$

with $p_1 = P(\gamma_1 \le Y_i < \gamma_2)$ and $p_2 = P(Y_i \ge \gamma_2)$ for $i = 1, 2, \ldots$. Consequently, this model also belongs to the defined class of shock models.

- **Generalized run shock model-I [Gong *et al.*, 2018]:** Consider two critical levels γ_1 and γ_2 such that $\gamma_1 < \gamma_2$. For given two positive integers r_1 and r_2, the system is assumed to fail if, at least r_1 consecutive shocks with magnitude above γ_1 or r_2 consecutive shocks with magnitude above γ_2 occur. It is obvious that $r_1 > r_2$. Let $p_1 = P(Y_i \le \gamma_1)$, $p_2 = P(\gamma_1 < Y_i \le \gamma_2)$ and $p_3 = P(Y_i > \gamma_2)$, $i = 1, 2, 3, \ldots$. Then $M \sim DPH(\boldsymbol{a_{grm1}}, A_{grm1})$, where $\boldsymbol{a_{grm1}} = (1, 0, \ldots, 0)$ and

$$A_{grm1} = \begin{pmatrix} E_{r_1}+S_{r_1} & T_{r_1} & & & \\ E_{r_1-1} & S_{r_1-1} & T_{r_1-1} & & \\ E_{r_1-2} & & S_{r_1-2} & \ddots & \\ \vdots & & & \ddots & T_{r_1-r_2+2} \\ E_{r_1-r_2+1} & & & & S_{r_1-r_2+1} \end{pmatrix},$$

$$E_j = \begin{pmatrix} p_1 & 0 & \ldots & 0 \\ p_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_1 & 0 & \ldots & 0 \end{pmatrix}_{j\times r_1}, \quad S_j = \begin{pmatrix} 0 & p_2 & \ldots & 0 \\ 0 & \vdots & \ddots & 0 \\ \vdots & 0 & \ldots & p_2 \\ 0 & 0 & \ldots & 0 \end{pmatrix}_{j\times j}, \quad T_j = \begin{pmatrix} p_3 & \ldots & 0 \\ \vdots & \ddots & 0 \\ 0 & \ldots & p_3 \\ 0 & \ldots & 0 \end{pmatrix}_{j\times j-1}.$$

Clearly, this model belongs to the defined class.

- **Generalized run shock model-II [Yalcin *et al.*, 2018]:** In this model, a system fails due to the occurrences of n non-overlapping runs of r consecutive critical shocks. This model can be useful for describing the performance of the n cold standby identical systems because, in this case, a system needs to be replaced with a new one after the occurrence of r consecutive critical shocks. For this model, $M \sim DPH(\boldsymbol{a_{grm2}}, A^{(n)}_{grm2})$, where

$$\boldsymbol{a_{grm2}} = (1, 0, \ldots, 0)_{1\times nr}, \quad A^{(n)}_{grm2} = \begin{pmatrix} A^{(n-1)}_{grm2} & S \\ 0 & T \end{pmatrix}, \quad n \ge 3,$$

and $A_{grm2}^{(n)}$ is an $nr \times nr$ matrix with

$$A_{grm2}^{(2)} = \begin{pmatrix} 1-p & p & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 1-p & 0 & p & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1-p & 0 & 0 & \dots & 0 & p & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1-p & p & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1-p & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1-p & 0 & \dots & 0 \end{pmatrix}_{2r \times 2r}.$$

Further, the matrices S and T are given by

$$S = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \vdots & 0 \\ p & 0 & 0 & \dots & 0 \end{pmatrix}_{(n-1)r \times r}, \quad T = \begin{pmatrix} 1-p & p & 0 & \dots & 0 \\ 1-p & 0 & p & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 1-p & 0 & 0 & \dots & 0 \end{pmatrix}_{r \times r}.$$

Thus, this model also belongs to our class.

The suggested in this chapter class of shock models is general, but obviously, it does not contain all existing models. The following examples illustrate this reasoning:

- **Cumulative shock model [A-Hameed and Proschan, 1973]:** In this model,

$$P(M = m) = P\left(\sum_{j=1}^{m-1} Y_j \le \gamma, \sum_{j=1}^{m} Y_j > \gamma\right) = F^{(m-1)}(\gamma) - F^{(m)}(\gamma),$$

  where $F^{(m)}(\cdot)$ is the cdf of $\sum_{j=1}^{m} Y_j$ (see Subsection 1.2.3 for more details). In general, M does not follow the discrete PH distribution and hence, the cumulative shock model is not a member of the defined class of shock models. However, if the distribution of M follows the discrete PH distribution, for a particular choice of $F^{(m)}(\cdot)$, then the cumulative shock model will also be a member of the defined class of shock models.

- **δ-shock model [Li and Kong, 2007]:** In this model, $P(M = m) = P(X_1 > \delta, X_2 > \delta, \dots, X_{m-1} > \delta, X_m \le \delta)$ (see Subsection 1.2.3 for mathematical details). Clearly, M and $\{X_i : i \in \mathbb{N}\}$ are *not independent*. Thus, the δ-shock model does not belong to our class.

6.3 Survival of Systems Under Shocks

In this section, we discuss some reliability indices (namely, the survival function, the failure rate function, the mean residual lifetime function and the mean lifetime) for the defined class of shock models. We also discuss some relevant stochastic comparisons.

6.3.1 Survival function

We start with three lemmas that will be used in proving the main results of this subsection. The proof of the first lemma follows from the fact that the inter-arrival times of the HPP are iid and follow the exponential distribution (see Eryilmaz [2017a]). The second lemma is borrowed from Osgood [2019].

Lemma 6.3.1 *Let $M \sim DPH(\boldsymbol{a}, A)$. Assume that shocks occur according to the HPP with intensity $\theta > 0$. Then the corresponding survival function for this model is given by $\bar{F}_L(t) = \boldsymbol{a}\exp\{-\theta t(I - A)\}\boldsymbol{e}$.* □

Before stating the next lemma, we give the formal definition of the Fourier transform of a function [see Osgood, 2019].

Definition 6.3.1 *Let $g : \mathbb{R} \to \mathbb{R}$. Then the Fourier transform of g, denoted by $\mathcal{F}[g](s)$, is given by*

$$\mathcal{F}[g](s) = \int_{-\infty}^{\infty} \exp\{-2\pi ist\}g(t)dt, \text{ for } s \in \mathbb{R},$$

provided that the integral exists.

Lemma 6.3.2 *[Osgood, 2019] The following results hold.*

(*i*) *Let $g(t) = \exp\{-ct\}I_0(t)$, where*

$$I_0(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

and c is a positive constant. Then $\mathcal{F}[g](s) = 1/(2\pi is + c)$;

(*ii*) *$\mathcal{F}[cg](s) = c\mathcal{F}[g](s)$, where c is a constant number (real or complex);*

(*iii*) *$\frac{d^k}{ds^k}\mathcal{F}[g](s) = \mathcal{F}[(-2\pi it)^k g(t)](s)$.*

Lemma 6.3.3 *Let $\epsilon \in \mathbb{C}$ such that $Re(\epsilon) > 0$. Then, for any $n \in \mathbb{N}$,*

$$\int_0^{\infty} \exp\{-(\epsilon x + 1)q\}q^{n-1}dq = \Gamma(n)\left(\frac{1}{1+\epsilon x}\right)^n, \quad x \geq 0.$$

Proof: Let $\epsilon = u + iv$, where $u > 0$ and $i = \sqrt{-1}$. Then

$$\begin{aligned}
\int_0^{\infty} \exp\{-(\epsilon x + 1)q\}q^{n-1}dq &= \int_0^{\infty} q^{n-1}\exp\{-(1+ux)q\}\exp\{-ivxq\}dq \\
&= \frac{1}{(-2\pi i)^{n-1}}\mathcal{F}((-2\pi iq)^{n-1}\exp\{-(1+ux)q\}I_0(q))(vx/2\pi) \\
&= \frac{1}{(-2\pi i)^{n-1}}\frac{d^{n-1}}{ds^{n-1}}\mathcal{F}(\exp\{-(1+ux)q\}I_0(q))(s)\Big|_{s=vx/2\pi} \\
&= \frac{1}{(-2\pi i)^{n-1}}\frac{d^{n-1}}{ds^{n-1}}\left(\frac{1}{2\pi is + (1+ux)}\right)\Big|_{s=vx/2\pi}
\end{aligned}$$

$$= \Gamma(n)\left(\frac{1}{2\pi is+(1+ux)}\right)^n\Big|_{s=vx/2\pi} = \Gamma(n)\left(\frac{1}{ivx+(1+ux)}\right)^n$$
$$= \Gamma(n)\left(\frac{1}{1+\epsilon x}\right)^n,$$

where the second equality follows from Definition 6.3.1 and Lemma 6.3.2(ii); the third equality follows from Lemma 6.3.2 (iii), and the fourth equality follows from Lemma 6.3.2(i). Moreover, the function $I_0(\cdot)$ is the same as in Lemma 6.3.2(i). □

It should be noted, that although some of our results look cumbersome, their structure is probabilistically rather simple. Besides, they are given in the matrix form that is useful for computations in applications (see later). In the following theorem, the survival function of a system under the considered shock process is derived. Note that, there are no assumptions on the structure of matrix A (see Lemma 6.3.1) for the shock processes with independent inter-arrival times (e.g., the HPP). However, when the shock processes with dependent inter-arrival times (e.g., PGGP,/Polya process) are considered, some sufficient conditions on the eigenvalues of the matrix A are implemented. These conditions are feasible and easy to verify. Moreover, they hold for aforementioned specific shock models (see examples in Section 6.4).

Theorem 6.3.1 *Let $M \sim DPH(\boldsymbol{a}, A)$, where A is a square matrix of size d. Further, let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be eigenvalues of the matrix $I-A$ with the algebraic multiplicity $d_1, d_2, \ldots, d_r$, respectively, and let P be a non-singular matrix such that $I-A = Pdiag(J_1, J_2, \ldots, J_r)P^{-1}$, where J_i is the ϵ_i-Jordan block with dimension d_i, $i = 1, 2, \ldots, r$, and $\sum_{i=1}^r d_i = d$.*

(i) *Assume that shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. If every eigenvalue of the matrix $I-A$ is real and positive, then the corresponding survival function for a system under the considered shock process is*

$$\bar{F}_L(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\boldsymbol{a}Pdiag(D_1(t), D_2(t), \ldots, D_r(t))P^{-1}\boldsymbol{e}, \tag{6.3.1}$$

where

$$D_i(t) = \begin{pmatrix} \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \frac{-\lambda t\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-1}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\ 0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} \end{pmatrix}_{d_i\times d_i},$$

for $t \geq 0$, $i = 1, 2, \ldots, r$;

(ii) *Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta \in \mathbb{N}$. If $Re(\epsilon_i) > 0$, for all $i = 1, 2, \ldots, r$, then*

$$\bar{F}_L(t) = \frac{1}{\Gamma(\beta)}Pdiag(\widetilde{D}_1(t), \widetilde{D}_2(t), \ldots, \widetilde{D}_r(t))P^{-1}\boldsymbol{e},$$

where

$$\widetilde{D}_i(t) = \begin{pmatrix} \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & -\Gamma(\beta+1)\left(\frac{t}{t\epsilon_i+b}\right)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & \dots & \frac{\Gamma(\beta+d_i-1)}{(d_i-1)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-1}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \\ 0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & \dots & \frac{\Gamma(\beta+d_i-2)}{(d_i-2)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-2}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \end{pmatrix}_{d_i\times d_i},$$

for $t \geq 0$, $i = 1, 2, \dots, r$. *Moreover, if* $Im(\epsilon_i) = 0$, *for all* $i = 1, 2, \dots, r$, *then the above result holds for all* $\beta \in \mathbb{R}$.

Proof:

(i) Consider

$$\bar{F}_L(t) = P\left(\sum_{i=1}^{M} X_i > t\right) = \int_0^{\infty} P\left(\sum_{i=1}^{M} X_i > t | Q = q\right) f_Q(q)dq, \tag{6.3.2}$$

where Q is the mixing distribution of the HPGGP. Note that, for given $Q = q$, the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$ is the same as the HPP with intensity $q\lambda$. Then, from Lemma 6.3.1, we get

$$P\left(\sum_{i=1}^{M} X_i > t | Q = q\right) = \boldsymbol{a}\exp\{-q\lambda t(I - A)\}\boldsymbol{e}. \tag{6.3.3}$$

By using Eq. (6.3.3) and Eq. (1.2.1) in Eq. (6.3.2), we get

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^{\infty} (\boldsymbol{a}\exp\{-q\lambda t(I - A)\}\boldsymbol{e}) \frac{q^{k-1}\exp\{-\alpha q\}}{(q + l)^{\nu}} dq. \tag{6.3.4}$$

It is given that $I - A$ is in the Jordan normal form, i.e., $I - A = Pdiag(J_1, J_2, \dots, J_r)P^{-1}$, for some r, with

$$J_i = \begin{pmatrix} \epsilon_i & 1 & 0 & \dots & 0 \\ 0 & \epsilon_i & 1 & \dots & 0 \\ 0 & 0 & \epsilon_i & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \dots & \epsilon_i \end{pmatrix}_{d_i\times d_i}, \quad \epsilon_i > 0, i = 1, 2, \dots, r.$$

Moreover, the dimension of d_i is equal to the multiplicity of the eigenvalue ϵ_i that belongs to this particular Jordan block, and $\sum_{i=1}^{r} d_i = d$. Therefore,

$$\exp\{-q\lambda t(I - A)\} = Pdiag(\exp\{-q\lambda tJ_1\}, \exp\{-q\lambda tJ_2\}, \dots, \exp\{-q\lambda tJ_r\})P^{-1}, \tag{6.3.5}$$

where

$$\exp\{-q\lambda tJ_i\} = \begin{pmatrix} \exp\{-q\lambda t\epsilon_i\} & -q\lambda t\exp\{-q\lambda t\epsilon_i\} & \dots & \frac{(-1)^{d_i-1}(q\lambda t)^{d_i-1}}{(d_i-1)!}\exp\{-q\lambda t\epsilon_i\} \\ 0 & \exp\{-q\lambda t\epsilon_i\} & \dots & \frac{(-1)^{d_i-2}(q\lambda t)^{d_i-2}}{(d_i-2)!}\exp\{-q\lambda t\epsilon_i\} \\ 0 & 0 & \dots & \frac{(-1)^{d_i-3}(q\lambda t)^{d_i-3}}{(d_i-3)!}\exp\{-q\lambda t\epsilon_i\} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \exp\{-q\lambda t\epsilon_i\} \end{pmatrix}_{d_i\times d_i},$$

for $i = 1, 2, \ldots, r$. Again, from Eq. (1.2.2), we can write

$$\int_0^\infty \exp\{-q(\lambda t\epsilon_i + \alpha)\}\frac{q^{n+k-1}}{(q+l)^\nu}dq = \frac{\Gamma_\nu(n+k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}},$$

for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N} \cup \{0\}$. Consequently,

$$\begin{aligned}
&\int_0^\infty \exp\{-q\lambda t J_i\}\exp\{-q\alpha\}\frac{q^{k-1}}{(q+l)^\nu}dq \\
&= \begin{pmatrix}
\frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \frac{-\lambda t\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-1}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\
0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}} \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \frac{\Gamma_\nu(k,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k-\nu}}
\end{pmatrix}_{d_i\times d_i} \\
&= D_i(t), \quad i = 1, 2, \ldots, r. \qquad (6.3.6)
\end{aligned}$$

Now, by using Eq. (6.3.5) and Eq. (6.3.6) in Eq. (6.3.4), we get

$$\bar{F}_L(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\boldsymbol{a}Pdiag(D_1(t), D_2(t), \ldots, D_r(t))P^{-1}\boldsymbol{e}$$

and hence, the result is proved.
(*ii*) By proceeding in the same line as in part (*i*), we get, from Remark 1.2.4 (*c*), that

$$\bar{F}_L(t) = \frac{1}{\Gamma(\beta)}\int_0^\infty (\boldsymbol{a}\exp\{-(qt/b)(I-A)\}\boldsymbol{e})\, q^{\beta-1}\exp\{-q\}dq, \qquad (6.3.7)$$

where

$$\exp\{-(qt/b)(I-A)\} = Pdiag(\exp\{-(qt/b)J_1\}, \exp\{-(qt/b)J_2\}, \ldots, \exp\{-(qt/b)J_r\})P^{-1}$$

and

$$\exp\{-(qt/b)J_i\} = \begin{pmatrix}
\exp\{-(qt/b)\epsilon_i\} & -(qt/b)\exp\{-(qt/b)\epsilon_i\} & \ldots & \frac{(-1)^{d_i-1}(qt/b)^{d_i-1}}{(d_i-1)!}\exp\{-(qt/b)\epsilon_i\} \\
0 & \exp\{-(qt/b)\epsilon_i\} & \ldots & \frac{(-1)^{d_i-2}(qt/b)^{d_i-2}}{(d_i-2)!}\exp\{-(qt/b)\epsilon_i\} \\
0 & 0 & \ldots & \frac{(-1)^{d_i-3}(qt/b)^{d_i-3}}{(d_i-3)!}\exp\{-(qt/b)\epsilon_i\} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \exp\{-(qt/b)\epsilon_i\}
\end{pmatrix}_{d_i\times d_i},$$

for $i = 1, 2, \ldots, r$. Again, if $\beta \in \mathbb{N}$, then we get, from Lemma 6.3.3, that

$$\int_0^\infty \exp\{-(\epsilon_i(t/b)+1)q\}q^{n+\beta-1}dq = \Gamma(\beta+n)\left(\frac{b}{b+\epsilon_i t}\right)^{\beta+n},$$

for all $i = 1, 2, \ldots, r$ and $n \in \mathbb{N} \cup \{0\}$. Consequently, we can write

$$\int_0^\infty \exp\{-(qt/b)J_i\}\exp\{-q\}q^{\beta-1}dq$$

$$= \begin{pmatrix} \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & -\Gamma(\beta+1)\left(\frac{t}{t\epsilon_i+b}\right)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & \cdots & \frac{\Gamma(\beta+d_i-1)}{(d_i-1)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-1}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \\ 0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} & \cdots & \frac{\Gamma(\beta+d_i-2)}{(d_i-2)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-2}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \Gamma(\beta)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta} \end{pmatrix}_{d_i\times d_i},$$

$$= \widetilde{D}_i(t), \quad i = 1, 2, \ldots, r.$$

By using the above equality in Eq. (6.3.7), we get

$$\bar{F}_L(t) = \frac{1}{\Gamma(\beta)} P diag(\widetilde{D}_1(t), \widetilde{D}_2(t), \ldots, \widetilde{D}_r(t)) P^{-1}\boldsymbol{e}$$

and hence, the result is proved for $\beta \in \mathbb{N}$. Now, if $Im(\epsilon_i) = 0$, then ϵ_i is real and positive for all $i = 1, 2, \ldots, r$. Then, by using part (i) and Remark 1.2.4c, we get the required result.□

The following corollary immediately follows from Theorem 6.3.1, Remark 1.2.4(c) and the fact that all Jordan blocks of a diagonalizable matrix are of size 1.

Corollary 6.3.1 *Let $I-A$ is a diagonalizable matrix such that $I-A = P diag(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)P^{-1}$.*

(i) Assume that shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. If ϵ_i's are real and positive, then

$$\bar{F}_L(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)}\right) \boldsymbol{a} P D(t) P^{-1}\boldsymbol{e},$$

where

$$D(t) = diag\left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_1 + \alpha)l)}{(\lambda t\epsilon_1 + \alpha)^{k-\nu}}, \frac{\Gamma_\nu(k, (\lambda t\epsilon_2 + \alpha)l)}{(\lambda t\epsilon_2 + \alpha)^{k-\nu}}, \ldots, \frac{\Gamma_\nu(k, (\lambda t\epsilon_d + \alpha)l)}{(\lambda t\epsilon_d + \alpha)^{k-\nu}}\right), \quad t \geq 0;$$

(ii) Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta \in \mathbb{N}$. If $Re(\epsilon_i) > 0$, for all $i = 1, 2, \ldots, d$, then

$$\bar{F}_L(t) = \boldsymbol{a} P \widetilde{D}(t) P^{-1}\boldsymbol{e},$$

where

$$\widetilde{D}(t) = diag\left(\left(\frac{b}{t\epsilon_1 + b}\right)^{\beta}, \left(\frac{b}{t\epsilon_2 + b}\right)^{\beta}, \ldots, \left(\frac{b}{t\epsilon_d + b}\right)^{\beta}\right), \quad t \geq 0.$$

Moreover, if $Im(\epsilon_i) = 0$, for all $i = 1, 2, \ldots, d$, then the above result holds for all $\beta \in \mathbb{R}$. □

In the next theorem, we give an alternate representation of the survival function of the above corollary. This will be used in proving the results given in the forthcoming subsections. The proof of part (ii) can be done in the same line as in part(i) and is therefore omitted.

Theorem 6.3.2 *Let $M \sim DPH(\boldsymbol{a}, A)$, where A is a matrix of size d. Further, let $I - A$ is a diagonalizable matrix such that $I - A = Pdiag(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)P^{-1}$.*

(i) Assume that shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. If ϵ_i's are real and positive, then

$$\bar{F}_L(t) = \sum_{i=1}^{d} c_i \left(\frac{\alpha}{\lambda t\epsilon_i + \alpha}\right)^{k-\nu} \frac{\Gamma_\nu(k, (\lambda t\epsilon_i + \alpha)l)}{\Gamma_\nu(k, \alpha l)}, \quad t \geq 0,$$

where c_i's are real constants such that $\sum_{i=1}^{d} c_i = 1$;

(ii) Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta \in \mathbb{N}$. If $Re(\epsilon_i) > 0$, for all $i = 1, 2, \ldots, d$, then

$$\bar{F}_L(t) = \sum_{i=1}^{d} g_i \left(\frac{b}{t\epsilon_i + b}\right)^{\beta}, \quad t \geq 0,$$

where g_i's are complex constants such that $\sum_{i=1}^{d} g_i = 1$. Moreover, if $Im(\epsilon_i) = 0$, for all $i = 1, 2, \ldots, d$, then the above result holds for all $\beta \in \mathbb{R}$,; here g_i's are real constants.

Proof (i): Let $C_1, C_2, \ldots, C_d$ be the column vectors of the matrix P, and let $R_1, R_2, \ldots, R_d$ be the row vectors of the matrix P^{-1}. Then, we can write

$$P = \begin{pmatrix} C_1 & C_2 & \ldots & C_d \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_d \end{pmatrix}.$$

Now,

$$\begin{aligned} \boldsymbol{a}PD(t)P^{-1}\boldsymbol{e} &= \boldsymbol{a}\begin{pmatrix} C_1 & C_2 & \ldots & C_d \end{pmatrix} D(t) \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_d \end{pmatrix} \boldsymbol{e} \\ &= \begin{pmatrix} \boldsymbol{a}C_1 & \ldots & \boldsymbol{a}C_d \end{pmatrix} diag\left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_1 + \alpha)l)}{(\lambda t\epsilon_1 + \alpha)^{k-\nu}}, \ldots, \frac{\Gamma_\nu(k, (\lambda t\epsilon_d + \alpha)l)}{(\lambda t\epsilon_d + \alpha)^{k-\nu}}\right) \begin{pmatrix} R_1\boldsymbol{e} \\ \vdots \\ R_d\boldsymbol{e} \end{pmatrix} \\ &= \sum_{i=1}^{d} \left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_i + \alpha)l)}{(\lambda t\epsilon_i + \alpha)^{k-\nu}}\right) (\boldsymbol{a}C_iR_i\boldsymbol{e}) = \sum_{i=1}^{d} c_i \left(\frac{\Gamma_\nu(k, (\lambda t\epsilon_i + \alpha)l)}{(\lambda t\epsilon_i + \alpha)^{k-\nu}}\right), \end{aligned}$$

where $c_i = \boldsymbol{a}C_iR_i\boldsymbol{e}$. Then, from Corollary 6.3.1, we get

$$\bar{F}_L(t) = \sum_{i=1}^{d} c_i \left(\frac{\alpha}{\lambda t\epsilon_i + \alpha}\right)^{k-\nu} \frac{\Gamma_\nu(k, (\lambda t\epsilon_i + \alpha)l)}{\Gamma_\nu(k, \alpha l)}.$$

Since all eigenvalues of $I - A$ are real, all entries of P are also real. Hence, c_i is a real constant, for all $i = 1, 2, 3, \ldots, d$. Further,

$$\sum_{i=1}^{d} c_i = \sum_{i=1}^{d} \boldsymbol{a} C_i R_i \boldsymbol{e} = \boldsymbol{a} \begin{pmatrix} C_1 & C_2 & \ldots & C_d \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_d \end{pmatrix} \boldsymbol{e} = \boldsymbol{a} P P^{-1} \boldsymbol{e} = 1.$$

Hence, the result is proved.

Remark 6.3.1 *In Theorem 6.3.2 (i), it is given that $\sum_{i=1}^{d} c_i = 1$. Thus, if $c_i \geq 0$, for all i, then L is a mixture of d random variables $V_1, V_2, \ldots, V_d$ with reliability functions given by*

$$\bar{F}_{V_i}(t) = \left(\frac{\alpha}{\lambda t \epsilon_i + \alpha}\right)^{k-\nu} \frac{\Gamma_\nu(k, (\lambda t \epsilon_i + \alpha) l)}{\Gamma_\nu(k, \alpha l)}, \quad i = 1, 2, \ldots, d.$$

Similarly, in Theorem 6.3.2 (ii), if $g_i \geq 0$, for all i, then L is a mixture of d Pareto distributed random variables with the sets of parameters $\{\beta, b/\epsilon_i\}$, $i = 1, 2, \ldots, d$. Note that, the Pareto distribution has the DFR property. Thus, if $g_i \geq 0$, for all i, then L is also DFR. In particular, if $I - A$ is a diagonal matrix then $g_i \geq 0$, for all i, and hence, L is DFR in this case.

6.3.2 Failure rate function

As discussed in Subsection 1.2.1, the failure rate is an important characteristic of a lifetime. Its shape can be the indication of, e.g., ageing or other stochastic properties. The proof of part (ii) of the following theorem, where the failure rate $r_L(t)$ for a system subject to the defined shock process is derived, can be performed in the same line as in part (i) and is therfore omitted.

Theorem 6.3.3 *Let $M \sim DPH(\boldsymbol{a}, A)$, where A is a square matrix of size d. Further, let $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ be eigenvalues of the matrix $I - A$ with the algebraic multiplicity $d_1, d_2, \ldots, d_r$, respectively, and let P be a non-singular matrix such that $I - A = P diag(J_1, J_2, \ldots, J_r) P^{-1}$, where J_i is the ϵ_i-Jordan block with dimension d_i, $i = 1, 2, \ldots, r$, and $\sum_{i=1}^{r} d_i = d$.*

(i) *Assume that shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. If every eigenvalue of the matrix $I - A$ is real and positive, then*

$$r_L(t) = \lambda \left(\frac{\boldsymbol{a} P diag(H_1(t), H_2(t), \ldots, H_r(t)) P^{-1} \boldsymbol{e}}{\boldsymbol{a} P diag(D_1(t), D_2(t), \ldots, D_r(t)) P^{-1} \boldsymbol{e}} \right),$$

where $D_i(t)$ is same as in Theorem 6.3.1, and $H_i(t) = E_i(t) - S_i(t)$,

$$E_i(t) = \begin{pmatrix} \frac{\epsilon_i \Gamma_\nu(k+1, (\lambda t \epsilon_i + \alpha) l)}{(\lambda t \epsilon_i + \alpha)^{k+1-\nu}} & \frac{-\lambda t \epsilon_i \Gamma_\nu(k+2, (\lambda t \epsilon_i + \alpha) l)}{(\lambda t \epsilon_i + \alpha)^{k+2-\nu}} & \cdots & \frac{(-\lambda t)^{d_i - 1} \epsilon_i \Gamma_\nu(k + d_i, (\lambda t \epsilon_i + \alpha) l)}{(d_i - 1)! (\lambda t \epsilon_i + \alpha)^{k+d_i-\nu}} \\ 0 & \frac{\epsilon_i \Gamma_\nu(k+1, (\lambda t \epsilon_i + \alpha) l)}{(\lambda t \epsilon_i + \alpha)^{k+1-\nu}} & \cdots & \frac{(-\lambda t)^{d_i - 2} \epsilon_i \Gamma_\nu(k + d_i - 1, (\lambda t \epsilon_i + \alpha) l)}{(d_i - 2)! (\lambda t \epsilon_i + \alpha)^{k+d_i-1-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\epsilon_i \Gamma_\nu(k+1, (\lambda t \epsilon_i + \alpha) l)}{(\lambda t \epsilon_i + \alpha)^{k+1-\nu}} \end{pmatrix}_{d_i \times d_i},$$

$$S_i(t) = \begin{pmatrix} 0 & \frac{-\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{-(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\ 0 & 0 & \cdots & \frac{-(-\lambda t)^{d_i-3}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-3)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}_{d_i\times d_i},$$

$t \geq 0$, $i = 1, 2, \ldots, r$;

(*ii*) *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$, $\beta \in \mathbb{N}$. *If* $Re(\epsilon_i) > 0$, *for all* $i = 1, 2, \ldots, r$, *then*

$$r_L(t) = \frac{1}{b}\left(\frac{\boldsymbol{a}Pdiag(\widetilde{H}_1(t), \widetilde{H}_2(t), \ldots, \widetilde{H}_r(t))P^{-1}\boldsymbol{e}}{\boldsymbol{a}Pdiag(\widetilde{D}_1(t), \widetilde{D}_2(t), \ldots, \widetilde{D}_r(t))P^{-1}\boldsymbol{e}}\right),$$

where $\widetilde{D}_i(t)$ *is the same as in Theorem 6.3.1, and* $\widetilde{H}_i(t) = \widetilde{E}_i(t) - \widetilde{S}_i(t)$,

$$\widetilde{E_i}(t) = \begin{pmatrix} \epsilon_i\Gamma(\beta+1)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} & -\epsilon_i\Gamma(\beta+2)\left(\frac{t}{t\epsilon_i+b}\right)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} & \cdots & \frac{\epsilon_i\Gamma(\beta+d_i)}{(d_i-1)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-1}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} \\ 0 & \epsilon_i\Gamma(\beta+1)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} & \cdots & \frac{\epsilon_i\Gamma(\beta+d_i-1)}{(d_i-2)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-2}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \epsilon_i\Gamma(\beta+1)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} \end{pmatrix}_{d_i\times d_i},$$

$$\widetilde{S}_i(t) = \begin{pmatrix} 0 & -\Gamma(\beta+1)\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} & \cdots & \frac{-\Gamma(\beta+d_i-1)}{(d_i-2)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-2}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} \\ 0 & 0 & \cdots & \frac{-\Gamma(\beta+d_i-2)}{(d_i-3)!}\left(\frac{-t}{t\epsilon_i+b}\right)^{d_i-3}\left(\frac{b}{t\epsilon_i+b}\right)^{\beta+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}_{d_i\times d_i},$$

$t \geq 0$,$i = 1, 2, \ldots, r$. *Moreover, if* $Im(\epsilon_i) = 0$, *for all* $i = 1, 2, \ldots, r$, *then the above result holds for all* $\beta \in \mathbb{R}$.

Proof : From Eq. (6.3.1), we get

$$f_L(t) = -\left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)}\right)\boldsymbol{a}Pdiag(D_1'(t), D_2'(t), \ldots, D_r'(t))P^{-1}\boldsymbol{e},$$

where $D_i'(t) = \frac{d}{dt}(D_i(t))$, for all $i = 1, 2, \ldots, r$. By using Eq. (1.2.2), we get, for every $n \in \mathbb{N} \cup \{0\}$,

$$\begin{aligned} \frac{d}{dt}\left(\frac{(\lambda t)^n\Gamma_\nu(k+n, (\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}\right) &= \frac{d}{dt}\left((\lambda t)^n\int_0^\infty \frac{q^{k+n-1}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq\right) \\ &= -\lambda\epsilon_i(\lambda t)^n\int_0^\infty \frac{q^{k+n}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq \\ &\quad +(n\lambda)\frac{\Gamma_\nu(k+n, (\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}(\lambda t)^{n-1} \\ &= -\lambda\epsilon_i(\lambda t)^n\frac{\Gamma_\nu(k+n+1, (\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n+1-\nu}} \end{aligned}$$

$$+(n\lambda)\frac{\Gamma_\nu(k+n,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+n-\nu}}(\lambda t)^{n-1}.$$

Consequently,

$$D_i'(t) = -\lambda(E_i(t) - S_i(t)),$$

where

$$E_i(t) = \begin{pmatrix} \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \frac{-\lambda t\epsilon_i\Gamma_\nu(k+2,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+2-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-1}\epsilon_i\Gamma_\nu(k+d_i,(\lambda t\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda t\epsilon_i+\alpha)^{k+d_i-\nu}} \\ 0 & \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{(-\lambda t)^{d_i-2}\epsilon_i\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \frac{\epsilon_i\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} \end{pmatrix}$$

and

$$S_i(t) = \begin{pmatrix} 0 & \frac{-\Gamma_\nu(k+1,(\lambda t\epsilon_i+\alpha)l)}{(\lambda t\epsilon_i+\alpha)^{k+1-\nu}} & \cdots & \frac{-(-\lambda t)^{d_i-2}\Gamma_\nu(k+d_i-1,(\lambda t\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda t\epsilon_i+\alpha)^{k+d_i-1-\nu}} \\ 0 & 0 & \cdots & \frac{-(-\lambda t)^{d_i-3}\Gamma_\nu(k+d_i-2,(\lambda t\epsilon_i+\alpha)l)}{(d_i-3)!(\lambda t\epsilon_i+\alpha)^{k+d_i-2-\nu}} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$i = 1, 2, \ldots, r$. Finally, the result follows from the fact that $r_L(t) = f_L(t)/\bar{F}_L(t)$. Hence, the result is proved. □

The following corollary immediately follows from Theorem 6.3.3, Remark 1.2.4(*c*) and the fact that all Jordan blocks of a diagonalizable matrix are of size 1.

Corollary 6.3.2 *Let $I-A$ is a diagonalizable matrix such that $I-A = P\,diag(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)P^{-1}$.*

(*i*) *Assume that shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$. If ϵ_i's are real and positive, then*

$$r_L(t) = \lambda\left(\frac{\boldsymbol{a}PH(t)P^{-1}\boldsymbol{e}}{\boldsymbol{a}PD(t)P^{-1}\boldsymbol{e}}\right),$$

where $D(t)$ is the same as in Corollary 6.3.1 and

$$H(t) = diag\left(\frac{\epsilon_1\Gamma_\nu(k+1,(\lambda t\epsilon_1+\alpha)l)}{(\lambda t\epsilon_1+\alpha)^{k+1-\nu}}, \frac{\epsilon_2\Gamma_\nu(k+1,(\lambda t\epsilon_2+\alpha)l)}{(\lambda t\epsilon_2+\alpha)^{k+1-\nu}}, \ldots, \frac{\epsilon_d\Gamma_\nu(k+1,(\lambda t\epsilon_d+\alpha)l)}{(\lambda t\epsilon_d+\alpha)^{k+1-\nu}}\right),$$

for $t \geq 0$;

(*ii*) *Assume that shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta \in \mathbb{N}$. If $Re(\epsilon_i) > 0$, for all $i = 1, 2, \ldots, d$, then*

$$r_L(t) = \frac{\beta}{b}\left(\frac{\boldsymbol{a}P\widetilde{H}(t)P^{-1}\boldsymbol{e}}{\boldsymbol{a}P\widetilde{D}(t)P^{-1}\boldsymbol{e}}\right),$$

where $\widetilde{D}(t)$ is the same as in Corollary 6.3.1 and

$$\widetilde{H}(t) = diag\left(\epsilon_1\left(\frac{b}{t\epsilon_1+b}\right)^{\beta+1}, \epsilon_2\left(\frac{b}{t\epsilon_2+b}\right)^{\beta+1}, \ldots, \epsilon_d\left(\frac{b}{t\epsilon_d+b}\right)^{\beta+1}\right), \quad t \geq 0.$$

Moreover, if $Im(\epsilon_i) = 0$, *for all* $i = 1, 2, \ldots, d$, *then the above result holds for all* $\beta \in \mathbb{R}$. □

From Remark 6.3.1 we already know that the cdf $F_L(t)$ has a DFR. The following intuitively clear theorem (as the failure rate of the Pareto distribution asymptotically tends to 0) shows that the failure rate that corresponds to $F_L(t)$ also tends to 0.

Theorem 6.3.4 *Let* $M \sim DPH(\boldsymbol{a}, A)$, *where* A *is a square matrix of size* d. *Further, let* $I - A$ *be a diagonalizable matrix such that* $I - A = Pdiag(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)P^{-1}$. *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$, $\beta \in \mathbb{N}$. *If* $Re(\epsilon_i) > 0$, *for all* $i = 1, 2, \ldots, d$, *then*

$$\lim_{t\to\infty} r_L(t) = 0.$$

Moreover, if $Im(\epsilon_i) = 0$, *for all* $i = 1, 2, \ldots, d$, *then this result holds for all* $\beta \in \mathbb{R}$.

Proof: From Theorem 6.3.2(ii), we get

$$\begin{aligned} r_L(t) = \beta\frac{\sum_{i=1}^d g_i\epsilon_i\left(\frac{1}{t\epsilon_i+b}\right)^{\beta+1}}{\sum_{i=1}^d g_i\left(\frac{1}{t\epsilon_i+b}\right)^{\beta}} &= \beta\sum_{i=1}^d g_i\epsilon_i\left(\frac{\left(\frac{1}{t\epsilon_i+b}\right)^{\beta+1}}{\sum_{j=1}^d g_j\left(\frac{1}{t\epsilon_j+b}\right)^{\beta}}\right) \\ &= \beta\sum_{i=1}^d g_i\epsilon_i\left(\frac{1}{\sum_{j=1}^d g_j\left(\frac{t\epsilon_i+b}{t\epsilon_j+b}\right)^{\beta}(t\epsilon_i+b)}\right). \end{aligned}$$

Taking limit $t \to \infty$ on both sides of the above expression, we get $\lim_{t\to\infty} r_L(t) = 0$.

6.3.3 Mean residual lifetime and mean lifetime

The mean residual life function is also an important reliability characteristic for a system operating under shocks (see Subsection 1.2.1). In the following theorem, we derive the corresponding expressions for the defined shock model. We prove only part (i), whereas part (ii) can be performed in a similar way and is therfore omitted.

Theorem 6.3.5 *Let* $M \sim DPH(\boldsymbol{a}, A)$, *where* A *is a square matrix of size* d. *Further, let* $\epsilon_1, \epsilon_2, \ldots, \epsilon_r$ *be eigenvalues of the matrix* $I - A$ *with the algebraic multiplicity* $d_1, d_2, \ldots, d_r$, *respectively, and let* P *be a non-singular matrix such that* $I - A = Pdiag(J_1, J_2, \ldots, J_r)P^{-1}$, *where* J_i *is the* ϵ_i*-Jordan block with dimension* d_i, $i = 1, 2, \ldots, r$, *and* $\sum_{i=1}^r d_i = d$.

(*i*) *Assume that shocks occur according to the HPGGP with the set of parameters* $\{\lambda, \nu, k, \alpha, l\}$, $k > 1$. *If every eigenvalue of the matrix* $I - A$ *is real and positive then*

$$E(L - t|L > t) = \frac{\boldsymbol{a}Pdiag(R_1(t), R_2(t), \ldots, R_r(t))P^{-1}\boldsymbol{e}}{\boldsymbol{a}Pdiag(D_1(t), D_2(t), \ldots, D_r(t))P^{-1}\boldsymbol{e}},$$

where $D_i(t)$ *is the same as in Theorem 6.3.1, and*

$$R_i(t) = \begin{pmatrix} \phi(0,t,\epsilon_i) & -\phi(1,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-1}\phi(d_i-1,t,\epsilon_i)}{(d_i-1)!} \\ 0 & \phi(0,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-2}\phi(d_i-2,t,\epsilon_i)}{(d_i-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \phi(0,t,\epsilon_i) \end{pmatrix}_{d_i \times d_i},$$

$$\begin{aligned} \phi(n,t,\epsilon_i) &= \frac{(\lambda t)^n \Gamma_\nu(k+n-1, (\alpha+\lambda t\epsilon_i)l)}{\lambda\epsilon_i(\alpha+\lambda t\epsilon_i)^{n+k-\nu-1}} \\ &\quad + \frac{(\lambda t)^n}{\lambda\epsilon_i}\sum_{i=1}^{n} \frac{n(n-1)\ldots(n-i+1)}{(\lambda t\epsilon_i)^i} \frac{\Gamma_\nu(n+k-i-1, (\alpha+\lambda t\epsilon_i)l)}{(\alpha+\lambda t\epsilon_i)^{n+k-\nu-i-1}}, \end{aligned}$$

for $t \geq 0$, $i = 1, 2, \ldots, r$ *and* $n \in \mathbb{N} \cup \{0\}$;

(*ii*) *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$, $\beta \in \mathbb{N}$, $\beta > 1$. *If* $Re(\epsilon_i) > 0$, *for all* $i = 1, 2, \ldots, r$, *then*

$$E(L - t|L > t) = \frac{\boldsymbol{a}Pdiag(\widetilde{R}_1(t), \widetilde{R}_2(t), \ldots, \widetilde{R}_r(t))P^{-1}\boldsymbol{e}}{\boldsymbol{a}Pdiag(\widetilde{D}_1(t), \widetilde{D}_2(t), \ldots, \widetilde{D}_r(t))P^{-1}\boldsymbol{e}},$$

where $\widetilde{D}_i(t)$ *is the same as in Theorem 6.3.1, and*

$$\widetilde{R_i}(t) = \begin{pmatrix} \psi(0,t,\epsilon_i) & -\psi(1,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-1}\psi(d_i-1,t,\epsilon_i)}{(d_i-1)!} \\ 0 & \psi(0,t,\epsilon_i) & \ldots & \frac{(-1)^{d_i-2}\psi(d_i-2,t,\epsilon_i)}{(d_i-2)!} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \psi(0,t,\epsilon_i) \end{pmatrix},$$

$$\begin{aligned} \psi(n,t,\epsilon_i) &= (b/\epsilon_i)\Gamma(\beta+n-1)\left(\frac{t}{b+t\epsilon_i}\right)^n \left(\frac{b}{b+t\epsilon_i}\right)^{\beta-1} \\ &\quad + (b/\epsilon_i)\left(\frac{t}{b+t\epsilon_i}\right)^n \sum_{i=1}^{n} \frac{b^i n(n-1)\ldots(n-i+1)}{(t\epsilon_i)^i\Gamma(\beta+n-i-1)} \left(\frac{b}{b+t\epsilon_i}\right)^{\beta-i-1}, \end{aligned}$$

for $t \geq 0$, $i = 1, 2, \ldots, r$, *and* $n \in \mathbb{N} \cup \{0\}$. *Moreover, if* $Im(\epsilon_i) = 0$, *for all* $i = 1, 2, \ldots, r$, *then the above result holds for all* $\beta \in \mathbb{R}$.

Proof (i): We have

$$E(L-t|L>t)=\frac{1}{\bar{F}_L(t)}\int_t^\infty \bar{F}_L(x)dx.$$

For any $n\in\mathbb{N}\cup\{0\}$, we get

$$\begin{aligned}
&\int_t^\infty \frac{(\lambda x)^n\Gamma_\nu(k+n,(\alpha+\lambda x\epsilon_i)l)}{(\alpha+\lambda x\epsilon_i)^{n+k-\nu}}dx\\
&=\int_t^\infty\int_0^\infty \frac{(\lambda x)^n q^{k+n-1}\exp\{-(\lambda x\epsilon_i+\alpha)q\}}{(q+l)^\nu}dqdx\\
&=\int_0^\infty \frac{\lambda^n q^{k+n-1}\exp\{-\alpha q\}}{(q+l)^\nu}\left(\int_t^\infty x^n\exp\{-\lambda x\epsilon_i q\}dx\right)dq\\
&=\frac{(\lambda t)^n}{\lambda\epsilon_i}\int_0^\infty \frac{q^{k+n-2}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq\\
&\quad+\frac{(\lambda t)^n}{\lambda\epsilon_i}\sum_{i=1}^n\frac{n(n-1)\dots(n-i+1)}{(\lambda t\epsilon_i)^i}\int_0^\infty\frac{q^{k+n-i-2}\exp\{-(\lambda t\epsilon_i+\alpha)q\}}{(q+l)^\nu}dq\\
&=\frac{(\lambda t)^n\Gamma_\nu(k+n-1,(\alpha+\lambda t\epsilon_i)l)}{\lambda\epsilon_i(\alpha+\lambda t\epsilon_i)^{n+k-\nu-1}}\\
&\quad+\frac{(\lambda t)^n}{\lambda\epsilon_i}\sum_{i=1}^n\frac{n(n-1)\dots(n-i+1)}{(\lambda t\epsilon_i)^i}\frac{\Gamma_\nu(n+k-i-1,(\alpha+\lambda t\epsilon_i)l)}{(\alpha+\lambda t\epsilon_i)^{n+k-\nu-i-1}}\\
&=\phi(n,t,\epsilon_i),
\end{aligned}\tag{6.3.8}$$

where the first equality follows from Eq. (1.2.2), and the third equality follows from the fact that

$$\int_t^\infty x^n\exp\{-\lambda x\epsilon_i q\}dx=\frac{\exp\{-t\epsilon_i q\lambda\}}{(\lambda\epsilon_i q)^{n+1}}\left[(t\lambda\epsilon_i q)^n+\sum_{i=1}^n n(n-1)\dots(n-i+1)(t\lambda\epsilon_i q)^{n-i}\right].$$

Upon using Eq. (6.3.8), we get

$$\begin{aligned}
\int_t^\infty\bar{F}_L(x)dx &= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\boldsymbol{a}P\left(\int_t^\infty diag(D_1(t),D_2(t),\dots,D_r(t))dx\right)P^{-1}\boldsymbol{e}\\
&= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\boldsymbol{a}Pdiag(R_1(t),R_2(t),\dots,R_r(t))P^{-1}\boldsymbol{e},
\end{aligned}$$

where

$$\begin{aligned}
R_i(t) &= \begin{pmatrix}
\int_t^\infty\left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx & \int_t^\infty\left(\frac{-\lambda x\Gamma_\nu(k+1,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k+1-\nu}}\right)dx & \dots & \int_t^\infty\left(\frac{(-\lambda x)^{d_i-1}\Gamma_\nu(k+d_i-1,(\lambda x\epsilon_i+\alpha)l)}{(d_i-1)!(\lambda x\epsilon_i+\alpha)^{k+d_i-1-\nu}}\right)dx\\
0 & \int_t^\infty\left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx & \dots & \int_t^\infty\left(\frac{(-\lambda x)^{d_i-2}\Gamma_\nu(k+d_i-2,(\lambda x\epsilon_i+\alpha)l)}{(d_i-2)!(\lambda x\epsilon_i+\alpha)^{k+d_i-2-\nu}}\right)dx\\
\vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \int_t^\infty\left(\frac{\Gamma_\nu(k,(\lambda x\epsilon_i+\alpha)l)}{(\lambda x\epsilon_i+\alpha)^{k-\nu}}\right)dx
\end{pmatrix}_{d_i\times d_i}\\
&= \begin{pmatrix}
\phi(0,t,\epsilon_i) & -\phi(1,t,\epsilon_i) & \dots & \frac{(-1)^{d_i-1}\phi(d_i-1,t,\epsilon_i)}{(d_i-1)!}\\
0 & \phi(0,t,\epsilon_i) & \dots & \frac{(-1)^{d_i-2}\phi(d_i-2,t,\epsilon_i)}{(d_i-2)!}\\
\vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \phi(0,t,\epsilon_i)
\end{pmatrix}_{d_i\times d_i},
\end{aligned}$$

$i = 1, 2, \ldots, r$. Hence, the result is proved. □

The following corollary immediately follows from Theorem 6.3.5, Remark 1.2.4(c) and the fact that all Jordan blocks of a diagonalizable matrix are of size 1.

Corollary 6.3.3 *Let $I-A$ be a diagonalizable matrix such that $I-A = Pdiag(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)P^{-1}$.*

(i) Assume that shocks occur on the system according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$, $k > 1$. If ϵ_i's are real and positive, then

$$E(L - t|L > t) = \frac{\boldsymbol{a}PR(t)P^{-1}\boldsymbol{e}}{\boldsymbol{a}PD(t)P^{-1}\boldsymbol{e}},$$

where $D(t)$ is the same as in Corollary 6.3.1, and

$$R(t) = diag(\phi(0, t, \epsilon_i), \phi(0, t, \epsilon_2), \ldots, \phi(0, t, \epsilon_d)),$$

$$\phi(0, t, \epsilon_i) = \frac{\Gamma_\nu(k-1, (\alpha + \lambda t\epsilon_i)l)}{\lambda\epsilon_i(\alpha + \lambda t\epsilon_i)^{k-\nu-1}}, \quad t \geq 0, i = 1, 2, \ldots, d;$$

(ii) Assume that shocks occur on the system according to the Pólya process with the set of parameters $\{\beta, b\}$, $\beta \in \mathbb{N}$, $\beta > 1$. If $Re(\epsilon_i) > 0$, for all $i = 1, 2, \ldots, d$, then

$$E(L - t|L > t) = \frac{\boldsymbol{a}P\widetilde{R}(t)P^{-1}\boldsymbol{e}}{\boldsymbol{a}P\widetilde{D}(t)P^{-1}\boldsymbol{e}},$$

where $\widetilde{D}(t)$ is the same as in Corollary 6.3.1, and

$$\widetilde{R}(t) = diag\left((b/\epsilon_i)\left(\frac{1}{\beta-1}\right)\left(\frac{b}{b+t\epsilon_i}\right)^{\beta-1}, \ldots, (b/\epsilon_i)\left(\frac{1}{\beta-1}\right)\left(\frac{b}{b+t\epsilon_i}\right)^{\beta-1}\right),$$

for $t \geq 0$. Moreover, if $Im(\epsilon_i) = 0$, for all $i = 1, 2, \ldots, d$, then the above result holds for all $\beta \in \mathbb{R}$. □

In the following theorem, the mean lifetime of the system is given for the defined shock model. Although it is a specific case for the mean residual lifetime, it is more efficient and simple to obtain this result directly and under weaker assumptions.

Theorem 6.3.6 *Let $M \sim DPH(\boldsymbol{a}, A)$. Assume that shocks occur according to the HPGGP with parameters $\{\lambda, \nu, k, \alpha, l\}$, $k > 1$. Then*

$$E(L) = \frac{\alpha(\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})}{\lambda}\frac{\Gamma_\nu(k-1, \alpha l)}{\Gamma_\nu(k, \alpha l)}.$$

Proof: We have

$$E(L) = E(E(L|Q)),$$

where Q is a mixing random variable of the HPGGP with parameter set $\{\lambda, \nu, k, \alpha, l\}$, $k > 1$. Now, for given $Q = q$, X_i's are iid random variables and follow the exponential distribution with parameter $q\lambda$. Then

$$E(L|Q=q) = E\left(\sum_{i=1}^{M} X_i | Q = q\right) = E(M)E(X_1|Q=q) = (\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})\frac{1}{q\lambda}$$

and consequently,

$$\begin{aligned} E(L) &= \frac{(\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})}{\lambda} E\left(\frac{1}{Q}\right) \\ &= \frac{(\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})}{\lambda} \int_0^\infty \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \frac{q^{k-2}\exp\{-\alpha q\}}{(q+l)^\nu} dq \\ &= \frac{\alpha(\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})}{\lambda} \frac{\Gamma_\nu(k-1,\alpha l)}{\Gamma_\nu(k,\alpha l)}. \end{aligned}$$

Hence, the result is proved.

Corollary 6.3.4 *Assume that shocks occur according to the Pólya process with the set of parameters* $\{\beta, b\}$, $\beta > 1$. *Then*

$$E(L) = \frac{b(\boldsymbol{a}(I-A)^{-1}\boldsymbol{e})}{\beta - 1}.$$

6.3.4 Stochastic comparisons

In this subsection, we study some stochastic comparisons (see Definition 1.2.14) for systems for the defined shock model.

Theorem 6.3.7 *Let* $L_1 = \Sigma_{i=1}^{M_1} X_i$ *and* $L_2 = \Sigma_{i=1}^{M_2} X_i$ *be the lifetimes of two systems subject to the same HPGGP of shocks with the set of parameters* $\{\lambda, \nu, k, \alpha, l\}$. *Further, let* $M_i \sim DPH(\boldsymbol{\gamma_i}, A_i)$, *where* $I - A_i$ *is a diagonal matrix of order* d *with positive and real eigenvalues* $\epsilon_1^{(i)}, \epsilon_2^{(i)}, \ldots, \epsilon_d^{(i)}$ *such that* $\epsilon_1^{(i)} \le \epsilon_2^{(i)} \le \cdots \le \epsilon_d^{(i)}$, $i = 1, 2$. *If* $\boldsymbol{\gamma_1} \ge_{st} \boldsymbol{\gamma_2}$ *and* $\epsilon_j^{(1)} \ge \epsilon_j^{(2)}$, *for all* $j = 1, 2, \ldots, d$, *then* $L_1 \le_{st} L_2$.

Proof: Let $\boldsymbol{\gamma_i} = (\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_d^{(i)})$, $i = 1, 2$. Then, from Theorem 6.3.2 (i), we have

$$\bar{F}_{L_1}(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \sum_{j=1}^{d} \frac{\theta_j^{(1)} \Gamma_\nu(k, (\lambda t \epsilon_j^{(1)} + \alpha) l)}{(\lambda t \epsilon_j^{(1)} + \alpha)^{k-\nu}}$$

and

$$\bar{F}_{L_2}(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \sum_{i=1}^{d} \frac{\theta_j^{(2)} \Gamma_\nu(k, (\lambda t \epsilon_j^{(2)} + \alpha) l)}{(\lambda t \epsilon_j^{(2)} + \alpha)^{k-\nu}}.$$

Clearly, both L_1 and L_2 are mixtures of d random variables with the survival functions given by

$$\bar{F}_j^{(i)}(t) = \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right) \frac{\Gamma_\nu(k, (\lambda t \epsilon_j^{(i)} + \alpha) l)}{(\lambda t \epsilon_j^{(i)} + \alpha)^{k-\nu}}$$

$$= \left(\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\right)\int_0^\infty \frac{q^{k-1}\exp\{-(\lambda t\epsilon_j^{(i)}+\alpha)q\}}{(q+l)^\nu}dq, \quad j=1,2,\ldots,d,$$

where the second equality follows from Eq. (1.2.2). Since $\epsilon_1^{(i)} \le \epsilon_2^{(i)} \le \cdots \le \epsilon_d^{(i)}$, we get $\bar{F}_1^{(i)}(t) \ge \bar{F}_2^{(i)}(t)\cdots \ge \bar{F}_d^{(i)}(t)$, $i=1,2$. Again, $\epsilon_j^{(1)} \ge \epsilon_j^{(2)}$ implies $\bar{F}_j^{(1)}(t) \le \bar{F}_j^{(2)}(t)$, for all $j=1,2,\ldots,d$. Moreover, we have that $\boldsymbol{\gamma_1} \ge_{st} \boldsymbol{\gamma_2}$. Then the result follows from Theorem 4.6 of Shojaee *et al.* [2021]. □

The following corollary immediately follows from Theorem 6.3.7.

Corollary 6.3.5 *The following results hold.*

(a) *Assume that* $A_1 = A_2$. *If* $\boldsymbol{\gamma_1} \ge_{st} \boldsymbol{\gamma_2}$ *then* $L_1 \le_{st} L_2$;

(b) *Assume that* $\boldsymbol{\gamma_1} = \boldsymbol{\gamma_2}$. *If* $\epsilon_j^{(1)} \ge \epsilon_j^{(2)}$, *for all* $j=1,2,\ldots,d$, *then* $L_1 \le_{st} L_2$;

(c) *Assume that shocks occur on the system according to the Pólya process with the set of parameters* $\{\beta, b\}$. *If* $\boldsymbol{\gamma_1} \ge_{st} \boldsymbol{\gamma_2}$ *and* $\epsilon_j^{(1)} \ge \epsilon_j^{(2)}$, *for all* $j=1,2,\ldots,d$, *then* $L_1 \le_{st} L_2$.

Theorem 6.3.8 *Let* $L_1 = \Sigma_{i=1}^M X_i^{(1)}$ *and* $L_2 = \Sigma_{i=1}^M X_i^{(2)}$ *be the lifetimes of two systems subject to different HPGGP shock processes with parameter sets* $\{\lambda_1,\nu_1,k_1,\alpha_1,l_1\}$ *and* $\{\lambda_2,\nu_2,k_2,\alpha_2,l_2\}$, *respectively, where* $X_i^{(1)}$ *and* $X_i^{(2)}$ *are the* i*-th inter-arrival times of shock processes that impact the first and the second systems, respectively. Further, let* $M \sim DPH(\boldsymbol{a}, A)$. *Suppose that* $\lambda_1 \le \lambda_2$, *and one of the following conditions holds.*

(a) $\alpha_1 = \alpha_2$, $k_2 \ge k_1$ *and* $k_2 - k_1 \ge \nu_2 - \nu_1$;

(b) $\alpha_2 < \alpha_1$ *and* $(\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2)^2 - 4(\alpha_1 - \alpha_2)(k_2 - k_1) \le 0$;

(c) $\alpha_2 < \alpha_1$ *and* $\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2 \ge 0$.

Then $L_2 \le_{st} L_1$.

Proof: For $j=1,2$, we have

$$\bar{F}_{L_j}(t) = P\left(\sum_{i=1}^M X_i^{(j)} > t\right) = \int_0^\infty P\left(\sum_{i=1}^M X_i^{(j)} > t | Q_j = q_j\right) f_{Q_j}(q_j)dq_j, \quad (6.3.9)$$

where Q_j is the mixing random variable of the HPGGP with parameter set $\{\lambda_j,\nu_j,k_j,\alpha_j,l_j\}$, $j=1,2$. Note that, for given $Q_j = q_j$, the HPGGP with parameter set $\{\lambda_j,\nu_j,k_j,\alpha_j,l_j\}$ is the same as the HPP with intensity $q_j\lambda_j$, $j=1,2$. Then, from Corollary 6.3.1, we get

$$P\left(\sum_{i=1}^M X_i^{(j)} > t | Q_j = q_j\right) = \boldsymbol{a}\exp\{-q_j\lambda_j t(I-A)\}\boldsymbol{e}.$$

By using the above equality in Eq. (6.3.9), we get

$$\bar{F}_{L_j}(t) = E(\boldsymbol{a}\exp\{-Q_j\lambda_j t(I-A)\}\boldsymbol{e}), \quad j=1,2.$$

Again, note that $\boldsymbol{a}\exp\{-x(I-A)\}\boldsymbol{e}$ is decreasing in $x>0$ and hence, for $\lambda_1 \leq \lambda_2$, we get

$$\begin{aligned}\bar{F}_{L_1}(t) &= E(\boldsymbol{a}\exp\{-Q_1\lambda_1 t(I-A)\}\boldsymbol{e}) \\ &\geq E(\boldsymbol{a}\exp\{-Q_1\lambda_2 t(I-A)\}\boldsymbol{e}). \end{aligned} \tag{6.3.10}$$

Now, if $Q_2 \geq_{st} Q_1$ holds then $\boldsymbol{a}\exp\{-Q_1\lambda_2 t(I-A)\}\boldsymbol{e} \geq_{st} \boldsymbol{a}\exp\{-Q_2\lambda_2 t(I-A)\}\boldsymbol{e}$ (see Theorem 1.A.3 (a) in Shaked and Shanthikumar [2007]) and hence, $E(\boldsymbol{a}\exp\{-Q_1\lambda_2 t(I-A)\}\boldsymbol{e}) \geq \boldsymbol{a}\exp\{-Q_2\lambda_2 t(I-A)\}\boldsymbol{e}$. Upon combining this and Eq. (6.3.10), we get

$$\bar{F}_{L_1}(t) \geq E(\boldsymbol{a}\exp\{-Q_2\lambda_2 t(I-A)\}\boldsymbol{e}) = \bar{F}_{L_2}(t)$$

provided that $Q_2 \geq_{st} Q_1$ holds. Again, $Q_2 \geq_{st} Q_1$ follows provided that one of the following conditions holds (see the proof of Proposition 5 in Cha and Mercier [2021]).

- $\alpha_1 = \alpha_2$, $k_2 \geq k_1$ and $k_2 - k_1 \geq \nu_2 - \nu_1$;
- $\alpha_2 < \alpha_1$ and $(\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2)^2 - 4(\alpha_1 - \alpha_2)(k_2 - k_1) \leq 0$;
- $\alpha_2 < \alpha_1$ and $\alpha_1 - \alpha_2 + k_2 - k_1 + \nu_1 - \nu_2 \geq 0$

Note that these conditions are given in the assumption and hence, the result is proved. □

The following corollary immediately follows from Theorem 6.3.8 and Remark 1.2.4(c).

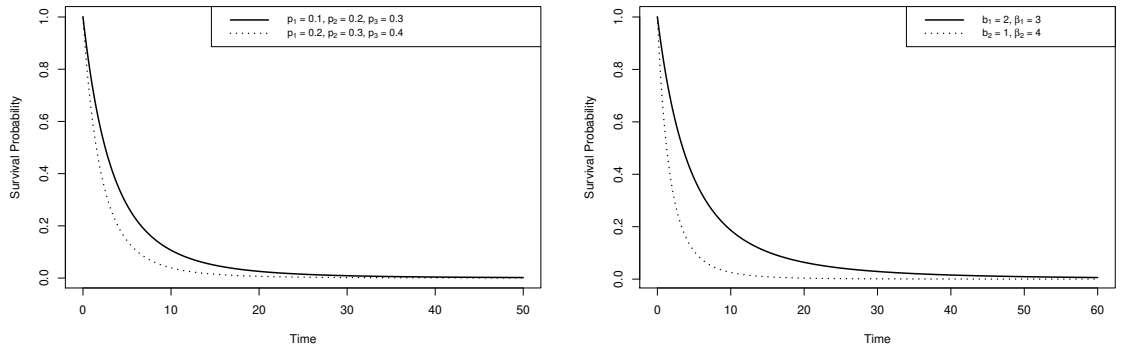
Corollary 6.3.6 *Assume that shocks occur on the first and the second systems according to the Pólya processes with parameter sets $\{\beta_1, b_1\}$ and $\{\beta_2, b_2\}$, respectively. If $b_1 \geq b_2$ and $\beta_1 \leq \beta_2$, then $L_2 \leq_{st} L_1$.*

6.4 Illustrative Examples

In this section, we discuss some numerical examples to illustrate the results given in Section 6.3.

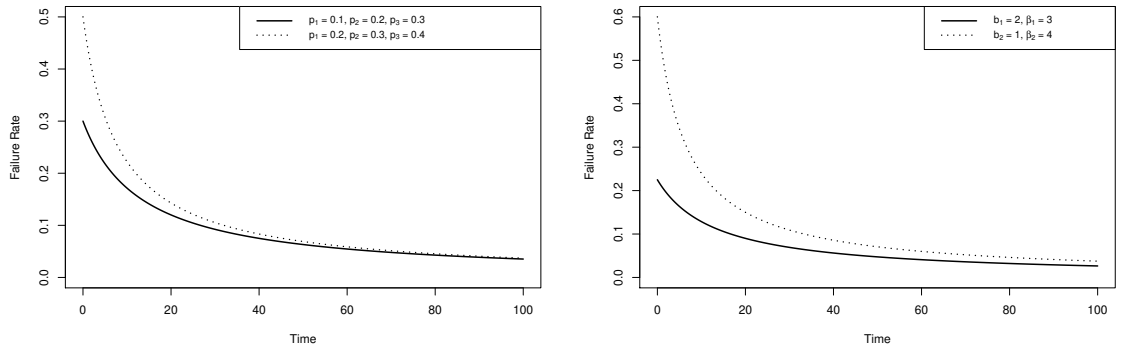
Example 6.4.1 *Consider the generalized extreme shock model-I as mentioned in Section 6.2. We assume that $n = 3$, $\theta_1 = 0.6$, $\theta_2 = 0.3$ and $\theta_3 = 0.1$. In Figures 6.1a, 6.2a and 6.3a, we plot the survival function, the failure rate function and the mean residual life function, respectively, for different p_i's and for fixed process parameters $b = 2$ and $\beta = 4$. Further, in Figures 6.1b, 6.2b, 6.3b we plot the survival function, the failure rate function and the mean residual life function, respectively, for different process parameters and for fixed $p_1 = 0.1$, $p_2 = 0.2$ and $p_3 = 0.3$. Clearly, Figures 6.1a and 6.1b demonstrate the results given in Corollaries 6.3.5 and 6.3.6, respectively. Figures 6.2a and 6.2b demonstrate the result given in Theorem 6.3.4. Moreover, from theses figure, we see that the failure rate of the system is DFR (see Remark 6.3.1). Further, Figures 6.3a and 6.3b show that the system's mean residual life function is nondecreasing.*

We plot Figures 6.4 and 6.5 that illustrate the impact of the dependent increments property of the HPGGP on different reliability measures derived in the previous sections. In Figure 6.4, we compare systems' survival functions for the HPP (the process with independent increments) and the HPGGP (the process with dependent increments) shock processes.



(a) (b)

Figure 6.1: Plot of the system's survival function for the generalized extreme shock model-I



(a) (b)

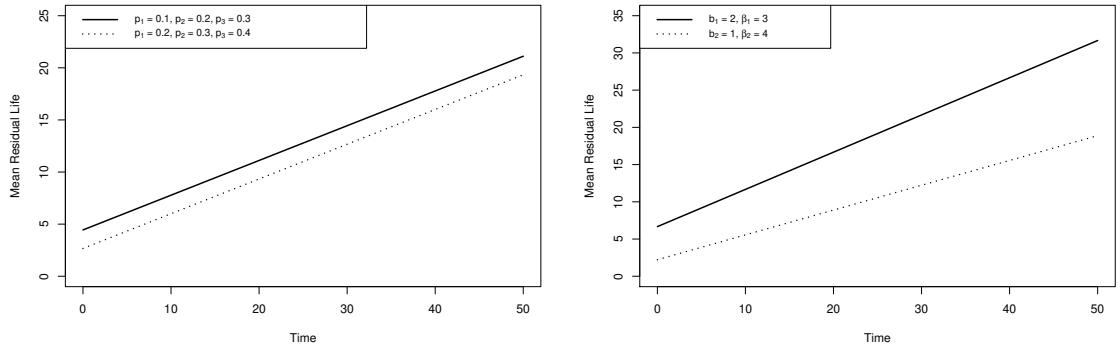
Figure 6.2: Plot of the system's failure rate function for the generalized extreme shock model-I

We clearly see that survival functions for dependent and independent increments cases are significantly different. Similarly, in Figure 6.5, we compare systems' failure rate functions (resp. mean residual lifetime functions) for the HPP and the HPGGP cases. This figure also shows a similar conclusion.

Example 6.4.2 *Consider the classical run shock model (with $r = 2$) as mentioned in Section 6.2. Then*

$$\boldsymbol{a_{rm}} = (1, 0), \quad A_{rm} = \begin{pmatrix} 1-p & p \\ 1-p & 0 \end{pmatrix}.$$

Clearly, $I - A_{rm}$ is diagonalizable and has positive and real eigenvalues for all $p \in [0, 1)$. For $p = 0.2$, eigenvalues of the matrix $I - A_{rm}$ are given by $\epsilon_1^{(1)} = 1.16568542$ and $\epsilon_2^{(1)} = 0.03431458$. Similarly, for $p = 0.3$, eigenvalues of the matrix $I - A_{rm}$ are given by $\epsilon_1^{(2)} =$



(a) (b)

Figure 6.3: Plot of the system's mean residual life function for the generalized extreme shock model-I

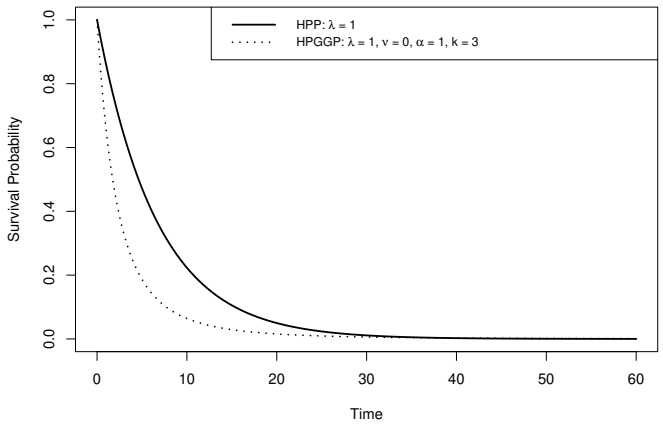
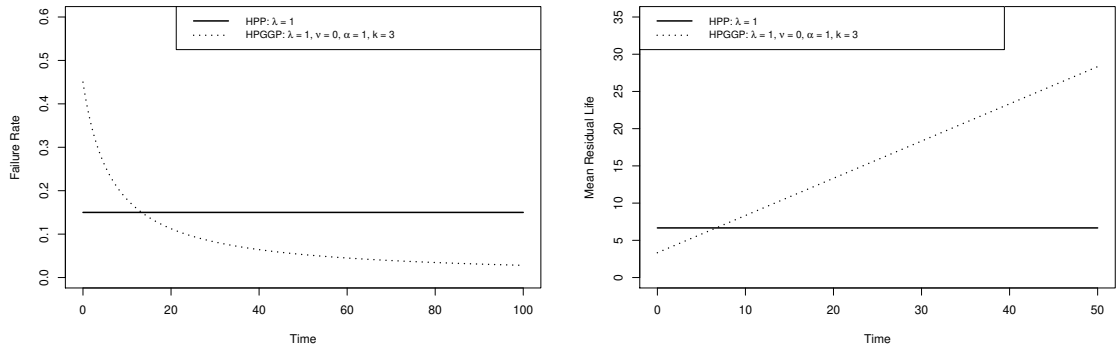


Figure 6.4: Plot of systems' survival functions for HPP and HPGGP shock processes



(a) Failure rate function (b) Mean residual life function

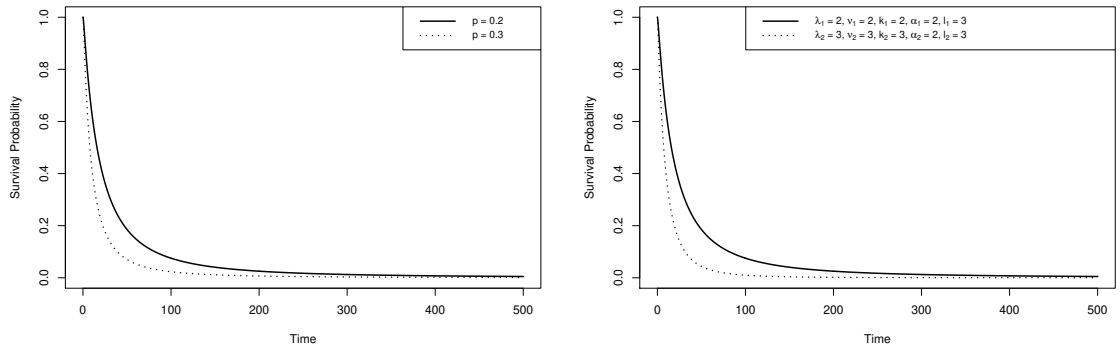
Figure 6.5: Plot of systems' failure rate and mean residual life functions for HPP and HPGGP shock processes

1.22662813 *and* $\epsilon_2^{(2)} = 0.07337187$. *Clearly, all eigenvalues, for* $p = 0.2$ *and* $p = 0.3$, *are real and positive. In Figures 6.6a, 6.7a and 6.8a, we plot the system's survival, failure rate and mean residual life functions, respectively, for different* p *values by assuming that shock process is the HPGGP with the set of parameters* $\{2, 2, 2, 2, 3\}$. *In Figure 6.6b, we plot the system's survival function for fixed* $p = 0.2$ *and for the HPGGP with different values of parameters* $\{\lambda, \nu, k, \alpha, l\}$. *In Figures 6.7b and 6.8b, we plot the system's failure rate and mean residual life functions for fixed* $p = 0.3$ *and for the HPGGP with different values of parameters* $\{\lambda, \nu, k, \alpha, l\}$. *Since* $\epsilon_j^{(1)} \leq \epsilon_j^{(2)}$ *for* $j = 1, 2$, *Figure 6.6a illustrates the result given in Theorem 6.3.7. Further, Figure 6.6b illustrates the result given in Theorem 6.3.8 as condition* (a) *of Theorem 6.3.8 holds. Figures 6.7a and 6.7b demonstrate that the failure rate of the system asymptotically tends to 0. Figures 6.8a and 6.8b show that the system's mean residual life function is nondecreasing.*

Example 6.4.3 *Consider the classical run shock model (with* $r = 3$*) as mentioned in Section 6.2. Then*

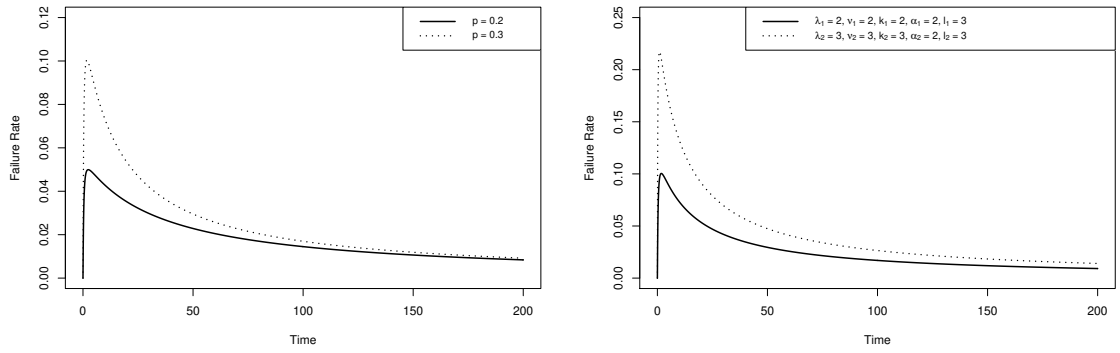
$$\boldsymbol{a_{rm}} = (1, 0, 0), \quad A_{rm} = \begin{pmatrix} 1-p & p & 0 \\ 1-p & 0 & p \\ 1-p & 0 & 0 \end{pmatrix}.$$

We assume $p = 0.2$ *and* 0.3. *For* $p = 0.2$, *eigenvalues of the matrix* $I - A_{rm}$ *are* $1.096736514 + 0.15117i$, $1.096736514 - 0.15117i$ *and* 0.006526972, *where* $i = \sqrt{-1}$. *Similarly, for* $p = 0.3$, *eigenvalues of the matrix* $I - A_{rm}$ *are* $1.13995689 + 0.211432i$, $1.13995689 - 0.211432i$ *and* 0.02008622. *Clearly, all eigenvalues, for both* $p = 0.2$ *and* $p = 0.3$, *are either real or complex with positive real parts. In Figures 6.9a, 6.10a and 6.11a, we plot the system's survival, failure rate and mean residual life functions, respectively, for different* p *values and for the Pólya shock process with parameters* $b = 2$ *and* $\beta = 4$. *In Figure 6.9b, we plot the system's survival function for fixed* $p = 0.2$ *and for the Pólya shock process with different values of parameters* b *and* β. *In Figures 6.10b and 6.11b, we plot the system's failure rate and*



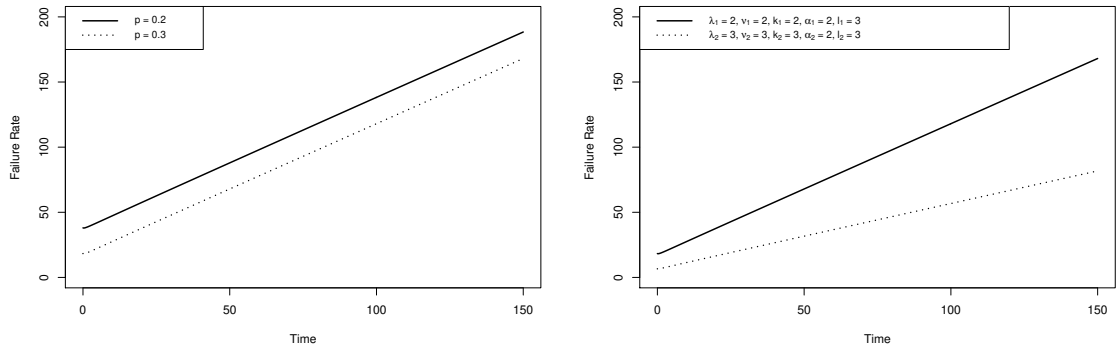
(a) (b)

Figure 6.6: Plot of the system's survival function for the run shock model ($r = 2$)



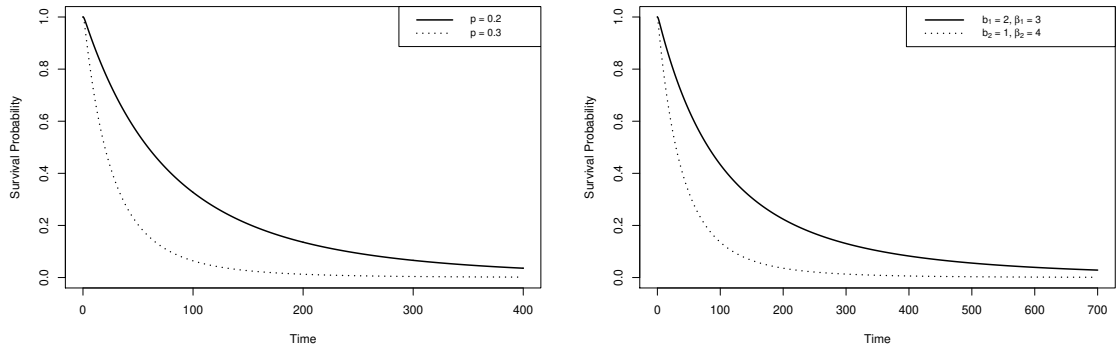
(a) (b)

Figure 6.7: Plot of the system's failure rate function for the run shock model ($r = 2$)



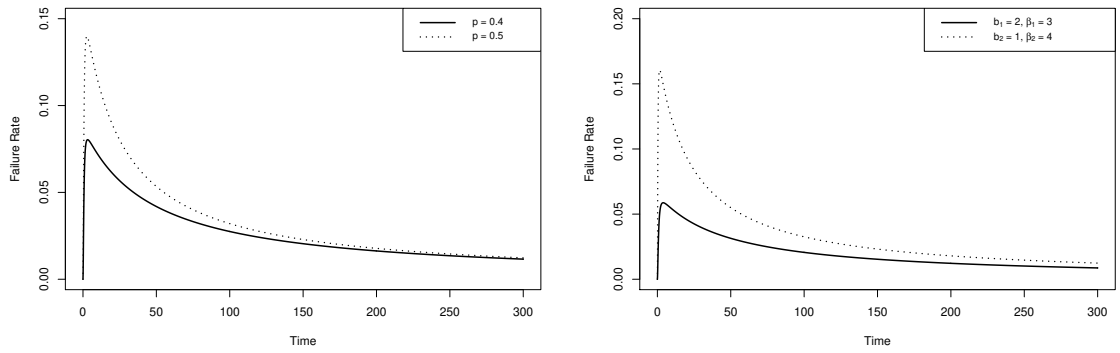
(a) (b)

Figure 6.8: Plot of the system's mean residual life function for the run shock model ($r = 2$)



(a) (b)

Figure 6.9: Plot of the system's survival function for the run shock model ($r = 3$)



(a) (b)

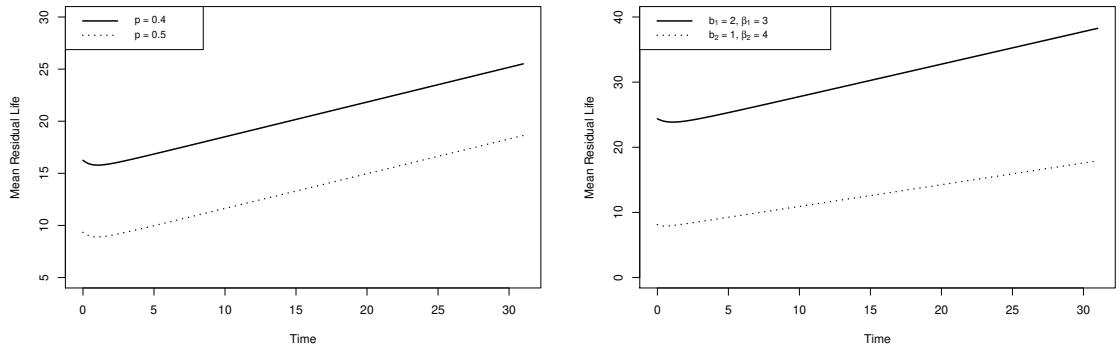
Figure 6.10: Plot of the system's failure rate function for the run shock model ($r = 3$)

mean residual life functions for fixed $p = 0.4$ and for the Pólya shock process with different values of parameters b and β. Figure 6.9a shows that an increment in the parameter p decreases the system's survivability. Further, Figure 6.9b illustrates the result given in Corollary 6.3.6. Figures 6.10a and 6.10b demonstrate the result given in Theorem 6.3.4. Further, Figure 6.11a shows that an increment in the parameter p decreases the system's mean residual life.

Example 6.4.4 *Consider the generalized run shock model-I with $k_1 = 2$ and $k_2 = 1$. Then*

$$\boldsymbol{a_{grm1}} = (1, 0) \text{ and } A_{grm1} = \begin{pmatrix} p_1 & p_2 \\ p_1 & 0 \end{pmatrix}.$$

We assume $p_1 = 0.7$ and $p_2 = 0.1$. In Figure 6.12, we plot the system's survival function for the Pólya shock process with different parameter values. This illustrates the result given



(a) (b)

Figure 6.11: Plot of the system's mean residual life function for the run shock model ($r = 3$)

in Corollary 6.3.6. In Figures 6.13a and 6.13b, we plot the system's failure rate and mean residual life functions for fixed $p_1 = 0.6$ *and* $p_2 = 0.3$. *Figure 6.13a demonstrates the result given in Theorem 6.3.4. From Figure 6.13b we can see that mean residual life function is non-decreasing.*

6.5 Application: Optimal Mission Duration

In this section, we consider the optimal mission duration policy (see Section 4.4 of Chapter 4) for a system subject to shocks, whereas its failure is defined in accordance with the classical run shock model. The same can be done for other models as well.

Assumptions:

1. Let a new system be installed at time $t = 0$, and let its lifetime be modeled by the classical run shock model. Further, let L and t (> 0) denote the lifetime of the system and the mission duration, respectively. The mission can be terminated at any time $\tau \in (0, t]$.

2. Shocks occur according to the Pólya process with parameter set $\{\beta, b\}$.

3. The profit $C(t)$ is obtained when the mission is completed (i.e., the system does not fail during the mission or the mission is not aborted in $[0, t]$)). The per time unit reward, when the system is operating, is c_p and the per time unit operational cost is c_0, where $c_0 < c_p$.

4. The penalty C_f is imposed if the system fails during the mission. In case of the premature termination, the fixed penalty C_{pt} $(C_{pt} < C_f)$ is administrated. Further, C_r is an additional reward for the mission completion.

5. The reward after the failure is discarded.

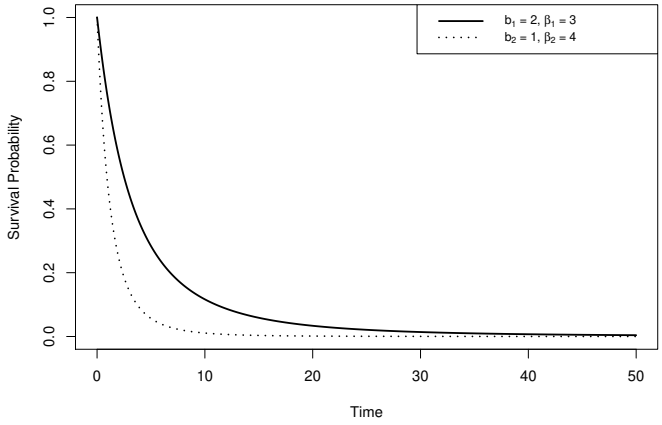
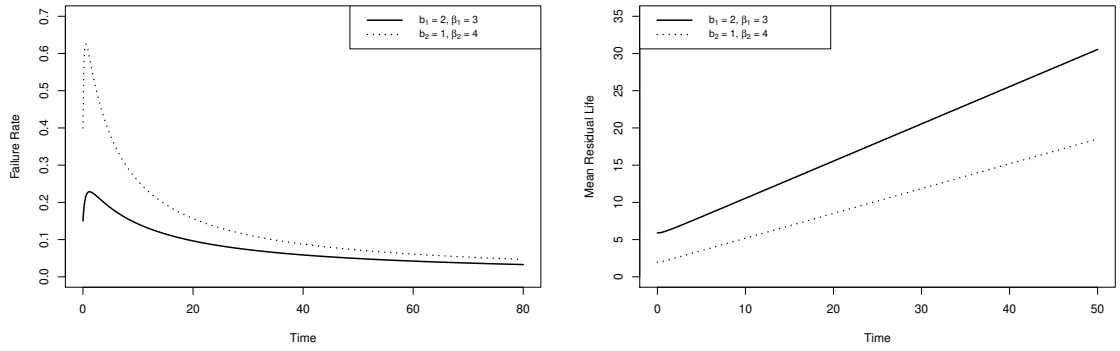


Figure 6.12: Plot of the system's survival function for generalized run shock model-I



(a) Failure rate function

(b) Mean residual life function

Figure 6.13: Plot of the system's failure rate function and mean residual life function for generalized run shock model-I

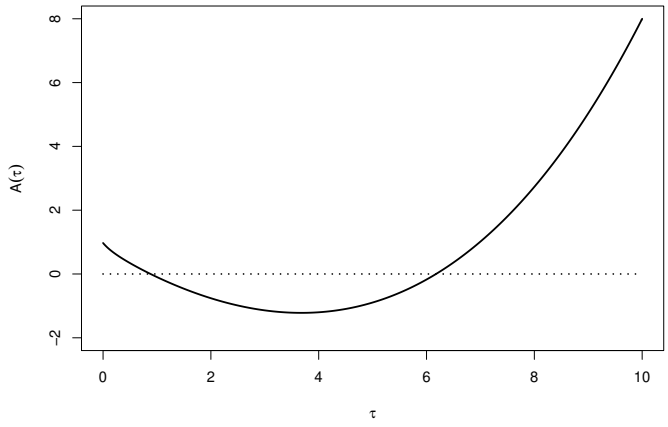


Figure 6.14: Profit comparison function $A(\tau)$ against $\tau \in [0, 10]$

Based on aforementioned assumptions, the profit $C(t)$ for the mission completion is given by [see Finkelstein and Levitin, 2017]

$$C(t) = (c_p - c_0)t + C_r.$$

Note that the mission is aborted at time τ if the total profit at termination exceeds the expected profit in case of mission continuation. The profit at termination at time τ is equal to $(c_p - c_0)\tau - C_{pt}$. On the other hand, the expected profit in the case of mission continuation is

$$\frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\left((c_p - c_0)\, t + C_r\right) - \left(1 - \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\right) C_f,$$

where $\bar{F}_L(t)/\bar{F}_L(\tau)$ is the probability that a system will not fail in the remaining mission time given that it is operable at time τ; here $\bar{F}_L(t)$ is the same as in Theorem 6.3.1. Thus, if for some τ, the expression

$$A(\tau) \stackrel{\text{def.}}{=} \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\left((c_p - c_0)\, t + C_r\right) - \left(1 - \frac{\bar{F}_L(t)}{\bar{F}_L(\tau)}\right) C_f - ((c_p - c_0)\tau - C_{pt})$$

is non-negative, then the mission should not be terminated at time τ. Clearly, $A(0) \geq 0$, as there is no need to terminate the mission that had just started. Since the expression of $A(\tau)$ is complicated, it is not analytically possible to find out the values of τ for which $A(\tau) \geq 0$. Thus, we consider the following numerical example.

Let us assume $t = 10, c_p = 2.5, c_0 = 0.5, C_r = 3, C_f = 8$ and $C_{pt} = 5$. Further, we assume that the run shock model parameters are $r = 2$ and $p = 0.25$, and the process parameters are $b = 1$ and $\beta = 5$. In Figure 6.14, we plot the profit comparison function $A(\tau)$ against $\tau \in [0, 10]$. This figure shows that $A(\tau)$ is in U-shaped. Further, it takes negative values in $\tau \in [0.89, 6.17]$. This implies that the mission should not be terminated in the interval $[0, 0.89)$ and $(6.17, 10]$, whereas it should be aborted just at $\tau = 3.68$ as it

is the optimal solution. In case the mission is not terminated at time $\tau = 3.68$, it may be terminated at any time in the interval $[0.89, 6.17]$. Further, if this is not done, then the mission should not be terminated at all because its termination in the interval $(6.17, 10]$ is not beneficial.