

Effect of nonlinearity in gain-loss assisted photonic systems hosting Exceptional Points

6.1 Introduction

In recent years, the realm of quantum mechanics has been significantly expanded with the introduction of Non-Hermitian quantum mechanics, offering a novel perspective to study open systems. These open systems, characterized by their interactions with the surrounding environment, have manifold applications, especially in the burgeoning field of quantum-inspired photonic systems. Nonlinearity is another pivotal player that significantly influences the behavior of these systems [Reinisch and Gazeau, 2016; Xia et al., 2021]. This type of nonlinearity is known for its role in processes such as second-harmonic generation and focusing-defocusing phenomena in classical optics [Boyd, 2008; Baldwin, 2012; Agrawal, 2012]. When combined with the exotic landscape of EPs , nonlinear effects introduce intriguing new paradigms in the manipulation of light within photonic systems [Benzaouia et al., 2022; Ramezanpour and Bogdanov, 2021; Bai et al., 2022; Ezawa, 2022]. Particularly, the interplay between nonlinearity and EPs results in phenomena such as nonreciprocal optical transmission [Laha et al., 2020b; Jalas et al., 2013; Nazari et al., 2014], frequency conversion [Pick et al., 2017], lasing [Ezawa, 2022] and enhanced sensitivity in sensing applications [Bai et al., 2022]. In non-Hermitian photonics, including optical gain and loss as nonconservative elements has allowed for a wide array of applications, such as nonlinear switching [Suchkov et al., 2016], extreme enhancements in sensing [Meng et al., 2019], and nonreciprocity enhancement [Laha et al., 2020b; Jalas et al., 2013]. These gain-loss assisted systems can be intricately designed to host EPs and modulate interactions among multiple supported modes. The chiral behavior exhibited in the presence of EPs is compelling but raises several questions for higher-order systems or systems with more than two interacting modes. More intriguingly, when nonlinearity is introduced into these higher-order systems, the resultant dynamics are far richer and more complex. For instance, Kerr-type focusing and defocusing nonlinearities can result in different dominating modes at the output, essentially destroying the inherent chirality of the device. This presents a scientific curiosity and a potential for practical applications in integrated photonic devices. This chapter aims to bridge the knowledge gaps by investigating the effects of Kerr nonlinearity on gain-loss assisted photonic systems hosting EPs . It particularly emphasizes the diverse mode conversion processes enabled by the interplay between EPs and nonlinearities. Through comprehensive analytical models, we aim to illuminate the complex dynamics governing these systems and offer novel design strategies for the next generation of integrated photonic devices.

In this chapter, the innovative realm of non-Hermitian photonics receives particular focus, especially concerning waveguides with embedded EPs and the interplay of local Kerr-type nonlinearity with these EPs [Doley et al., 2020; Dey et al., 2020b]. The chapter is divided into two main sections to illuminate distinct yet interconnected phenomena observed in optical waveguides with complex gain-loss profiles. The first section explores a dual-mode optical waveguide characterized by a length-dependent gain-loss variation

designed to host a second-order EP in its parameter plane [Doley et al., 2020]. This section elucidates how dynamically encircling this EP allows for chirality-driven asymmetric mode conversion, whereby the direction of light propagation determines a specific dominating output mode independent of input modes. The investigation includes the impact of local Kerr-type nonlinearity, revealing an intriguing unidirectional asymmetric mode conversion facilitated by the interplay between nonlinearity and the EP . The second section focuses on a more complex three-mode supported planar waveguide structure featuring local Kerr nonlinearity [Dey et al., 2020b]. Initially, in the absence of nonlinearity, an unbalanced gain-loss profile leads to an EP between two coupled modes, leaving the third mode unaffected. The chiral behavior of the device is probed by dynamically encircling this EP . However, upon the introduction of nonlinearity, a dramatic change is observed; the chirality is disrupted, and all modes converge into a specific dominating mode, irrespective of input. An analytical model is constructed to account for this anomalous mode collapsing and to investigate the role of focusing and defocusing nonlinearities in determining different dominating output modes. By traversing through these sections, the chapter aims to comprehensively understand how the complex interplay between gain-loss profiles, EP s, and nonlinearity can lead to various mode-conversion phenomena in optical waveguides. Both chiral and non-chiral behaviors are examined, offering a complete picture of the possibilities inherent in non-Hermitian photonics in the presence of local Kerr-type nonlinearities.

6.2 Effect of a local nonlinearity on the light dynamics around an exceptional point

6.2.1 An analytical model analogous to a dual-mode planar waveguide

The appearance of an EP in a dual-mode planar waveguide can analytically be realized by considering a two-level non-Hermitian Hamiltonian as

$$\mathcal{H} = \begin{pmatrix} \beta_p + i\alpha_p & \eta \\ \eta^* & \beta_q + i\alpha_q \end{pmatrix}, \quad (6.1)$$

where β_p and β_q are the real propagation constants with the respective decay rates α_p and α_q . For simplification, two new terms $\tilde{\beta}_j = \beta_j + i\alpha_j$, ($j = p, q$) are defined as complex propagation constants. The off-diagonal terms η and its complex conjugate η^* appear as perturbation to modulate the coupling between $\tilde{\beta}_p$ and $\tilde{\beta}_q$. The eigenvalues of the Hamiltonian $\mathcal{H}$ can then be expressed as

$$\beta_{\pm} = \frac{\tilde{\beta}_p + \tilde{\beta}_q}{2} \pm \sqrt{\left(\frac{\tilde{\beta}_p - \tilde{\beta}_q}{2}\right)^2 + |\eta|^2}. \quad (6.2)$$

In Eq. (6.2), two individual replacements are introduced: $\tilde{\beta}_{av} = (\tilde{\beta}_p + \tilde{\beta}_q)/2$ and $\delta\tilde{\beta} = (\tilde{\beta}_p - \tilde{\beta}_q)/2$, which lead to a revised form of $\beta_{\pm}$ as

$$\beta_{\pm} = \tilde{\beta}_{av} \pm \sqrt{(\delta\tilde{\beta})^2 + |\eta|^2}. \quad (6.3)$$

At this juncture, two coupled eigenvalues coalesce with $\beta_+ = \beta_-$, setting the condition for the occurrence of an EP . Thus, in the complex η -plane, an EP appears at the critical point $\eta_c = \pm i\delta\tilde{\beta}c$, and the position of the identified EP can be accessed by real and imaginary parts of $\delta\tilde{\beta}$ (where, $\delta\tilde{\beta} \equiv \delta\beta + i\delta\alpha$) with the fulfillment of the conditions $\beta_p = \beta_q$ and $\eta = |\alpha_p - \alpha_q|/2$.

While implementing this analytical scheme in a physical dual-mode waveguide system, the coupling terms (η and η^*) can be introduced in the form of optical gain-loss. Considering the intensity-dependent local nonlinearity in the waveguide, the parameters β_j and α_j ($j = p, q$) in the corresponding Hamiltonian $\mathcal{H}$

vary as a function of signal intensity (I), namely, $\beta_j = \beta_j(I)$ and $\alpha_j = \alpha_j(I)$. A threshold intensity I_{th} can be defined, where only for $I > I_{\text{th}}$ does the effect of nonlinearity come into play. In such cases, along with the introduced gain-loss, the nonlinearity factor indeed influences the overall interaction phenomenon. No effect of nonlinearity occurs for $I < I_{\text{th}}$. The interplay between gain-loss and nonlinearity will be discussed later in context.

6.2.2 Optical waveguide hosting an asymmetric mode conversion scheme

A planar waveguide is designed, as shown schematically in Fig. 6.1 (a). The z -axis is considered the propagation direction for the light beam, and x -axis is the transverse direction. The region $-W/2 < x < W/2$ of the waveguide consists of a core of passive refractive index $n_h = 1.5$, surrounded by cladding with passive refractive index $n_l = 1.46$. By normalising $k = 1$, the width of the structure, $W = 40\lambda/2\pi = 40$ and the length, $L = 10 \times 10^3$ are set in dimensionless units. With this set of characteristic parameters, the waveguide supports only the fundamental mode (FM) and the first higher-order mode (HOM). The non-Hermitian characteristic is introduced into the system through a specific transverse distribution of the gain-loss profile, where the complex profile of $n(x)$ for a specific cross-section of the waveguide is expressed as:

$$n(x) = \begin{cases} n_l + i\gamma & \text{for } W/6 \leq |x| \leq W/2, \\ n_h - i\gamma & \text{for } -W/6 \leq x \leq 0, \\ n_h + i\tau\gamma & \text{for } 0 \leq x \leq W/6. \end{cases} \quad (6.4)$$

Here, γ and τ are the two control parameters that represent the gain coefficient and the loss-to-gain ratio, respectively. The overall refractive index profile for a specific cross-section of the waveguide, as represented by Eq. (6.4), is shown in the upper panel of Fig. 6.1 (b). Non-Hermiticity in the designed waveguide can be modulated by tuning these parameters independently according to the Kramers-Kronig causality relation at a single operating frequency [Phang et al., 2015]. A prototype of the proposed planar waveguide structure can be fabricated by thin film deposition of glass material ($n_h = 1.50$) over a thick silica-glass substrate ($n_l = 1.46$). Achieving the patterned gain-loss profile is possible through a standard photolithography technique or by controlled doping of gain or lossy materials using state-of-the-art fabrication techniques.

In the proposed waveguide, the values of the propagation constants (β) of the quasi-guided modes are calculated by solving the scalar modal equation $[\partial_x^2 + n^2(x)\omega^2 - \beta^2]\psi(x) = 0$, which corresponds to a steady-state mode profile $\psi(x)$. The intensity profile for the propagating modes of the waveguide for the chosen set of parameters is shown in the lower panel of Fig. 6.1 (b) where the green line represents the fundamental mode and the black line represents the higher order mode.

With the onset of gain-loss, the supported modes (say, ψ_{FM} and ψ_{HOM}) are mutually coupled. To encounter an EP , the concept of ARC phenomena [Ghosh and Chong, 2016; Gandhi et al., 2020] between the corresponding propagation constants (say, β_{FM} and β_{HOM}) is exploited, with crossing and anticrossing of their real and imaginary parts, i.e., $\text{Re}(\beta)$ and $\text{Im}(\beta)$. By varying the parameter γ within a chosen range $[0, 0.015]$, such ARC -interactions phenomena between β_{FM} and β_{HOM} are studied for different τ -values. After examining several cases, a specific $\tau = 3.179$ is set, at which β_{FM} and β_{HOM} coalesce in the β -plane near $\gamma \approx 0.008$, as shown in Fig. 6.1 (c), indicating the presence of an EP in the (γ, τ) -plane. Near this point, two topologically different ARC phenomena are observed. Thus, an EP is numerically identified at $\sim (\gamma_{\text{EP}} = 0.008, \tau_{\text{EP}} = 3.179)$. To encircle the identified EP , a closed parameter space in the (γ, τ) -plane is considered, following the equations:

$$\gamma(\phi) = \gamma_0 \sin\left(\frac{\phi}{2}\right); \quad \tau(\phi) = \tau_{\text{EP}} + a \sin(\phi). \quad (6.5)$$

Here γ_0 and a are two characteristics parameters, and ϕ , ($0 \leq \phi \leq 2\pi$) is a tunable angle variable that governs the variation of γ and τ around the EP . To ensure the presence of EP inside the parametric loop, $\gamma_0 > \gamma_{\text{EP}}$ must be considered. The variation of ϕ from 0 to 2π gives the CW progression of γ and τ . Conversely, the

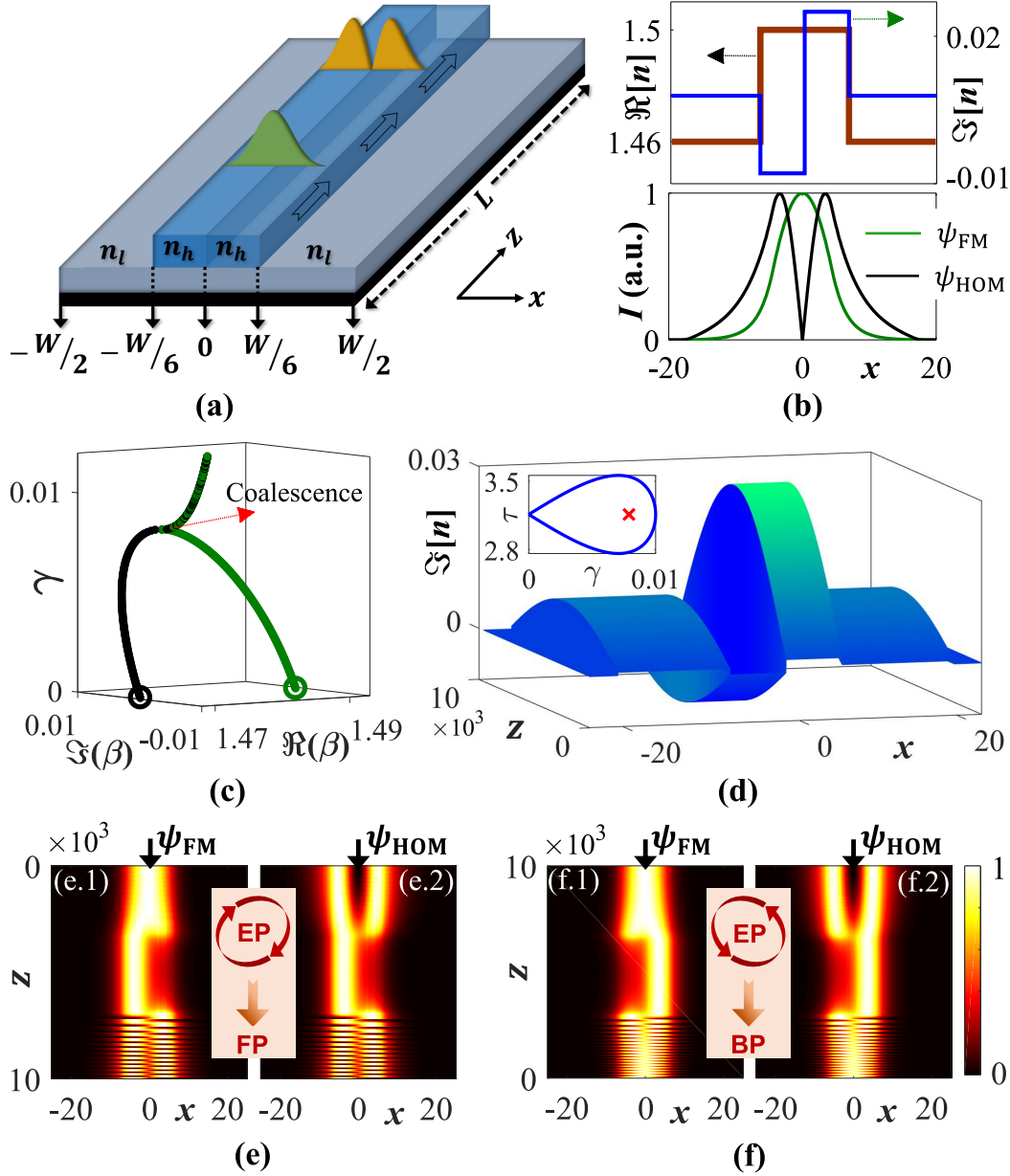


Figure 6.1: Waveguide hosting a dynamical EP-encirclement scheme. (a) Schematic of the designed optical waveguide with transverse x -axis. Propagation is considered along the z -axis. (b) Transverse refractive index profile $n(x)$ showing $\Re(n)$ (solid brown line) and $\Im(n)$ (upper panel) at specific $\gamma = 0.0095$ and $\tau = 3.1799$. Normalized output field intensity profiles of the supported modes (ψ_{FM} and ψ_{HOM}). (c) Trajectories of complex β values with respect to γ for a specific value of $\tau = 3.1799$, where the two β s coalesce near $\gamma = 0.0095$. (d) Length dependent variation of $\Im(n)$ after mapping the chosen parameter as shown in the inset. Chosen topological structure of the waveguide with simultaneous variation of γ and τ around the identified EP (inset). (e) Beam evolution for the CW dynamical encirclement scheme. (e.1) Evolution of ψ_{FM} to ψ_{HOM} . (e.2) Evolution of ψ_{HOM} to ψ_{HOM} . (f) Beam evolution for the CCW dynamical encirclement scheme. (f.1) Evolution of ψ_{FM} to ψ_{FM} . (f.2) Evolution of ψ_{HOM} to ψ_{FM} .

variation of ϕ from 2π to 0 indicates the CCW progression of γ and τ . Unlike the conventional circular loops, this specific shape of the parametric loop facilitates the device to achieve passive modes at the input and output interface [Ghosh and Chong, 2016; Gandhi et al., 2020; Dey et al., 2022]. To consider the dynamical encirclement and realize the actual beam propagation through the waveguide around an EP , this parameter space (given by Eq. (6.5)) needs to be mapped along the z -direction through the complete length

(L) of the waveguide. For this mapping, $\phi = 2\pi z/L$ is chosen to set $\phi = 0$ at $z = 0$ and $\phi = 2\pi$ at $z = L$. Therefore, the length-dependent variation of γ and τ is expressed as

$$\gamma(z) = \gamma_0 \sin \left[\frac{\pi z}{L} \right]; \quad \tau(z) = \tau_{EP} + a \sin \left[\frac{2\pi z}{L} \right]. \quad (6.6)$$

Such a length-dependent parameter space has been shown in Fig. 6.1 (d) with $\gamma_0 = 0.009 (> \gamma_{EP})$ and $a = 0.5$. The shape of the corresponding closed-loop in the (γ, τ) -plane (following Eq. (6.5)) has been shown in the inset. Thus, a complete encirclement around the EP following Eq. (6.5) is equivalent to one complete pass of light along the length of the waveguide; where the CW encirclement is equivalent to the propagation from $z = 0$ to $z = L$, i.e., the forward propagation, and the CCW encirclement is equivalent to the propagation from $z = L$ to $z = 0$, i.e., the backward propagation. Now, such propagations of the modes through the waveguide should follow the $TDSE$, where time is the quantum-mechanical counterpart of the z -axis. To study modal propagation, scalar beam propagation is used to solve the equation $2i\omega\partial_z\psi(x, z) = -[\partial_x^2 + \Delta n^2(x, z)\omega^2]\psi(x, z)$ ($\Delta n^2(x, y) \equiv n^2(x, z) - n_l^2$), taking into account the paraxial approximation and the adiabatic variation of $\text{Im}(n)$ in the z -axis.

The light dynamics in the designed waveguide following the dynamical EP encirclement scheme [described in Fig. 6.1 (d)] are depicted in Figs. 6.1 (e) and 6.1 (f). Light is launched at $z = 0$ for the implementation of a CW encirclement scheme, resulting in the conversion of both ψ_{FM} and ψ_{HOM} to ψ_{HOM} at $z = L$, as illustrated in plots (e.1) and (e.2) of Fig. 6.1 (e). In this scenario, one NAT corresponds to ψ_{HOM} . For the CCW encirclement scheme, light is launched at $z = L$, leading to the conversion of both ψ_{FM} and ψ_{HOM} to ψ_{FM} , as evident in Fig. 6.1 (f). Here, ψ_{HOM} converts adiabatically to ψ_{FM} while ψ_{FM} undergoes a NAT . A breakdown in adiabaticity in modal evolutions due to the dynamical variation of the control parameters (γ and τ) around the EP is observed [Gilyar et al., 2013; Uzdin and Moiseyev, 2012; Ghosh and Chong, 2016], resulting in a chirality-driven asymmetric mode conversion phenomenon. In this setup, irrespective of the choice of inputs, light converts into a specific mode depending on the direction of propagation.

6.2.3 Effects of nonlinearity on beam dynamics

In conjunction with optical gain-loss in the waveguide, local Kerr nonlinearity is introduced as an additional parameter to control the interactions between the two supported modes. The Kerr nonlinearity takes the form $\Delta n_{NL}(x, y) = \sigma n_2 I$, where n_2 is the nonlinear-coefficient, I represents the signal intensity, and $\sigma = +1$ signifies focusing nonlinearity while $\sigma = -1$ indicates defocusing nonlinearity. In a prototype of the proposed waveguide, the local nonlinearity distribution can be realized through the intensity (I) of the injected light signal. By varying the signal power, changes in nonlinearity amounts are attainable. More broadly, nonlinearity amounts can be quantified in relation to $\Delta n (= n_h - n_l)$ via the percentage of nonlinearities as $(\Delta n_{NL}/\Delta n) \times 100\%$. During numerical investigations into the dynamics of the eigenmodes, the appropriate value of the nonlinear coefficient (n_2) for silica-based materials is selected, and the light signal is launched with an initial power of 80 watts.

While introducing nonlinearity below 1%, no observable effect of nonlinearity on the overall phenomenon of asymmetric mode conversion is evident, as described in the preceding section. In this context, the EP -aided nonadiabatic corrections predominantly control the light dynamics through the waveguide, with the conversion of the eigenmodes being direction-dependent regarding light propagation. A starkly different scenario emerges when increasing nonlinearity above this threshold value. Implementing a CW encirclement scheme and considering the propagation of light from $z = 0$ to $z = L$, a choice of $\sigma = 1$ is made for focusing nonlinearity. Upon reaching a nonlinearity of 3%, both modes collapse to ψ_{HOM} , evident in Fig. 6.2 (a.1) and Fig. 6.2 (a.2). Plots (b.1) and (b.2) of Fig. 6.2 (b) depict the normalized field intensities at two distinct ports of the waveguide, corresponding to the beam propagation displayed in plots (a.1) and (a.2) of Fig. 6.2 (a). In these diagrams, solid and dotted curves of their respective colors (green for ψ_{FM} and black for ψ_{HOM}) delineate input and output intensities. Subsequently, with $\sigma = -1$ chosen for defocusing

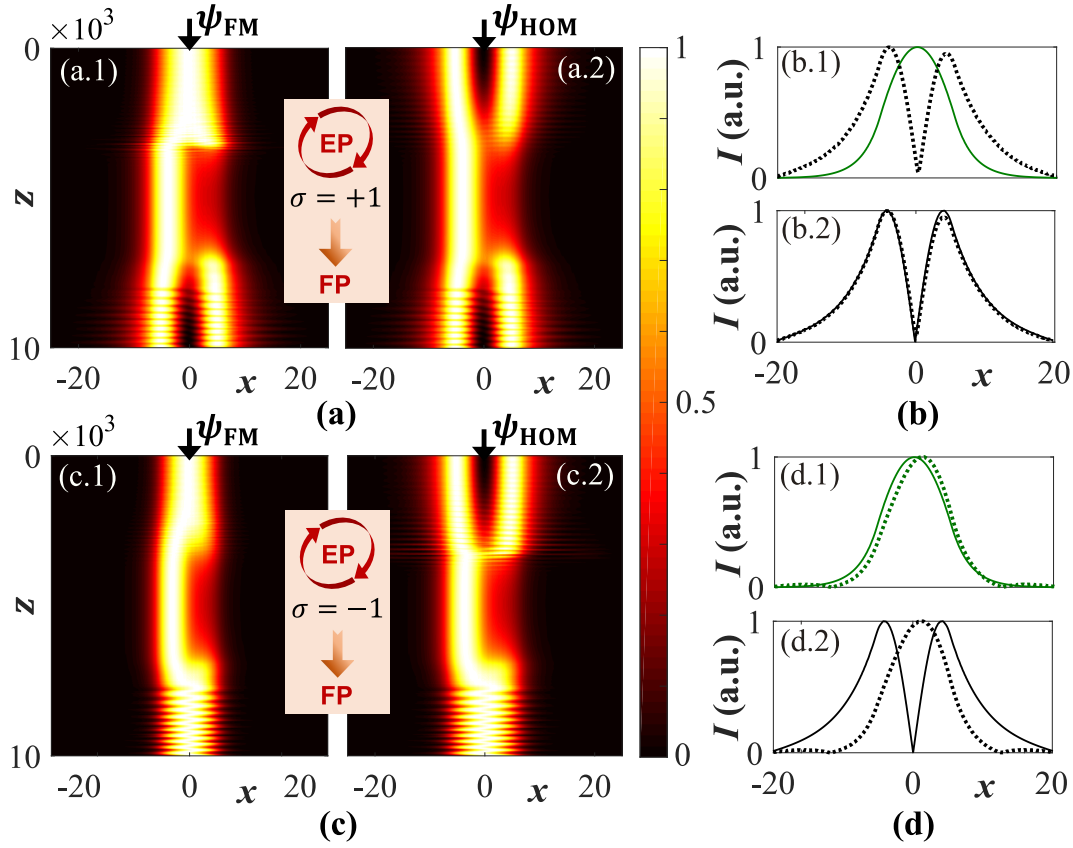


Figure 6.2: Beam propagation results in presence of 3% nonlinearity. (a) The conversions $\psi_{\text{FM}} \rightarrow \psi_{\text{HOM}}$ (plot a.1) and $\psi_{\text{HOM}} \rightarrow \psi_{\text{HOM}}$ (plot a.2) in the presence of focusing nonlinearity (FN) while encircling the *EP* in the *CW* direction. (b) Plots (b.1) and (b.2) show the normalized field intensities at $z = 0$ (solid curves indicating the inputs) and $z = L$ (dotted curves indicating the outputs) corresponding to the beam propagation shown in plots (a.1) and (a.2), respectively. (c) Beam propagation results showing the conversions $\psi_{\text{FM}} \rightarrow \psi_{\text{FM}}$ (plot c.1) and $\psi_{\text{HOM}} \rightarrow \psi_{\text{FM}}$ (plot c.2) in the presence of defocusing nonlinearity (DFN) while encircling the *EP* in the *CW* direction. (d) Plots (d.1) and (d.2) show the normalized field intensities at $z = 0$ (solid curves indicating the inputs) and $z = L$ (dotted curves indicating the outputs) corresponding to the beam propagation shown in plots (c.1) and (c.2), respectively.

nonlinearity, the modal dynamics with the onset of the identical nonlinearity level is investigated. This exploration results in the conversion of both modes to ψ_{FM} , visible in Fig. 6.2 (c.1) and Fig. 6.2 (c.2). The associated normalized output intensities, correlating to the beam propagation demonstrated in plots (c.1) and (c.2) of Fig. 6.2 (c), are illustrated by the dotted green and black lines in Fig. 6.2 (d.1) and Fig. 6.2 (d.2). An *CCW* encirclement scheme is implemented to alter the direction of light propagation to examine the modal dynamics. For a defocusing type nonlinearity ($\sigma = -1$) and increasing the nonlinearity amount to 5%, it becomes evident that both eigenmodes collapse to ψ_{FM} , as depicted in plots (a.1) and (a.2) of Fig. 6.3 (a). Plots (b.1) and (b.2) of Fig. 6.3 (b) illustrate the normalized input (solid green and black lines for FM and HOM, respectively) and output intensities (dotted lines in corresponding colors) for the propagations detailed in Fig. 6.3 (a.1) and (a.2). Introducing the same level of focusing nonlinearity ($\sigma = 1$) leads to the conversion of both eigenmodes to ψ_{HOM} , observable in plots (c.1) and (c.2) of Fig. 6.3 (c). Similarly, the associated input-output intensities are presented in plots (d.1) and (d.2) of Fig. 6.3 (d).

Thus, an unconventional dynamical *EP*-encirclement scheme in the presence of local Kerr-nonlinearity (above a specific threshold) in the spatial distribution of the refractive index of the waveguide, results in a nonchiral, unidirectional asymmetric mode conversion scheme, as illustrated in Figs. 6.2 and 6.3. This approach is entirely different from the *EP*-aided chirality-driven asymmetric-mode-conversion scheme pre-

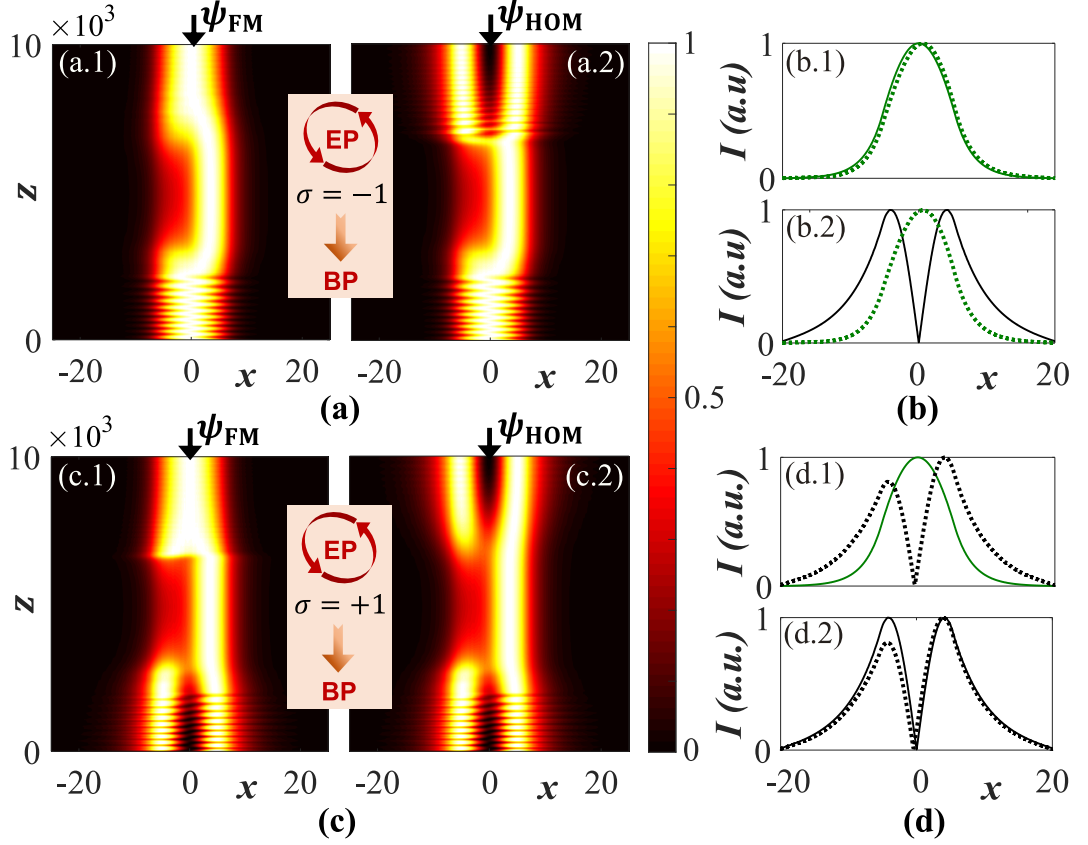


Figure 6.3: Beam propagation results in presence of 5% nonlinearity. (a) The conversions $\psi_{\text{FM}} \rightarrow \psi_{\text{FM}}$ (plot a.1) and $\psi_{\text{HOM}} \rightarrow \psi_{\text{FM}}$ (plot a.2) in the presence of DFN while encircling the *EP* in the *CCW* direction. (b) Plots (b.1) and (b.2) show the normalized field intensities at $z = 0$ (solid curves indicating the inputs) and $z = L$ (dotted curves indicating the outputs) corresponding to the beam propagation shown in plots (a.1) and (a.2), respectively. (c) Beam propagation results showing the conversions $\psi_{\text{FM}} \rightarrow \psi_{\text{HOM}}$ (plot c.1) and $\psi_{\text{HOM}} \rightarrow \psi_{\text{HOM}}$ (plot c.2) in the presence of FN while encircling the *EP* in the *CCW* direction. (d) Plots (d.1) and (d.2) show the normalized field intensities at $z = 0$ (solid curves indicating the inputs) and $z = L$ (dotted curves indicating the outputs) corresponding to the beam propagation shown in plots (c.1) and (c.2), respectively.

sented in Figs. 6.1 (e) and 6.1 (f) in the absence of nonlinearity. In the absence of nonlinearity, the proposed waveguide exhibits a chirality-driven asymmetric mode conversion scheme. At the output, the waveguide delivers ψ_{HOM} during propagation in the forward direction, whereas ψ_{FM} during backward propagation, irrespective of the input choice. Here, the dominating output is governed by the device's chirality, i.e., the direction of light propagation. For this particular configuration, a threshold nonlinearity amount at 1% is identified. Below this 1% nonlinearity level, the waveguide retains the standard *EP*-aided chiral behavior. However, exceeding this threshold leads the waveguide to facilitate one-way asymmetric mode conversion. Nonlinearity above this threshold significantly alters transmission levels during light propagation in two different directions [Choi et al., 2017]. In the presence of 3% local nonlinearity, the waveguide provides high transmission in the forward direction and reduced transmission in the backward direction. During forward propagation, the same amount of focusing type nonlinearity results in ψ_{HOM} output, while defocusing type leads to ψ_{FM} . Thus, without altering the light's propagation direction, a different dominating mode at the same output port emerges, illustrating a nonchiral asymmetric mode conversion phenomenon. Considering backward propagation under these conditions, similar outputs for focusing and defocusing nonlinearities are observed, but with reduced transmission for 3% nonlinearity. An average of 2 *dB* difference between forward and backward transmissions is detected. Further optimization of nonlinearity by adjusting the signal intensity reveals that at 5% nonlinearity, the waveguide provides high transmission in the backward direction

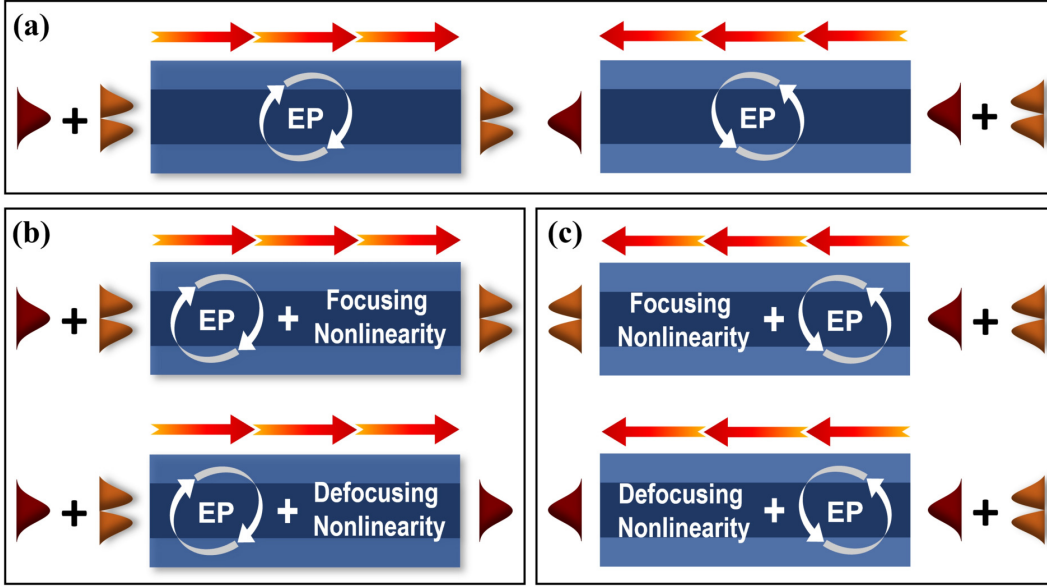


Figure 6.4: Schematic representation of the overall scheme explored in this work. (a) The chirality-driven asymmetric mode conversion scheme in the presence of nonlinearity below the threshold level (or in the absence of nonlinearity). Nonchiral unidirectional asymmetric mode conversion schemes (b) for 3% nonlinearity (high transmission in the forward direction) and (c) for 5% nonlinearity (high transmission in the backward direction). The group of red arrows indicate the direction of light propagation.

and reduced transmission in the forward. Under these conditions, during backward propagation, focusing type nonlinearity leads to ψ_{HOM} , while defocusing results in ψ_{FM} at the output. These observations are schematically presented in Fig. 6.4. Fig. 6.4 (a) illustrates the chirality-driven asymmetric mode conversion scheme in the absence of nonlinearity (or below the threshold level), and Figs. 6.4 (b) and 6.4 (c) showcase two nonchiral unidirectional asymmetric mode conversions for distinct nonlinearity levels, where high transmission is observed in two different directions. In the proposed scheme, the nonchiral asymmetric mode conversion is independent of the direction of light propagation (or direction of EP encirclement) but primarily hinges on the type of nonlinearity. A breakdown in the device's chirality is observed for a specific nonlinearity type. In this context, the waveguide delivers ψ_{HOM} in the presence of focusing nonlinearity and ψ_{FM} with defocusing nonlinearity, irrespective of the direction of light propagation. However, to achieve high transmission, two different nonlinearity amounts must be selected when considering forward and backward propagation and also the choice of inputs.

6.2.4 Analytical approach toward the nonadiabatic corrections in the beam dynamics

The analytical treatment behind such nonadiabatic state evolution of one of the two eigenstates is presented here [Gilary et al., 2013]. It is assumed that the 2×2 Hamiltonian $\mathcal{H}(t)$, given in Eq. (6.1), depends on two generic time-dependent potential parameters $\kappa_1(t)$ and $\kappa_2(t)$ that includes the simultaneous effect of gain-loss and nonlinearity. However, the effect of nonlinearity is noticeable only when its amount exceeds the threshold level (1% for the proposed waveguide configuration). Instead of $\beta_{\pm}$, this discussion considers the physical eigenvalues $\beta_{\text{FM}}^{\text{ad}}(\kappa_1, \kappa_2)$ and $\beta_{\text{HOM}}^{\text{ad}}(\kappa_1, \kappa_2)$ with corresponding eigenvectors $\psi_{\text{FM}}^{\text{ad}}(\kappa_1, \kappa_2)$ and $\psi_{\text{HOM}}^{\text{ad}}(\kappa_1, \kappa_2)$ to represent the proposed waveguide. The time dependence of the potential parameters $\kappa_1(t)$ and $\kappa_2(t)$ governs the dynamical nonadiabatic corrections in the solutions of the $TDSE$ associated with the Hamiltonian $\mathcal{H}(t)$. Such nonadiabatic correction terms due to dynamical encirclement around the EP can be written as:

$$\mathcal{M}_{F \rightarrow H}^{\text{NA}} = C_{F \rightarrow H} \exp \left\{ -i \oint_0^T \Delta \beta_{F,H}^{\text{ad}}[\kappa_1, \kappa_2] dt \right\}, \quad (6.7a)$$

$$\mathcal{M}_{H \rightarrow F}^{\text{NA}} = C_{H \rightarrow F} \exp \left\{ +i \oint_0^T \Delta \beta_{F,H}^{\text{ad}}[\kappa_1, \kappa_2] dt \right\}, \quad (6.7b)$$

with the pre-exponent terms

$$C_{F \rightarrow H} = \left\langle \psi_{\text{FM}}^{\text{ad}}(\kappa_1, \kappa_2) \left| \sum_{j=1}^2 \dot{\kappa}_j \frac{\partial}{\partial \kappa_j} \right| \psi_{\text{HOM}}^{\text{ad}}(\kappa_1, \kappa_2) \right\rangle, \quad (6.8a)$$

$$C_{H \rightarrow F} = \left\langle \psi_{\text{HOM}}^{\text{ad}}(\kappa_1, \kappa_2) \left| \sum_{j=1}^2 \dot{\kappa}_j \frac{\partial}{\partial \kappa_j} \right| \psi_{\text{FM}}^{\text{ad}}(\kappa_1, \kappa_2) \right\rangle. \quad (6.8b)$$

The factor $\Delta \beta_{F,H}^{\text{ad}}$ (in the exponent terms of Eq. (6.7)) can be expressed as

$$\begin{aligned} \Delta \beta_{F,H}^{\text{ad}}(\kappa_1, \kappa_2) &= \beta_{\text{FM}}^{\text{ad}}(\kappa_1, \kappa_2) - \beta_{\text{HOM}}^{\text{ad}}(\kappa_1, \kappa_2) \\ &\equiv \text{Re}[\Delta \beta_{F,H}^{\text{ad}}(\kappa_1, \kappa_2)] - i \Delta \gamma_{F,H}^{\text{ad}}(\kappa_1, \kappa_2) \end{aligned} \quad (6.9)$$

In Eq. (6.7), $\mathcal{M}_{F \rightarrow H}^{\text{NA}}$ and $\mathcal{M}_{H \rightarrow F}^{\text{NA}}$ represent the nonadiabatic correction factor for the conversion of $|\psi_{\text{FM}}^{\text{ad}}\rangle$ to $|\psi_{\text{HOM}}^{\text{ad}}\rangle$ and vice-versa, respectively. T is the duration of EP -encirclement and $|\Delta \gamma_{F,H}^{\text{ad}}|$ is the relative gain between the supported modes. Since the pre-exponent terms given by Eq. (6.8) contain the time derivative of the two potential parameters $\dot{\kappa}_j$ ($j = 1, 2$), the divergence in T due to the exponential terms of $\mathcal{M}_{F \rightarrow H}^{\text{NA}}$ and $\mathcal{M}_{H \rightarrow F}^{\text{NA}}$ surpasses the decay of T^{-1} incorporated in $C_{F \rightarrow H}$ and $C_{H \rightarrow F}$, respectively. Thus, for the situations, $\Delta \gamma_{F,H}^{\text{ad}} > 0$ and $T \rightarrow \infty$, the correction terms $\mathcal{M}_{F \rightarrow H}^{\text{NA}} \rightarrow 0$ and $\mathcal{M}_{H \rightarrow F}^{\text{NA}} \rightarrow \infty$. Here, the situation $\mathcal{M}_{F \rightarrow H}^{\text{NA}} \rightarrow 0$ yields the adiabatic evolution of $|\psi_{\text{FM}}^{\text{ad}}\rangle$, and the situation $\mathcal{M}_{H \rightarrow F}^{\text{NA}} \rightarrow \infty$ yields the nonadiabatic evolution of $|\psi_{\text{HOM}}^{\text{ad}}\rangle$. Therefore, under a slow parametric variation encircling an EP , out of the two considered modes, only one mode with an overall lower decay rate undergoes with the adiabatic evolution whereas the other one evolves nonadiabatically.

Considering the proposed waveguide configuration in the absence of nonlinearity (or in the presence of nonlinearity below the threshold amount), and while taking into account the CW dynamical encirclement scheme, it is observed that $\mathcal{M}_{F \rightarrow H}^{\text{NA}} \rightarrow 0$ and $\mathcal{M}_{H \rightarrow F}^{\text{NA}} \rightarrow \infty$. Accordingly, the mode ψ_{FM} evolves adiabatically and converts to ψ_{HOM} , whereas ψ_{HOM} evolves nonadiabatically and remains in ψ_{HOM} , as illustrated in Fig. 6.1 (e). Similarly, for the CCW dynamical encirclement scheme, the results are $\mathcal{M}_{F \rightarrow H}^{\text{NA}} \rightarrow \infty$ and $\mathcal{M}_{H \rightarrow F}^{\text{NA}} \rightarrow 0$, leading to the nonadiabatic evolution of ψ_{FM} ($\rightarrow \psi_{\text{FM}}$) and the adiabatic conversion of ψ_{HOM} to ψ_{FM} , as shown in Fig. 6.1 (f).

In the presence of nonlinearity above the threshold level, the desired outputs for the nonchiral unidirectional asymmetric-mode-conversions shown in Fig. 6.2 and 6.3 can also be predicted analytically. The application of different types of nonlinearity in the optical waveguide modifies the refractive index profile locally, which in turn changes the β values of the quasi-guided modes. Accordingly, the sign of relative-gain factor $\Delta \gamma_{F,H}^{\text{ad}}$ in Eq. (6.9) also gets modified. For both optimized nonlinearity levels and focusing-type nonlinearity, it is observed that $\Delta \gamma_{F,H}^{\text{ad}} > 0$, irrespective of the direction of light propagation. This criterion yields the conversions of both ψ_{FM} and ψ_{HOM} to ψ_{HOM} , as illustrated in Figs. 6.2 (a,b) and 6.3 (c,d), respectively. In this scenario, ψ_{FM} evolves adiabatically and ψ_{HOM} behaves nonadiabatically. Conversely, with defocusing-type nonlinearity and for both optimized nonlinearity-amounts, it is found that $\Delta \gamma_{F,H}^{\text{ad}} < 0$, irrespective of the direction of light propagation. As a result, ψ_{FM} behaves nonadiabatically and remains in ψ_{FM} , whereas ψ_{HOM} evolves adiabatically and converts to ψ_{FM} , as shown in Figs. 6.2 (c,d) and 6.3 (a,b), respectively.

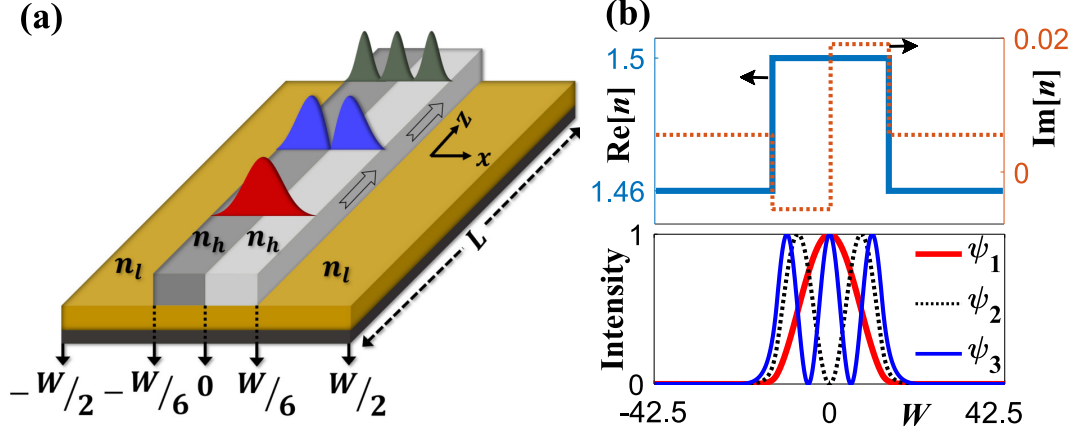


Figure 6.5: (a) Schematic of the designed optical waveguide with transverse x -axis (considering propagation along z -axis). (b) (Upper panel) Transverse refractive index profile $n(x)$ showing $\text{Re}(n)$ (solid blue line) and $\text{Im}(n)$ a specific $\gamma = 0.005$ and $\tau = 3.45$ (dotted brown line). (Lower panel) Normalized field-intensity profiles of the supported modes ψ_j ($j = 1, 2, 3$).

6.3 Nonlinearity-induced anomalous mode collapse and nonchiral asymmetric mode switching around multiple exceptional points

Beyond the dual-mode systems previously reported in [Ghosh and Chong, 2016; Doppler et al., 2016; Doley et al., 2020], achieving mode switching or retrieval in few-mode or multimode systems could offer compactness, enhancing feasibility for integrated devices. Recent investigations, such as the one presented in [Zhang and Chan, 2019], have detailed nonchiral state dynamics around multiple EP s, determined by the choice of the starting point during encirclement in the parameter space. The prospect of hosting nonchiral state dynamics in a multimode yet planar waveguide geometry presents an intriguing avenue for device applications, especially when coupled with a dynamically tunable parameter. This section focuses on a gain-loss assisted, three-mode supported planar waveguide structure complemented by local Kerr nonlinearity [Dey et al., 2020b]. A specific topological structure of the waveguide, accompanied by appropriate gain-loss variation, is designed to modulate the interactions among the three supported modes. An EP between two coupled modes is observed by adjusting the unbalanced gain-loss profile without nonlinearity, leaving the third mode unaffected. The chiral characteristics of the device are then analyzed by dynamically encircling this identified EP . It's established that the device exhibits immutable chiral behavior. Based on the encircling direction and corresponding EP -aided nonadiabatic corrections, one dominant mode from the pair of coupled modes persists, coexisting with the noninteracting mode. However, introducing nonlinearity into the optical medium prompts the previously unaffected mode to interact with the other two. Multiple second-order EP s emerge by altering the gain-loss profile further, analytically linking the three coupled modes. When enclosing at least two EP s within the dynamic parametric loop, the dynamics of the coupled eigenmodes come into play. Despite the input mode choice, all modes converge into a specific dominating mode. Notably, this process disrupts the device's chirality. To elucidate this unexpected mode collapsing behavior, an analytical model is introduced. The section delves deeper into the impact of nonlinearity on nonadiabatic mode conversions around multiple EP s. The findings reveal that, even in the absence of chirality, equal amounts of focusing nonlinearity (FN) and defocusing nonlinearity (DFN) yield different dominating output modes.

6.3.1 Design of an optical waveguide

To emulate a non-Hermitian system, as illustrated in Fig. 6.5 (a), a step-index planar optical waveguide is considered, possessing a core and a cladding with refractive indices $n_h = 1.5$ and $n_l = 1.46$,

respectively. The wave vector is to 1, the total width is set at $W = 85\lambda/2\pi = 85$, and the operating length is fixed at $L = 15 \times 10^3$ in a dimensionless unit. To incorporate non-Hermiticity into the passive waveguide, a transverse unbalanced gain-loss profile is introduced as follows:

$$n(x) = \begin{cases} n_h - i\gamma, & -W/6 \leq x \leq 0 \\ n_h + i\tau\gamma, & 0 \leq x \leq W/6 \\ n_l + i\gamma, & W/6 \leq |x| \leq W/2. \end{cases} \quad (6.10)$$

Eq. (6.10) delineates the overall refractive index profile for a specific cross-section of the waveguide, as illustrated in the upper panel of Fig. 6.5 (b); here, two independent parameters γ and τ signify the gain-coefficient and loss-to-gain ratio, respectively. Adhering to the Kramers-Kronig causality relation at a single operating frequency [Phang et al., 2015], it is possible to independently tune γ and τ along the longitudinal direction to modulate the overall non-Hermiticity. Given the selected parameter set, the waveguide accommodates three quasi-guided linearly polarized modes, depicted in the lower panel of Fig. 6.5 (b), named LP_{01} , LP_{11} , and LP_{02} . In this context, these modes are denoted as ψ_j where ($j = 1, 2, 3$), and the corresponding propagation constants β_j , ($j = 1, 2, 3$) are computed using the scalar modal equation $[\partial_x^2 + n^2(x)\omega^2 - \beta^2]\psi(x) = 0$. To manage the interactions between the supported modes, in conjunction with optical gain-loss, a local Kerr-nonlinearity is introduced, taking the form $\Delta n_{NL}(x, z) = \sigma n_2 I$ (with n_2 indicating the nonlinear-coefficient, I signifying signal-intensity, and $\sigma = \pm 1$ for focusing and defocusing nonlinearity, respectively). The actual nonlinearity level is quantified as $(\Delta n_{NL}/\Delta n) \times 100\%$ with $\Delta n = (n_h - n_l)$.

6.3.2 Chiral light propagation scheme in the absence of nonlinearity

To identify an $EP2$ between two coupled modes, the transition between two topologically dissimilar ARC phenomena between the β -values of the corresponding modes with crossing/anticrossing in $\text{Re}[\beta]$ and $\text{Im}[\beta]$ in the vicinity of a particular singular point is recognized [Ghosh and Chong, 2016]. In the absence of nonlinearity and by varying γ within the range from 0 to 0.01, the dynamics of β_j ($j = 1, 2, 3$) are tracked for different τ values, which leads to the numerical identification of an $EP2$ at $(\gamma_{EP} = 0.0023, \tau_{EP} = 3.315)$ (designated as $EP^{(1)}$, as marked in Fig. 6.6 (a)), where ψ_1 and ψ_2 are analytically connected. In contrast, ψ_3 remains unaffected. To dynamically encircle the identified $EP2$, an enclosed parametric loop with a length-dependent distribution of γ and τ is chosen, following by the equations

$$\gamma(\phi) = \gamma_0 \sin \left[\frac{\pi z}{L} \right]; \quad \tau(\phi) = \tau_{EP} + a \sin \left[\frac{2\pi z}{L} \right] \quad (6.11)$$

Here, γ_0 and a are two characteristic parameters; for a proper enclosure of the $EP2$, it is essential that $\gamma_0 > \gamma_{EP}$ and $a > 0$. The shape of the parametric loop in the (γ, τ) -plane is displayed in Fig. 6.6 (a) with chosen values $\gamma_0 = 0.007$ and $a = 0.12$. The corresponding distribution of the $\text{Im}[n(x, z)]$ is presented in Fig. 6.6 (b). Given the defined shape of the parameter space in the (γ, τ) -plane, γ equals 0 for both $z = 0$ and $z = L$. Consequently, the complete profile of $\text{Im}[n(x, z)]$ from $z = 0$ to $z = L$ encloses the $EP2$ dynamically, and passive modes are obtained at the input and output interfaces, avoiding any loss-dominated modes. One direction of propagation indicates CW encirclement, while the opposite direction indicates CCW encirclement. The propagation of the modes through the waveguide is governed by the $TDSE$ using z as the time axis. Under the paraxial approximation and the variation of $\text{Im}(n)$ within the adiabatic limit, the scalar beam-propagation method is employed to solve the equation $2i\omega\partial_z\psi(x, z) = -[\partial_x^2 + \Delta n^2(x, z)\omega^2]\psi(x, z)$ (with $\Delta n^2(x, z) \equiv n^2(x, z) - n_l^2$) for analyzing modal propagations.

Following the dynamical encirclement scheme described in Fig. 6.6, the beam propagations of ψ_j , ($j = 1, 2, 3$) are illustrated in Fig. 6.7. For a CW encirclement scheme, the light is launched at $z = 0$, as

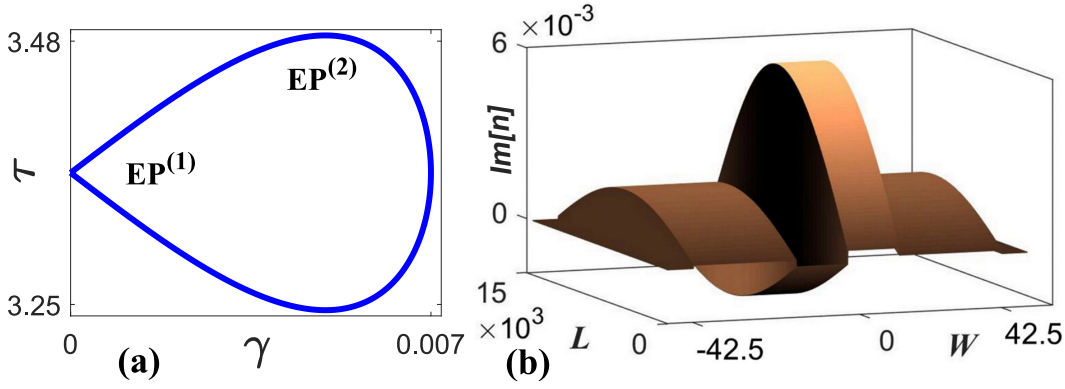


Figure 6.6: (a) Chosen topological structure of the waveguide with simultaneous variation of γ and τ around two embedded EPs . Here, $EP^{(2)}$ appears inside the loop only in the presence of nonlinearity. In absence of nonlinearity, the loop encloses only $EP^{(1)}$. (b) Length-dependent variation of $\text{Im}(n)$ after mapping the chosen parameter space as shown in (a).

depicted in Fig. 6.7 (a). In this scenario, both ψ_1 and ψ_2 associated with $EP^{(1)}$ convert essentially to ψ_2 at $z = L$. Consequently, one NAT corresponds to ψ_2 . Notably, ψ_3 remains unaffected by the presence of $EP^{(1)}$ and persists as ψ_3 at $z = L$. Launching the light at $z = L$ implements an CCW encirclement scheme; the related beam propagation results are given in Fig. 6.7 (b). In this configuration, ψ_1 undergoes a NAT , ψ_2 adiabatically converts to ψ_1 at $z = 0$, yet ψ_3 remains unchanged. Fig. 6.7 (c) displays a comparative study among the output field intensities under different launching conditions, considered for commonly excited field intensities at the input (as shown in Fig. 6.7 (c.1)). When encircling the $EP2$ in the CW direction, a combination of ψ_2 and ψ_3 is obtained at the output, with associated output field intensities presented in Fig. 6.7 (c.2). Conversely, encircling the $EP2$ in the CCW direction yields a combination of ψ_1 and ψ_3 at the output, with relevant output field intensities in Fig. 6.7 (c.3). Due to the breakdown in adiabaticity during the evolutions of the coupled modes, the chiral transmission behavior becomes evident for the dynamical encirclement scheme enclosing only $EP^{(1)}$, even in the presence of the non-interacting ψ_3 . Given that the parametric loop encloses only $EP^{(1)}$, the overall loss distribution predominantly impacts the coupled modes ψ_1 and ψ_2 but spares ψ_3 . During transmission, one of the coupled modes that evolve with a higher average loss compared to the other exhibits nonadiabatic behavior [Ghosh and Chong, 2016; Gilary et al., 2013].

6.3.3 Nonchiral light propagation scheme in the presence of nonlinearity

The investigations described above were conducted in the absence of nonlinearity when ψ_1 and ψ_2 interact, with the simultaneous presence of non-interacting ψ_3 . With the introduction of intensity-dependent Kerr-induced nonlinearity in the spatial index distribution, the dynamics of the eigenmodes warrant further investigation. The onset of nonlinearity locally modifies the real refractive index profile, altering the complex β -values of all modes. The light dynamics through the waveguide for various nonlinearity levels were explored by varying the input intensity. Observations reveal that the proposed device demonstrates similar chirality-driven light dynamics for a nonlinearity level up to 1.2%, as depicted in Fig. 6.7. However, upon reaching a nonlinearity of 1.2% in the optical medium, the corresponding local modifications in $n(x)$ significantly influence the dynamics of ψ_3 , initiating its interaction with the other two modes ψ_1 and ψ_2 . Such interactions persist up to 5% nonlinearity. Therefore, the simultaneous interactions among the three modes can be modulated by adjusting the nonlinearity level within the 1.2% to 5% range. Within this range, specific signal intensity was optimized to introduce 2.5% nonlinearity in the spatial index distribution of the waveguide, facilitating a study of the mutual interaction among all the modes to host multiple second-order EPs . By examining the $ARCs$ between β_j , ($j = 1, 2, 3$) in the presence of nonlinearity, an $EP2$ in the (γ, τ) -plane was identified at $\sim (0.0067, 3.385)$ (denoted as $EP^{(2)}$), in addition to the previously described $EP^{(1)}$. In this context, the concurrent presence of $EP^{(1)}$ and $EP^{(2)}$ analytically links all the three supported

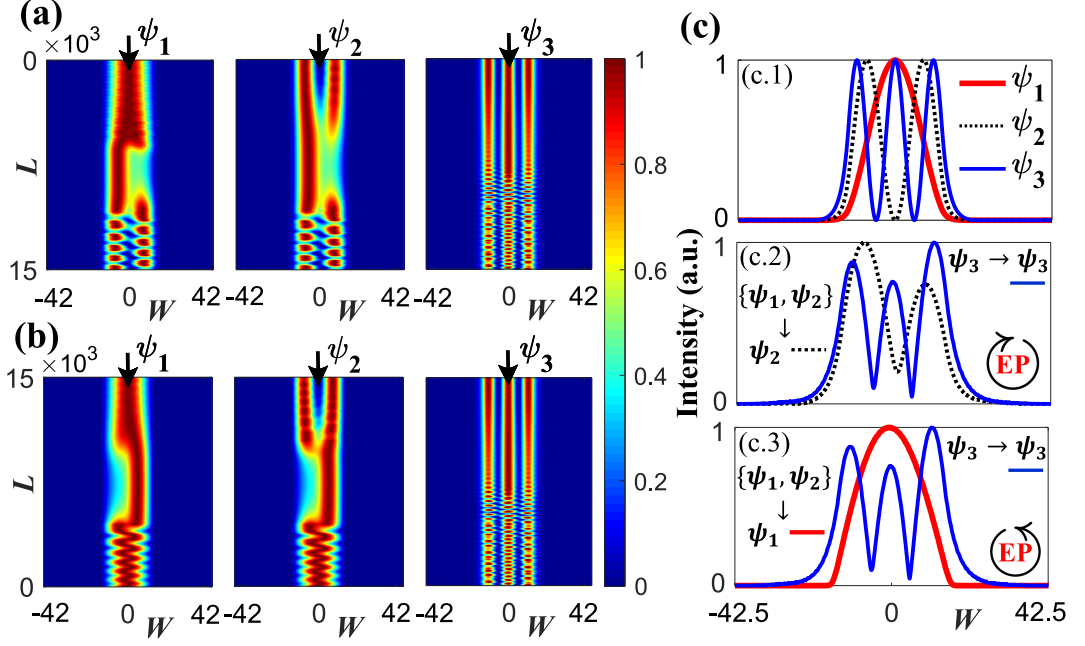


Figure 6.7: (a) Modal propagations of ψ_j ($j = 1, 2, 3$) with dynamical encirclement of $EP^{(1)}$ in CW direction showing adiabatic conversion of $\psi_1 \rightarrow \psi_2$ and nonadiabatic evolution of $\psi_2 (\rightarrow \psi_2)$. (b) Nonadiabatic evolution of $\psi_1 (\rightarrow \psi_1)$ and adiabatic conversion of $\psi_2 \rightarrow \psi_1$, while encircling the $EP^{(1)}$ in CCW direction. For both (a) and (b) ψ_3 remains unaffected. (c) (c.1) Supported field intensities. (c.2) Output field intensities at $z = L$ for CW EP -encirclement process (considering input at $z = 0$). (c.2) Output field intensities at $z = 0$ for CCW EP -encirclement process (considering input at $z = L$). We re-normalize the modal intensities at each z for clear visualization and hence the overall intensity variations are essentially scaled.

modes ψ_j , ($j = 1, 2, 3$). Both $EP^{(1)}$ and $EP^{(2)}$ are dynamically encircled simultaneously inside the length-dependent parametric loop shown in Fig. 6.6, and the beam propagation results are depicted in Fig. 6.8. The topological structure of the waveguide is fixed in such a manner that both scenarios, i.e., with and without nonlinearity, can be considered. Specifically, only in the presence of the chosen nonlinearity ψ_3 does interact. In its absence, ψ_3 behaves as a noninteracting state, and accordingly $EP^{(2)}$ disappears. The value $\sigma = +1$ is chosen to consider focusing nonlinearity (FN), and considering the light propagation from $z = 0$ to $z = L$, a CW encirclement scheme is implemented as shown in Fig. 6.8 (a). Here, it's evident that all three interacting modes collapse into ψ_2 , given that ψ_1 evolves adiabatically, while the other two modes (ψ_2 and ψ_3) evolve nonadiabatically. Changing the direction of light propagation to implement an CCW encirclement scheme results in similar modal dynamics, as observed in Fig. 6.8 (b). Thus, the findings in Fig. 6.8 (a) and (b) introduce a new nonchiral behavior in modal dynamics for three interacting modes around multiple second-order EP s. In the absence of chirality, a different mode other than ψ_2 can be switched or retrieved as a unique behavior. For $\sigma = -1$, an equivalent amount of de-focusing nonlinearity (DFN) is considered, and the dynamics of ψ_j ($j = 1, 2, 3$) are illustrated in Fig. 6.8 (c) for a CW encirclement along the same parametric loop described in Fig. 6.6. As observed in Fig. 6.8 (c), all coupled modes collapse into ψ_1 at $z = L$ due to the adiabatic evolution of ψ_2 and nonadiabatic evolutions of ψ_1 and ψ_3 . With de-focusing nonlinearity, nonchiral behavior in modal dynamics is also evident when considering the CCW encirclement scheme. Fig. 6.8 (d) presents a comparative study depicting the output field intensities under different launching conditions in the presence of nonlinearity. The commonly excited field intensities (normalized) at the input are shown in Fig. 6.8 (d.1). Figs. 6.8 (d.2) and (d.3) display the normalized output field intensities for CW and CCW encirclement, respectively, in the presence of FN , while Fig. 6.8 (d.4) showcases the same for CW encirclement in the presence of DFN .

Such unconventional light dynamics are primarily controlled by the combined effects of the dynamical

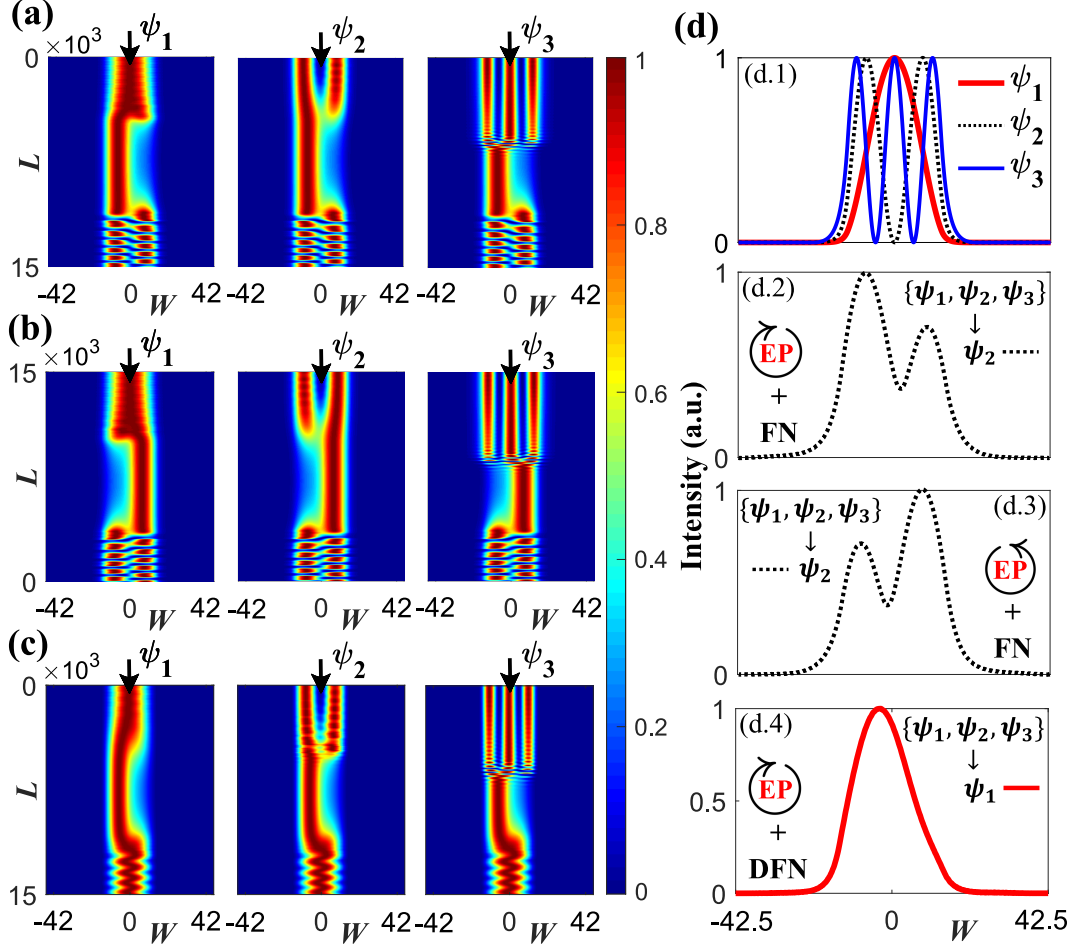


Figure 6.8: (a) In presence of focusing nonlinearity (FN), the propagations of ψ_j ($j = 1, 2, 3$) following dynamical encirclement around two identified EPs ($EP^{(1)}$ and $EP^{(2)}$) in CW direction where all the modes are collapsed to ψ_2 . (b) Similar propagation characteristic of ψ_j ($\rightarrow \psi_2$) showing the nonchiral behavior, while both the EPs have been dynamically encircled in the CCW direction. (c) Propagation characteristic of ψ_j following a CW multiple EPs -encirclement process, however, in the presence of de-focusing nonlinearity (DFN) in the optical medium where all the modes are collapsed to ψ_1 . (d) (d.1) Supported field intensities. In the presence of FN , the output field intensities (d.2) for CW and (d.3) CCW encirclement process. (d.4) In the presence of DFN , the output field intensities, while, multiple EPs are encircled in the CW direction.

EP -encirclement process and the presence of nonlinearity. By varying the input power, light dynamics have been investigated for various degrees of nonlinearity. It has been observed that the described phenomena persist within the nonlinearity range from 1.2% to 5%. Within this range, slight shifts in the EP locations in the presence of nonlinearity, combined with the operating parametric loop, don't influence the light dynamics, as depicted in Fig. 6.8. However, if the input power is increased further, pushing the nonlinearity beyond 5%, the overall modifications in the index profile might prevent both EPs from properly encircling, which could disrupt the overall light dynamics. In this context, it can be suggested that rather than introducing nonlinearity, manipulation in the positions of multiple EPs can also be achieved by altering the real refractive index of the waveguide. Yet, dynamically adjusting the refractive index of any waveguide geometry is indeed challenging during device operation. For this reason, nonlinearity was selected as a tuning parameter for the dynamic modifications of the refractive index concerning the input power of the signal.

6.3.4 An analytical treatment to describe unconventional light propagation schemes

This anomalous dynamics of the three interacting modes in the presence of multiple *EPs* can be analytically addressed as follows. For instance, consider the 3×3 Hamiltonian $\mathcal{H}$ corresponding to the designed waveguide, which depends on three time-dependent parameters $\mu_j(t)$ (for $j = 1, 2, 3$; analogous to γ , τ , and Δn_{NL}). Under the adiabatic limit, the evaluations of the eigenfunctions of $\mathcal{H}$ follow *TDSE*. Presenting generic mathematics behind nonadiabatic dynamics during conversion between two eigenmodes involves considering the conversions between ψ_m^{ad} and ψ_n^{ad} with eigenvalues β_m^{ad} and β_n^{ad} . For a three-mode supported waveguide, possible pairs are $(m, n) \in 1, 2, 3, m \neq n$. The corresponding dynamical nonadiabatic correction terms around multiple *EPs* are as follows,

$$\Omega_{m \rightarrow n}^{\text{NA}} = \vartheta_{m \rightarrow n} \exp \left\{ -i \int_0^T \Delta \beta_{m,n}^{\text{ad}}[\mu_j(t)] dt \right\}, \quad (6.12a)$$

$$\Omega_{n \rightarrow m}^{\text{NA}} = \vartheta_{n \rightarrow m} \exp \left\{ +i \int_0^T \Delta \beta_{m,n}^{\text{ad}}[\mu_j(t)] dt \right\}; \quad (6.12b)$$

with

$$\vartheta_{m \rightarrow n} = \left\langle \psi_m^{\text{ad}}[\mu_j(t)] \left| \sum_{j=1}^3 \dot{\mu}_j \frac{\partial}{\partial \mu_j} \right| \psi_n^{\text{ad}}[\mu_j(t)] \right\rangle, \quad (6.13a)$$

$$\vartheta_{n \rightarrow m} = \left\langle \psi_n^{\text{ad}}[\mu_j(t)] \left| \sum_{j=1}^3 \dot{\mu}_j \frac{\partial}{\partial \mu_j} \right| \psi_m^{\text{ad}}[\mu_j(t)] \right\rangle, \quad (6.13b)$$

$$\begin{aligned} \text{and } \Delta \beta_{m,n}^{\text{ad}}[\mu_j(t)] &= \beta_m^{\text{ad}}[\mu_j(t)] - \beta_n^{\text{ad}}[\mu_j(t)] \\ &\equiv \text{Re}[\Delta \beta_{m,n}^{\text{ad}}[\mu_j(t)]] - i \Delta \gamma_{m,n}^{\text{ad}}[\mu_j(t)]. \end{aligned} \quad (6.13c)$$

In Eqs. (6.12) and 6.13, the suffixes $m \rightarrow n$ and $n \rightarrow m$ indicate the conversions $|\psi_m^{\text{ad}}\rangle \rightarrow |\psi_n^{\text{ad}}\rangle$ and vice-versa, respectively. In Eq. (6.12), T is the *EP*-encirclement duration and in Eq. (6.13c), $|\Delta \gamma_{m,n}^{\text{ad}}|$ represents the relative gain between two considering modes. Considering a situation where $\Delta \gamma_{m,n}^{\text{ad}} > 0$, then $T \rightarrow \infty$ yields $\Omega_{m \rightarrow n}^{\text{NA}} \rightarrow 0$ and $\Omega_{n \rightarrow m}^{\text{NA}} \rightarrow \infty$. As the pre-exponent terms in Eqs. (6.12a) and 6.12b contain the time derivative of three potential parameters, i.e., $\dot{\mu}_j$ as given in Eqs. (6.13a) and (6.13b), the exponential divergence in T of the exponent term of $\Omega_{n \rightarrow m}^{\text{NA}}$ beats the T^{-1} suppression associated with $\vartheta_{n \rightarrow m}$. Thus, for a slow parametric evolution around the *EP* within the adiabatic limit, between two considered eigenmodes, only one of them having lower decay rate evolves adiabatically and the other one behaves non-adiabatically; i.e., at the present condition $|\psi_m^{\text{ad}}\rangle$ evolves adiabatically and converted to $|\psi_n^{\text{ad}}\rangle$, whereas $|\psi_n^{\text{ad}}\rangle$ follows *NAT*.

With appropriate choices of m and n , the possible adiabatic and nonadiabatic conversions between three interacting modes ψ_j ($j = 1, 2, 3$) in the designed waveguide can be studied. When individually considering two different types of nonlinearity ($\sigma = \pm 1$) in the optical medium of the waveguide, the modified refractive index profile changes the order of β -values of the supported modes. This depends on the type of nonlinearity, and accordingly, the signs of relative gain $\Delta \gamma_{m,n}^{\text{ad}}$ for different choices of m and n are modified. In the presence of focusing nonlinearity ($\sigma = +1$), $\Delta \gamma_{1,2}^{\text{ad}} > 0$ and $\Delta \gamma_{3,2}^{\text{ad}} > 0$ are obtained, which gives the conversions $\{\psi_1, \psi_2, \psi_3\} \rightarrow \psi_2$, regardless of the direction of encirclement, as can be seen in Figs. 6.8 (a) and (b). Conversely, $\Delta \gamma_{2,1}^{\text{ad}} > 0$ and $\Delta \gamma_{3,1}^{\text{ad}} > 0$ are obtained when considering de-focusing nonlinearity ($\sigma = -1$). This results in the conversions $\{\psi_1, \psi_2, \psi_3\} \rightarrow \psi_1$, as shown in Fig. 6.8 (c), where ψ_2 and ψ_3 evolve adiabatically, and ψ_1 behaves nonadiabatically.

6.4 Summary

This chapter presents a comprehensive exploration of gain-loss assisted dual-mode optical waveguides hosting dynamical *EP* encirclement schemes, emphasizing the critical role of Kerr nonlinearity on the associated light dynamics around the *EPs*. Notably, in the absence of nonlinearity, the waveguide demonstrates chirality-driven asymmetric-mode-conversion. However, the introduction of local Kerr nonlinearity beyond a certain threshold leads to local modifications in the spatial index distribution, enabling a distinctive one-way asymmetric-mode-conversion scheme. Here, beyond the inherent device chirality, light converts to a dominant mode regardless of the propagation direction. Remarkably, identical magnitudes of focusing ($\sigma = 1$) and defocusing ($\sigma = -1$) nonlinearities result in different dominating outputs in the same direction. Transitioning to the exploration of topologically robust and compact nonlinearity-induced anomalous mode collapsing phenomena, the framework of a three-mode supported planar gain-loss assisted optical waveguide is scrutinized. This waveguide, intriguingly, does not manifest the chiral property when confronted with multiple second-order *EPs*. In scenarios devoid of nonlinearity, the waveguide's chiral property remains evident, even with the concurrent existence of a noninteracting mode. Contrarily, the onset of nonlinearity brings about unique mode dynamics, where all coupled modes collapse into a specific dominant mode, irrespective of input choices. Furthermore, the different directions of encirclement around multiple *EPs* remain incapable of altering the output due to the loss of chiral property. Here, the absence of chirality, combined with either focusing or defocusing nonlinearity of equivalent magnitude, culminates in distinct dominating outputs for identical parametric encirclement processes around multiple *EPs*. Underlying principles of these explorations and findings hint at broader applicability, potentially extending to systems supporting more than three states. Given that light conversion into a specific state is determined by the type of nonlinearity independent of propagation direction, this scheme opens avenues for achieving nonreciprocal light transmission in multimode systems. Collectively, the insights and methodologies elucidated in this chapter pave the way for innovative chip-scale integrated photonic devices poised to revolutionize next-generation communication systems.