Chapter 3

On general δ-shock model [1]

In this chapter, we introduce a general δ-shock model when the recovery time depends on both the arrival times and the magnitudes of shocks. Moreover, we also consider a more general and flexible shock process, namely, the PGGP. For the defined survival model, we derive the relationships for the survival function and the mean lifetime and study some relevant stochastic properties. As an application, an example of the corresponding optimal replacement policy is discussed.

3.1 Introduction

The δ-shock model is one of the popular shock models which has a wide range of applications in reliability, finance and related fields. From the brief literature review done in Chapter 2, we conclude that all published research on the δ-shock modeling has been carried out under the assumption of a constant recovery time (i.e, δ is fixed). However, this assumption is too restrictive and unrealistic to describe many real-life scenarios. Indeed, δ can obviously depend on other parameters, namely, magnitude of shocks, arrival times of shocks, etc. Note that, different systems, depending on their degradation states, have different recovery times even if they experience the same magnitude of shocks. Moreover, a system needs more (less) recovery time if a shock with large (small) magnitude occurs. For example, the recovery time of a bridge from the damage of an earthquake depends on the magnitude of a shock along with the age of the bridge (apart from other factors). If the magnitude of an earthquake is large, then the damage of the bridge is also substantial and hence, it takes more time to recover. Further, due to deterioration, the recovery time of the bridge from the damage caused by an earthquake increases as the age of the bridge increases. Moreover, if the magnitude of an earthquake exceeds a fixed threshold level, then the bridge entirely collapses. In Chapter 2, the first step in considering a more general model was made when δ was assumed to depend only on the arrival time of a shock. In this chapter, we proceed a step further by assuming δ as a function of not only the arrival time but also the magnitude of a shock.

[1] One paper based on this chapter has appeared as under:

- "On general δ-shock model", *TEST*, 31, 994-1029, 2022.

Another key observation from the brief literature review done in Chapter 2 is that most of the studies on the δ-shock model have been done by assuming that shocks occur according to the HPP or the NHPP. Both of these processes have independent increments, which is indeed a very restrictive assumption in many applications. For example, if there is a larger number of shocks in the past, we may expect the same in the future as well (positive dependence).

Keeping the above research gaps in mind, we focus in this chapter on proposing and investigating a *new general δ-shock model* that takes into account the discussed deficiency of existing approaches. To be more specific, what we consider here and what mostly constitutes the novelty and contribution of this chapter, is as follows:

(a) We assume that the recovery time (δ) of a system from the damage of a shock depends on *both* the arrival times and the magnitudes of shocks;

(b) We consider a *more general shock process* (namely, the PGGP) which has the dependent increments property. This process contains the HPP, the NHPP, the Pólya process and the GPP as the particular cases [Cha and Mercier, 2021].

(c) We derive the *distribution of a fatal shock* (for a specific case) that causes the system's failure.

The rest of the chapter is organized as follows. In Section 3.2, we describe the model. In Section 3.3, we derive the survival function and the mean lifetime of a system for the defined model. Further, we study some stochastic comparisons results. In Section 3.4, we study a special case where the recovery time is assumed as a linear function of arrival times and magnitudes of shocks. For this special case, we derive the reliability function, the mean lifetime and the distribution for a fatal shock. Finally, in Section 3.5, as an example of a possible application, the optimal replacement policy for the proposed model is considered.

3.2 Model Formulation

Let L be a random variable representing the lifetime of a system which has started its operation at time $t = 0$. Assume that the system is subject to external shocks that arrive at random times. Let $0 = T_0 < T_1 < T_2 < \cdots < T_n$ be the sequence of random variables representing the arrival times of n shocks, and let $X_i = T_i - T_{i-1}$ be the random variable representing the time-lag between the i-th and the $(i-1)$-th shocks ($i \geq 1$). Further, let Y_i be the random variable representing the magnitude of the i-th shock, $i = 1, 2, \ldots$. Assume that the system has the maximum threshold γ (> 0) on shocks' magnitudes, i.e., the system fails if the magnitude of a single shock is larger than γ. In what follows, we give a list of model assumptions.

Assumptions:

1. Shocks occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$.

2. The shock process and $\{Y_i : i \in \mathbb{N}\}$ are independent.

3. The shock process and $\{X_i : i \in \mathbb{N}\}$ are independent with $\{Y_i : i \in \mathbb{N}\}$.

4. $\{Y_i : i \in \mathbb{N}\}$ is a sequence of iid random variables. However, the general case may be considered in the same line by assuming the corresponding joint density function.

5. $\delta : [0, \infty] \times [0, \infty] \to [\delta_0, \infty)$ is a recovery function such that

   (a) $\delta(0, 0) = \delta_0$, where δ_0 is a positive constant;

   (b) $\delta(t, y)$ is an increasing function in t, for each fixed y;

   (c) $\delta(t, y)$ is an increasing function in y, for each fixed t;

   (d) $\delta(t, y)$ is a continuous function of t and y.

Now, we assume that a system fails at i-th shock, $i \in \mathbb{N}$, in one of the following ways:

(*i*) $X_1 > \delta_0, Y_1 \leq \gamma, X_2 > \delta(T_1, Y_1), Y_2 \leq \gamma, \ldots, X_i > \delta(T_{i-1}, Y_{i-1}), Y_i > \gamma$;

(*ii*) $X_1 > \delta_0, Y_1 \leq \gamma, X_2 > \delta(T_1, Y_1), Y_2 \leq \gamma, \ldots, X_{i-1} > \delta(T_{i-1}, Y_{i-1}), Y_{i-1} \leq \gamma, X_i < \delta(T_{i-1}, Y_{i-1})$.

Then, the probability of the event "the system survives n shocks in $[0, t)$" is given by

$$\begin{aligned} P(L > t|N(t) = n) &= P(X_1 > \delta_0, Y_1 \leq \gamma, X_2 > \delta(T_1, Y_1), Y_2 \leq \gamma, \ldots, X_n > \delta(T_{n-1}, Y_{n-1}), \\ &\quad Y_n < \gamma|N(t) = n), \end{aligned}$$

or equivalently,

$$\begin{aligned} P(L > t|N(t) = n) &= P(T_1 > \delta_0, Y_1 \leq \gamma, T_2 > T_1 + \delta(T_1, Y_1), Y_2 \leq \gamma, \ldots, T_n > T_{n-1} + \\ &\quad \delta(T_{n-1}, Y_{n-1}), Y_n < \gamma|N(t) = n). \end{aligned}$$

Let $g : [0, \infty) \times [0, \infty) \to [\delta_0, \infty)$ be a function given by $g(t, y) = t + \delta(t, y)$. Note that, for each fixed y, $g(\cdot, y)$ is a strictly increasing, surjective and continuous function. Consequently, $g(\cdot, y)$ is invertible, for each fixed y. Further, for each fixed y, let the inverse function of $g(\cdot, y)$ be given by $g^{-1}(\cdot, y) = h(\cdot, y)$. Then $g(h(t, y), y) = h(g(t, y), y) = t$, for all t, for each fixed y. For $n \in \mathbb{N}$, we write $g^n(t, y_1, y_2, \ldots, y_n) = g(g^{n-1}(t, y_1, y_2, \ldots, y_{n-1}), y_n)$ to mean that g is composed with itself $(n - 1)$ times, where $g^0(t) = t$ for all $t \geq 0$.

3.3 Reliability Characteristics of the Model

In this section, we first discuss some reliability indices (namely, the reliability function and the mean lifetime) for the defined model and then provide some results on relevant stochastic comparisons. We assume the general form of the recovery function and hence, the results discussed in this section are quite general in nature and contain relatively cumbersome expressions. More practical results for a specific case (i.e., the linear recovery function) of this model are given in the next section.

3.3.1 Reliability indices

In the following theorem, we derive the reliability function for the defined model.

Theorem 3.3.1 *Let shocks occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. *Then the survival function of a system for the general* δ*-shock model with the recovery function* $\delta(t, y)$ *is given by*

$$\bar{F}_L(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)(\alpha + \Lambda(t))^{k-\nu}} \left[\Gamma_\nu(k, (\alpha + \Lambda(t))l) + \sum_{n=1}^{K_0(t)} \frac{\Gamma_\nu(k+n, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^n} u(t, n) \right],$$

where

$$u(t,n) = \overbrace{\underbrace{\int\int\cdots\int}_{A_0(t,n)}}^{n \text{ times}} \left\{ \int_{g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1})}^{t} \int_{g^{n-2}(\delta_0, y_1, y_2, \ldots, y_{n-2})}^{h(t_n, y_{n-1})} \cdots \int_{g(\delta_0, y_1)}^{h(t_3, y_2)} \int_{\delta_0}^{h(t_2, y_1)} \prod_{i=1}^{n} \lambda(t_i) \right.$$
$$\left. dt_1 dt_2 \ldots dt_n \right\} \prod_{i=1}^{n} f_{Y_1}(y_i) dy_1 dy_2 \ldots dy_n$$

and

$$A_0(t,n) = \{(y_1, y_2, \ldots, y_n) : g^{n-1}(\delta_0, y_1, y_2, \ldots, y_{n-1}) < t, 0 \leq y_i \leq \gamma, 1 \leq i \leq n\},$$
$$K_0(t) = max\{n \geq 1 | g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}) < t\}.$$

Proof: We have

$$\bar{F}_L(t) = P(L > t) = P(L > t, N(t) = 0) + \sum_{n=1}^{\infty} P(L > t, N(t) = n). \tag{3.3.1}$$

From Theorem 1.2.4, we have

$$P(L > t, N(t) = 0) = \frac{\alpha^{k-\nu} \Gamma_\nu(k, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)(\alpha + \Lambda(t))^{k-\nu}}. \tag{3.3.2}$$

Further, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
&P(L > t | N(t) = n) \\
&= P(T_1 > \delta_0, Y_1 \leq \gamma, T_2 > T_1 + \delta(T_1, Y_1), \ldots, T_n > T_{n-1} + \delta(T_{n-1}, Y_{n-1}), Y_n < \gamma | N(t) = n) \\
&= P(T_1 > \delta_0, Y_1 \leq \gamma, T_2 > g(T_1, Y_1), Y_2 \leq \gamma, \ldots, T_n > g(T_{n-1}, Y_{n-1}), Y_n < \gamma | N(t) = n).
\end{aligned}$$

Note that the system survives n shocks in $[0, t)$ provided $t > T_n > g(T_{n-1}, Y_{n-1}) > g^2(T_{n-2}, Y_{n-2}, Y_{n-1}) > \cdots > g^{n-1}(\delta_0, Y_1, Y_2, \ldots, Y_{n-1}) \geq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})$. This implies that, if $t \leq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})$, then the probability of the event "the system survives n shocks till time t" is equal to zero, i.e.,

$$P(L > t | N(t) = n) = 0, \text{ for } t \leq g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}).$$

For $1 \leq n \leq K_0(t)$, we have

$$\begin{aligned}
& P(L > t|N(t) = n) \\
= & \overbrace{\int\int\cdots\int}^{n \text{ times}}_{A_0(t,n)} P(T_1 > \delta_0, Y_1 \leq \gamma, \ldots, T_n > g(T_{n-1}, Y_{n-1}), Y_n < \gamma | N(t) = n, Y_1 = y_1, \ldots, Y_n = y_n) \\
& \times f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n|N(t) = n) dy_1 \ldots dy_n \\
= & \overbrace{\int\int\cdots\int}^{n \text{ times}}_{A_0(t,n)} P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1})|N(t) = n) \\
& \times f_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) dy_1 \ldots dy_n \\
= & \overbrace{\int\int\cdots\int}^{n \text{ times}}_{A_0(t,n)} P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1})|N(t) = n) \\
& \times \prod_{i=1}^{n} f_{Y_1}(y_i) dy_1 \ldots dy_n, \qquad (3.3.3)
\end{aligned}$$

where the second equality follows from Assumptions 2 and 3, and the third equality holds due to Assumption 4. Now, for $0 \leq y_i \leq \gamma,\ 1 \leq i \leq n$, consider

$$\begin{aligned}
& P(T_1 > \delta_0, T_2 > g(T_1, y_1), \ldots, T_n > g(T_{n-1}, y_{n-1})|N(t) = n) \\
= & \int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})} \cdots \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} f_{(T_1,\cdots,T_{N(t)}|N(t))}(t_1, \cdots, t_n|n) \\
& \times dt_1 dt_2 \ldots dt_n \\
= & \int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})} \cdots \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} \left(n! \prod_{i=1}^{n} \frac{\lambda(t_i)}{\Lambda(t)} \right) dt_1 \ldots dt_n,
\end{aligned}$$

where the last equality follows from Theorem 1.2.4(b). On using the above expression in Eq. (3.3.3), we get

$$P(L > t|N(t) = n) = \frac{n!}{(\Lambda(t))^n} u(t, n),$$

which, by Theorem 1.2.4(a), gives

$$\begin{aligned}
P(L > t, N(t) = n) &= P(L > t|N(t) = n) P(N(t) = n) \\
&= \frac{\alpha^{k-\nu}}{(\alpha + \Lambda(t))^{k+n-\nu}} \frac{\Gamma_\nu(k + n, (\alpha + \Lambda(t))l)}{\Gamma_\nu(k, \alpha l)} u(t, n). \qquad (3.3.4)
\end{aligned}$$

Finally, by using Eq. (3.3.2) and Eq. (3.3.4) in Eq. (3.3.1), we get the required result. □

In the next theorem, we derive the mean lifetime of the system for the defined model.

Theorem 3.3.2 *Let shocks occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. *Then the mean lifetime of a system for the general* δ*-shock model with the recovery function* $\delta(t, y)$ *is given by*

$$\begin{aligned} E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^\infty \frac{\Gamma_\nu(k, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^{k-\nu}} dt \\ &\quad + \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})}^{\infty} \left(\frac{\Gamma_\nu(k+n, (\alpha + \Lambda(t))l)}{(\alpha + \Lambda(t))^{k+n-\nu}} \right) u(t, n) dt, \end{aligned}$$

provided the expectation exists; here $u(t, n)$ *is the same as in Theorem 3.3.1.*

Proof: We have

$$\begin{aligned} E(L) &= \int_0^\infty \bar{F}_L(t) dt \\ &= \int_0^{\delta_0} \bar{F}_L(t) dt + \int_{\delta_0}^{g(\delta_0, 0)} \bar{F}_L(t) dt + \int_{g(\delta_0, 0)}^{g^2(\delta_0, 0, 0)} \bar{F}_L(t) dt + \ldots \\ &= \int_0^{\delta_0} P(L > t, N(t) = 0) dt + \int_{\delta_0}^{g(\delta_0, 0)} [P(L > t, N(t) = 0) + P(L > t, N(t) = 1)] \, dt \\ &\quad + \int_{g(\delta_0, 0)}^{g^2(\delta_0, 0, 0)} [P(L > t, N(t) = 0) + P(L > t, N(t) = 1) + P(L > t, N(t) = 2)] \, dt + \ldots \\ &= \int_0^\infty P(L > t, N(t) = 0) dt + \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n \text{ times}})}^{\infty} P(L > t, N(t) = n) dt. \end{aligned}$$

By using Eq. (3.3.2) and Eq. (3.3.4) in the above expression, we get the required result.

3.3.2 Stochastic comparisons

In this subsection, we study some stochastic comparison results for systems subject to external shocks. In the following theorems, we compare the lifetimes of two systems with respect to the usual stochastic order (see Definition 1.2.14). Here, we assume that both systems are subject to the same external shocks. The proofs of the first two theorems immediately follow from Theorem 3.3.1 and are therefore omitted.

Theorem 3.3.3 *Let* L_1 *and* L_2 *be the lifetimes of two systems with the same recovery function* $\delta(\cdot, \cdot)$. *Further, let* γ_1 *and* γ_2 *be the maximum thresholds on shocks' magnitudes for the first and the second systems, respectively. Assume that both systems are subject to the same external shocks that occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0$, $\nu \geq 0, k, \alpha, l > 0$. *If* $\gamma_1 \leq \gamma_2$ *then* $L_1 \leq_{st} L_2$.

Theorem 3.3.4 *Let* L_1 *and* L_2 *be the lifetimes of two systems with the same recovery function* $\delta(\cdot, \cdot)$. *Further, let* $\{Y_{1n} | \, n \in \mathbb{N}\}$ *and* $\{Y_{2n} | \, n \in \mathbb{N}\}$ *be two sequences of iid random variables representing the magnitudes of shocks that are experienced by the first and the second systems, respectively. Assume that both systems are subject to the same external*

shocks that occur according to the PGGP with the set of parameters $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0,\ \nu \geq 0, k, \alpha, l > 0$. *Further, assume that* $Y_{1n} \sim U[0, \xi_1]$ *and* $Y_{2n} \sim U[0, \xi_2]$, *for all* $n \in \mathbb{N}$, *for some* $\xi_1, \xi_2 > 0$. *If* $Y_{1n} \geq_{st} Y_{2n}$, *for all* $n \in \mathbb{N}$, *then* $L_1 \leq_{st} L_2$.

Theorem 3.3.5 *Let* L_1 *and* L_2 *be the lifetimes of two systems with the recovery functions* $\delta_1(\cdot,\cdot)$ *and* $\delta_2(\cdot,\cdot)$, *respectively. Assume that both systems are subject to the same external shocks that occur according to the PGGP with the set of parameters* $\{\lambda(t), \nu, k, \alpha, l\}$, $\lambda(t) > 0$ *for all* $t \geq 0,\ \nu \geq 0, k, \alpha, l > 0$. *Further, assume that* $\delta_1(0,0) = \delta_2(0,0) = \delta_0$. *If* $\delta_1(t,y) \leq \delta_2(t,y)$, *for all* $(t,y) \in [0,\infty) \times [0,\infty)$, *then* $L_2 \leq_{st} L_1$.

Proof: Since $\delta_1(t,y) \leq \delta_2(t,y)$, we get $g_1(t,y) \leq g_2(t,y)$, for all $(t,y) \in [0,\infty) \times [0,\infty)$. We know that both $g_1(\cdot,y)$ and $g_2(\cdot,y)$ are bijective and strictly increasing functions on $[0,\infty)$, for fixed $y \in [0,\infty)$. Let $h_1(\cdot,y)$ and $h_2(\cdot,y)$ be the inverse functions of $g_1(\cdot,y)$ and $g_2(\cdot,y)$, respectively, for fixed $y \in [0,\infty)$. Then, for fixed $y \in [0,\infty)$, the inequality "$g_1(t,y) \leq g_2(t,y)$, for all $t \in [0,\infty)$" implies $h_2(t,y) \leq h_1(t,y)$ for all $t \in [\delta_0,\infty)$. Now, from Theorem 3.3.1, we have

$$\bar{F}_{L_i}(t) = \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\Lambda(t))^{k-\nu}} \left[\Gamma_\nu(k,(\alpha+\Lambda(t))l) + \sum_{n=1}^{K_{i0}(t)} \frac{\Gamma_\nu(k+n,(\alpha+\Lambda(t))l)}{(\alpha+\Lambda(t))^n} u_i(t,n)\right],$$

where

$$K_{i0}(t) = max\{n \geq 1 | g_i^{n-1}(\delta_0, \underbrace{0,0,\ldots,0}_{n-1 \text{ times}}) < t\},$$

and

$$\begin{aligned} u_i(t,n) \quad = \quad & \overbrace{\int\int\cdots\int}^{n \text{ times}} \quad \Bigg\{ \\ & \{(y_1,y_2,\ldots,y_n) : g_i^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1}) < t, 0 \leq y_j \leq \gamma, 1 \leq j \leq n\} \\ & \int_{g_i^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g_i^{n-2}(\delta_0,y_i,y_2,\ldots,y_{n-2})}^{h_i(t_n,y_{n-1})} \cdots \int_{g_i(\delta_0,y_1)}^{h_i(t_3,y_2)} \int_{\delta_0}^{h_i(t_2,y_1)} \left(\prod_{j=1}^{n} \lambda(t_j)\right) \\ & dt_1 dt_2 \ldots dt_n \Bigg\} f_{Y_1,Y_2,\ldots,Y_n}(y_1,y_2,\ldots,y_n) dy_1 dy_2 \ldots dy_n, \quad i = 1,2. \end{aligned}$$

Since, for fixed $y \in [0,\infty)$, $g_1(t,y) \leq g_2(t,y)$, for all $t \in [0,\infty)$, and $h_2(t,y) \leq h_1(t,y)$ for all $t \in [\delta_0,\infty)$, we get

$$\begin{aligned} & \int_{g_1^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \cdots \int_{g_1(\delta_0,y_1)}^{h_1(t_3,y_2)} \int_{\delta_0}^{h_1(t_2,y_1)} \left(\prod_{i=1}^{n} \lambda(t_i)\right) dt_1 dt_2 \ldots dt_n \\ & \geq \int_{g_2^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \cdots \int_{g_2(\delta_0,y_1)}^{h_2(t_3,y_2)} \int_{\delta_0}^{h_2(t_2,y_1)} \left(\prod_{i=1}^{n} \lambda(t_i)\right) dt_1 dt_2 \ldots dt_n, \end{aligned}$$

which further implies that $u_1(t,n) \geq u_2(t,n)$, for all $t \in [0,\infty)$ and $n \in \mathbb{N}$. Again, $K_{10}(t) \geq K_{20}(t)$, for all $t \in [0,\infty)$. Thus, $\bar{F}_{L_2}(t) \leq \bar{F}_{L_1}(t)$, for all $t \in [0,\infty)$, and hence, the result is proved.

3.4 A Special Case: δ-Shock Model with the Linear Recovery Function

In this section, we assume the linear (with respect to both arrival times and magnitudes of shocks) recovery function, i.e., $\delta(t,y)=\delta_0+\sigma_1 t+\sigma_2 y$, for all $(t,y)\in[0,\infty)\times[0,\infty)$, for some $\delta_0>0$ and $\sigma_1,\sigma_2\geq 0$. Here, we specifically study the reliability function, the distribution for a fatal shock and the mean lifetime for the defined model by assuming that shocks occur according to the HPGGP.

3.4.1 Reliability function

Theorem 3.4.1 *Let shocks occur according to the HPGGP with the set of parameters* $\{\lambda,\nu,k,\alpha,l\}$, $\nu\geq 0$, $\lambda,k,\alpha,l>0$. *Assume that* $Y_i\sim U[0,\xi]$, $i\in\mathbb{N}$, *for some* $\xi>0$. *Then the survival function of a system for the general δ-shock model with the recovery function* $\delta(t,y)=\delta_0+\sigma_1 t+\sigma_2 y$ *is given by*

$$\begin{aligned}\bar{F}_L(t) &= \frac{\alpha^{k-\nu}}{(\alpha+\lambda t)^{k-\nu}}\frac{\Gamma_\nu(k,(\alpha+\lambda t)l)}{\Gamma_\nu(k,\alpha l)}\\ &\quad+\frac{\gamma\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}}\sum_{n=1}^{S_0(t)}\left(\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n}\left(\frac{\lambda}{\xi}\right)^n\frac{v(t,n)}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\right),\end{aligned}$$

where

$$v(t,n)=\underbrace{\overbrace{\int\int\cdots\int}^{n-1\ times}}_{B_0(t,n)}\left(t-g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})\right)^n dy_1dy_2\ldots dy_{n-1},$$

$$B_0(t,n)=\{(y_1,y_2,\ldots,y_{n-1}):g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})<t,\ \ 0\leq y_i\leq\gamma,\ 1\leq i\leq(n-1)\},$$

$$g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})=\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)+\sigma_2\left[(1+\sigma_1)^{n-2}y_1+(1+\sigma_1)^{n-3}y_2+\cdots+y_{n-1}\right],$$

and

$$S_0(t)=\left\lfloor\frac{\ln\left(\frac{\delta_0+\sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)}\right\rfloor.$$

Proof: Given that the recovery function is $\delta(t,y)=\delta_0+\sigma_1 t+\sigma_2 y$, for all $(t,y)\in[0,\infty)\times[0,\infty)$, which implies $g(t,y)=\delta_0+(1+\sigma_1)t+\sigma_2 y$, for all $(t,y)\in[0,\infty)\times[0,\infty)$, and $h(t,y)=(t-\delta_0-\sigma_2 y)/(1+\sigma_1)$, for all $(t,y)\in[\delta_0,\infty)\times[0,\infty)$. Consequently,

$$g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})=\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)+\sigma_2\left[(1+\sigma_1)^{n-2}y_1+(1+\sigma_1)^{n-3}y_2+\cdots+y_{n-1}\right].$$

Then

$$K_0(t) = max\left\{n\geq 1|g^{n-1}(\delta_0,\underbrace{0,0,\ldots,0}_{n-1\text{ times}})<t\right\}$$

$$\begin{aligned} &= max\left\{n \geq 1 | \delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right) < t\right\} \\ &= max\left\{n \geq 1 | n < \left(\frac{\ln\left(\frac{\delta_0+\sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)}\right)\right\} \\ &= \left\lfloor \frac{\ln\left(\frac{\delta_0+\sigma_1 t}{\delta_0}\right)}{\ln(1+\sigma_1)} \right\rfloor = S_0(t). \end{aligned} \tag{3.4.1}$$

Since $Y_i \sim U[0, \xi]$, for all $i \in \mathbb{N}$, for some $\xi > 0$, we have

$$f_{Y_1,\ldots Y_n}(y_1, \ldots, y_n) = \prod_{i=1}^{n} f_{Y_i}(y_i) = \frac{1}{\xi^n}, \text{ for all } 0 \leq y_1, \ldots, y_n \leq \xi, n \in \mathbb{N}. \tag{3.4.2}$$

Now, consider the following multiple-integration.

$$\int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t} \int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})} \cdots \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 \ldots dt_n.$$

We solve the above integration by iterative process. Let us first consider the following integration.

$$\begin{aligned} \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 dt_2 &= \int_{\delta_0[1+(1+\sigma_1)]+\sigma_2 y_1}^{\frac{t_3-\sigma_2 y_2-\delta_0}{1+\sigma_1}} \int_{\delta_0}^{\frac{t_2-\sigma_2 y_1-\delta_0}{1+\sigma_1}} dt_1 dt_2 \\ &= \frac{1}{2(1+\sigma_1)^3}[t_3 - \delta_0(1 + (1+\sigma_1) + (1+\sigma_1)^2) - \sigma_2(y_2 + (1+\sigma_1)y_1)]^2 \\ &= \frac{1}{2(1+\sigma_1)^3}[t_3 - g^2(\delta_0, y_1, y_2)]^2. \end{aligned}$$

Similarly, we get

$$\begin{aligned} &\int_{g^2(\delta_0,y_1,y_2)}^{h(t_4,y_3)} \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 dt_2 dt_3 \\ &= \int_{\delta_0[1+(1+\sigma_1)+(1+\sigma_1)^2]+\sigma_2(y_2+(1+\sigma_1)y_1)}^{\frac{t_4-\sigma_2 y_3-\delta_0}{1+\sigma_1}} \int_{\delta_0[1+(1+\sigma_1)]+\sigma_2 y_1}^{\frac{t_3-\sigma_2 y_2-\delta_0}{1+\sigma_1}} \int_{\delta_0}^{\frac{t_2-\sigma_2 y_1-\delta_0}{1+\sigma_1}} dt_1 dt_2 dt_3 \\ &= \frac{1}{2(1+\sigma_1)^3} \int_{\delta_0[1+(1+\sigma_1)+(1+\sigma_1)^2]+\sigma_2(y_2+(1+\sigma_1)y_1)}^{\frac{t_4-\sigma_2 y_3-\delta_0}{1+\sigma_1}} [t_3 - g^2(\delta_0, y_1, y_2)]^2 dt_3 \\ &= \frac{1}{3!(1+\sigma_1)^6}[t_4 - g^3(\delta_0, y_1, y_2)]^3. \end{aligned}$$

By proceeding in a similar manner, we get

$$\begin{aligned} \int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})} \cdots \int_{g(\delta_0,y_1)}^{h(t_3,y_2)} \int_{\delta_0}^{h(t_2,y_1)} dt_1 \ldots dt_{n-1} &= \frac{1}{(n-1)!(1+\sigma_1)^{\frac{n(n-1)}{2}}} \\ &\quad \times [t_n - g^{n-1}(\delta_0, y_1, \ldots, y_{n-1})]^{n-1}. \end{aligned}$$

Consequently,

$$\int_{g^{n-1}(\delta_0,y_1,y_2,\ldots,y_{n-1})}^{t}\int_{g^{n-2}(\delta_0,y_1,y_2,\ldots,y_{n-2})}^{h(t_n,y_{n-1})}\cdots\int_{g(\delta_0,y_1)}^{h(t_3,y_2)}\int_{\delta_0}^{h(t_2,y_1)}dt_1\ldots dt_n$$
$$=\frac{1}{n!(1+\sigma_1)^{\frac{n(n-1)}{2}}}[t-g^{n-1}(\delta_0,y_1,\ldots,y_{n-1})]^n. \tag{3.4.3}$$

By using Eq. (3.4.2) and Eq. (3.4.3) in the expression of $u(t,n)$ given in Theorem 3.3.1, we get

$$u(t,n) \;=\; \left(\frac{\gamma}{\xi^n}\right)(\lambda)^n\left(\frac{1}{n!(1+\sigma_1)^{\frac{n(n-1)}{2}}}\right)v(t,n).$$

Finally, the result follows from Theorem 3.3.1 by using the above equality along with Eq. (3.4.1). □

The following corollary immediately follows from Theorem 3.4.1.

Corollary 3.4.1 *Assume that $\sigma_2 \to 0$ (i.e., the recovery function depends only on the arrival times of shocks) in Theorem 3.4.1. Then*

$$\bar{F}_L(t)=\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\sum_{n=0}^{S_0(t)}\frac{(\lambda\gamma)^n}{\xi^n n!}\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n,$$

where $S_0(t)$ is the same as in Theorem 3.4.1. □

Some special cases of the above corollary are as follows. When $\lambda=1/b$, $\nu=0, k=\beta, \alpha=1$ (i.e., the Pólya process with the set of parameters $\{\beta,b\}$), we have

$$\bar{F}_L(t)=\left(\frac{b}{t+b}\right)^\beta\sum_{n=0}^{S_0(t)}\left(\frac{\gamma}{\xi}\right)^n\frac{\Gamma(\beta+n)}{\Gamma(\beta)n!}\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\left[\frac{t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}{t+b}\right]^n.$$

Further, when $\nu=0, \alpha=k$ and $k\to\infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t)=\exp\{-\lambda t\}\sum_{n=0}^{S_0(t)}\left(\frac{\gamma}{\xi}\right)^n\left(\frac{\lambda^n}{n!}\right)\frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}}\left[t-\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n.$$

The next corollary also follows from Theorem 3.4.1.

Corollary 3.4.2 *Assume that $\sigma_1 \to 0$ (i.e., the recovery function depends only on the magnitudes of shocks) in Theorem 3.4.1. Then*

$$\bar{F}_L(t)=\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}}\left[\Gamma_\nu(k,(\alpha+\lambda t)l)+\gamma\sum_{n=1}^{\left\lfloor\frac{t}{\delta_0}\right\rfloor}\left(\frac{\lambda}{\xi}\right)^n\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n}w(t,n)\right],$$

where

$$w(t,n)=\begin{cases}\left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\frac{(t-n\delta_0)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0<t\le n\delta_0+\gamma\sigma_2\\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\frac{(t-n\delta_0)^{(2n-1)}-\binom{n-1}{1}(t-n\delta_0-\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+\gamma\sigma_2<t\le n\delta_0+2\gamma\sigma_2\\ \vdots & \vdots\\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\sum_{i=0}^{k}(-1)^i\binom{n-1}{i}\frac{(t-n\delta_0-i\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+k\gamma\sigma_2<t\le n\delta_0+(k+1)\gamma\sigma_2\\ \vdots & \vdots\\ \left(\frac{n!}{\sigma_2^{n-1}}\right)\left\{\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\frac{(t-n\delta_0-i\gamma\sigma_2)^{(2n-1)}}{(2n-1)!}\right\}, & \text{if } n\delta_0+(n-1)\gamma\sigma_2<t.\end{cases}$$ □

The following observations can be made from Corollary 3.4.2. When $\lambda=1/b$, $\nu=0,k=\beta,\alpha=1$ (i.e., the Pólya process with the set of parameters $\{\beta,b\}$), we have

$$\bar{F}_L(t)=\left(\frac{b}{t+b}\right)^{\beta}\left[1+\gamma\sum_{n=1}^{\lfloor\frac{t}{\delta_0}\rfloor}\frac{\Gamma(\beta+n)}{\Gamma(\beta)n!}\left(\frac{1}{\xi}\right)^n\left(\frac{1}{t+b}\right)^n w(t,n)\right].$$

Further, when $\nu=0,\alpha=k$ and $k\to\infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t)=\exp\{-\lambda t\}\left[1+\gamma\sum_{n=1}^{\lfloor\frac{t}{\delta_0}\rfloor}\frac{1}{n!}\left(\frac{\lambda}{\xi}\right)^n w(t,n)\right].$$

Below we give another corollary of Theorem 3.4.1.

Corollary 3.4.3 *Assume that $\sigma_1\to 0$ and $\sigma_2\to 0$ (i.e., the recovery function is constant) in Theorem 3.4.1. Then*

$$\bar{F}_L(t)=\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}}\left[\Gamma_\nu(k,(\alpha+\lambda t)l)+\sum_{n=1}^{\lfloor\frac{t}{\delta_0}\rfloor}\left(\frac{\gamma\lambda}{\xi}\right)^n\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n}(t-n\delta_0)^n\right].$$ □

Some special cases of Corollary 3.4.3 are as follows. When $\lambda=1/b$, $\nu=0,k=\beta,\alpha=1$ (i.e., the Pólya process with the set of parameters $\{\beta,b\}$), we have

$$\bar{F}_L(t)=\left(\frac{b}{t+b}\right)^{\beta}\sum_{n=0}^{\lfloor\frac{t}{\delta_0}\rfloor}\frac{\Gamma(\beta+n)}{\Gamma(\beta)n!}\left(\frac{\gamma}{\xi}\right)^n\left(\frac{t-n\delta_0}{t+b}\right)^n.$$

Further, when $\nu=0,\alpha=k$ and $k\to\infty$ (i.e., the HPP with the intensity λ), we have

$$\bar{F}_L(t)=\exp\{-\lambda t\}\sum_{n=0}^{\lfloor\frac{t}{\delta_0}\rfloor}\frac{\lambda^n}{n!}\left(\frac{\gamma}{\xi}\right)^n(t-n\delta_0)^n.$$

3.4.2 Distribution for a fatal shock

In this subsection, we first derive the distribution for a fatal shock and then, find out the expected number of shocks that the system may experience by its failure. Let M be a random variable representing the fatal shock that causes the system's failure. Due to mathematical complexity, it is not possible to obtain the distribution of M for the case when shocks occur according to the HPGGP. Thus, in this subsection, we derive all results under the assumption that shocks occurs according to the Pólya process. As mentioned in Remark 1.2.4, the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega, 1/\omega, l\}$ is indeed the Pólya process with the set of parameters $\{\beta/\omega, b/\omega\}$, regardless of l. Here, we follow the terminology of the HPGGP for notational convenience. In what follows, we introduce some notion which will be be used in the subsequent study.

Let C be any finite sequence with the cardinality n ($\in \mathbb{N} \cup \{0\}$), where the elements of C may occur multiple times. Define $S_{C,i}$ as the collection of all subsequences of C containing exactly i ($\in \mathbb{N} \cup \{0\}$) elements, where $0 \leq i \leq n$. Clearly, the cardinality of $S_{C,i}$ is $\binom{n}{i}$. Further, define $S^l_{C,i}$ as the l-th element of $S_{C,i}$, where $1 \leq l \leq \binom{n}{i}$. We write $\prod_{i=1}^{0} a_i = 1$, for any a_i's. Further, we denote the null set by $\{\}$.

In the following theorem, we obtain the distribution and the expectation of M. The proof of the second part of this theorem immediately follows from the first part and is therefore omitted.

Theorem 3.4.2 *Let shocks occur according to the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega,\ 1/\omega, l\}$, where $\omega, \beta, l, b > 0$ and β/ω is not an integer. Assume that $Y_i \sim U[0, \xi]$, $i \in \mathbb{N}$, for some $\xi > 0$. Then the probability mass function (pmf) and the mean of M, for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$, are given by*

$$P(M = m) = \Upsilon_{m-1} - \Upsilon_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \Upsilon_m,$$

respectively, where $\Upsilon_0 = 1$ and, for $m = 1, 2, \ldots,$

$$\begin{aligned}\Upsilon_m \;=\; & \frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m \sigma_2^{m-1} (1+\sigma_1)^{(m-1)(m-1)} \left\{\prod_{i=1}^{m-1} (i\omega - \beta)\right\}} \\ & \times \left[\sum_{i=0}^{m-1} (-1)^{m-1-i} \sum_{l=1}^{\binom{m-1}{i}} \left(\omega\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \left(\sum_{s \in S^l_{A_{m-1},i}} s\right) + b\right)^{-\frac{\beta}{\omega}+m-1}\right]\end{aligned}$$

and

$$A_0 = \{\}, \quad A_{m-1} = \{\omega\sigma_2\gamma, \omega\sigma_2\gamma(1+\sigma_1), \ldots, \omega\sigma_2\gamma(1+\sigma_1)^{m-2}\}, \quad m = 2, 3, \ldots.$$

Proof: Let $\mathcal{X}$ be a structure random variable with the probability density given by

$$dH(\chi) = \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma\left(\frac{\beta}{\omega}\right)} \chi^{\left(\frac{\beta}{\omega}\right)-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\} d\chi. \tag{3.4.4}$$

Given that shocks occur according to the HPGGP with the set of parameters $\{1/b, 0, \beta/\omega, 1/\omega, l\}$, which is indeed the Pólya process with the set of parameters $\left\{\frac{\beta}{\omega}, \frac{b}{\omega}\right\}$. Moreover, on condition $\mathcal{X} = \chi$, the Pólya process is the same as the HPP with intensity χ [see Beichelt, 2006, p. 130]. Consequently, the inter-arrival times of the given HPGGP are iid random variables and follow the exponential distribution with parameter χ. Then, the conditional probability density function of the random vector $(X_1, X_2, \ldots, X_m)$, given $\mathcal{X} = \chi$, is given by

$$f_{X_1,\ldots,X_m|\mathcal{X}}(x_1, \ldots, x_m|\chi) = \prod_{i=1}^{m} \chi \exp\{-\chi x_i\}, \quad 0 < x_1, \ldots, x_m < \infty. \tag{3.4.5}$$

Now,

$$\begin{aligned} P(M=1) &= 1 - P(X_1 > \delta_0, Y_1 \le \gamma) \qquad (3.4.6)\\ &= 1 - F_{Y_1}(\gamma) \int_0^\infty P(X_1 > \delta_0|\mathcal{X} = \chi) dH(\chi) \\ &= 1 - F_{Y_1}(\gamma) \int_0^\infty \exp\{-\chi\delta_0\} \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})} \chi^{\frac{\beta}{\omega}-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\} d\chi \\ &= 1 - \frac{\gamma}{\xi}\left(\frac{b}{b+\omega\delta_0}\right)^{\frac{\beta}{\omega}} = \Upsilon_0 - \Upsilon_1. \qquad (3.4.7) \end{aligned}$$

Further, for $m = 2, 3, \ldots$, we have

$$\begin{aligned} P(M=m) &= P(X_1 > \delta_0, Y_1 \le \gamma, \ldots, X_{m-1} > g(T_{m-2}, Y_{m-2}), Y_{m-1} \le \gamma) \\ &\quad - P(X_1 > \delta_0, Y_1 \le \gamma, \ldots, X_m > g(T_{m-1}, Y_{m-1}), Y_m \le \gamma) \\ &= \zeta_{m-1} - \zeta_m \text{ (say)}, \qquad (3.4.8) \end{aligned}$$

where

$$\zeta_1 = P(X_1 > \delta_0, Y_1 \le \gamma) = \frac{\gamma}{\xi}\left(\frac{b}{b+\omega\delta_0}\right)^{\frac{\beta}{\omega}} = \Upsilon_1. \tag{3.4.9}$$

and

$$\begin{aligned} \zeta_m &= P(X_1 > \delta_0, Y_1 \le \gamma, X_2 > g(T_1, Y_1), \ldots, X_m > g(T_{m-1}, Y_{m-1}), Y_m \le \gamma) \\ &= \underbrace{\int_0^\gamma \int_0^\gamma \cdots \int_0^\gamma}_{m \text{ times}} \Big[P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1})) \\ &\quad \times f_{Y_1,Y_2,\ldots,Y_m}(y_1, y_2, \ldots, y_m)\Big] dy_1 dy_2 \ldots dy_m, \quad m = 2, 3, \ldots. \qquad (3.4.10) \end{aligned}$$

Now, we proceed to find the value of ζ_m. Consider

$$\begin{aligned}
&P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1})) \\
&= \int_0^\infty P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1}) | \mathcal{X} = \chi) dH(\chi) \\
&= \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})} \int_0^\infty P(X_1 > \delta_0, X_2 > \delta_0 + \sigma_1 X_1 + \sigma_2 y_1, \ldots, X_m > \delta_0 + \sigma_1(X_1 + \cdots + X_{m-1}) \\
&\quad + \sigma_2 y_{m-1} | \mathcal{X} = \chi) \chi^{\frac{\beta}{\omega}-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\} d\chi, \qquad (3.4.11)
\end{aligned}$$

where the second equality follows from Eq. (3.4.4). Again, consider

$$\begin{aligned}
&P(X_1 > \delta_0, X_2 > \delta_0 + \sigma_1 X_1 + \sigma_2 y_1, \ldots, X_m > \delta_0 + \sigma_1(X_1 + \cdots + X_{m-1}) + \sigma_2 y_{m-1} | \mathcal{X} = \chi) \\
&= \int_{\delta_0}^\infty \int_{\delta_0+\sigma_1 x_1 + \sigma_2 y_1}^\infty \cdots \int_{\delta_0+\sigma_1(x_1+\cdots+x_{m-1})+\sigma_2 y_{m-1}}^\infty \left(\prod_{i=1}^m \chi \exp\{-\chi x_i\}\right) dx_m dx_{m-1} \ldots dx_2 dx_1 \\
&= \frac{1}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \exp\left\{-\chi\delta_0\left[1 + (1+\sigma_1) + \cdots + (1+\sigma_1)^{m-1}\right]\right\} \\
&\quad \times \exp\left\{-\chi\sigma_2\left[y_{m-1} + (1+\sigma_1)y_{m-2} + \cdots + (1+\sigma_1)^{m-2} y_1\right]\right\} \\
&= \frac{\exp\left\{-\chi\left(\delta_0\left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right) + \sigma_2(y_{m-1} + (1+\sigma_1)y_{m-2} + \cdots + (1+\sigma_1)^{m-2} y_1)\right)\right\}}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}, \qquad (3.4.12)
\end{aligned}$$

where the first equality follows from Eq. (3.4.5). Then, by using Eq. (3.4.12) in Eq. (3.4.11), we get

$$\begin{aligned}
&P(X_1 > \delta_0, X_2 > g(T_1, y_1), \ldots, X_m > g(T_{m-1}, y_{m-1})) \\
&= \frac{\left(\frac{b}{\omega}\right)^{\frac{\beta}{\omega}}}{\Gamma(\frac{\beta}{\omega})} \int_0^\infty \left[\frac{\exp\left\{-\chi\left(\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right) + \sigma_2(y_{m-1} + (1+\sigma_1)y_{m-2} + \cdots + (1+\sigma_1)^{m-2}y_1)\right)\right\}}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}\right. \\
&\quad \left. \times \chi^{\frac{\beta}{\omega}-1} \exp\left\{-\left(\frac{b}{\omega}\right)\chi\right\}\right] d\chi \\
&= \frac{1}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \left(\frac{b}{\omega\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right) + \omega\sigma_2 y_{m-1} + \omega\sigma_2(1+\sigma_1)y_{m-2} + \cdots + \omega\sigma_2(1+\sigma_1)^{m-2}y_1 + b}\right)^{\frac{\beta}{\omega}}. \qquad (3.4.13)
\end{aligned}$$

Since $Y_i \sim U[0, \xi]$, $i \in \mathbb{N}$, for some $\xi > 0$, we have

$$f_{Y_1, Y_2, \ldots, Y_m}(y_1, y_2, \ldots, y_m) = \left(\frac{1}{\xi}\right)^m, \quad 0 < y_1, y_2, \ldots, y_m < \infty. \qquad (3.4.14)$$

By using Eq. (3.4.13) and Eq. (3.4.14) in Eq. (3.4.10), we get

$$\zeta_m = \left(\frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m (1+\sigma_1)^{\frac{m(m-1)}{2}}}\right) I_m, \qquad (3.4.15)$$

where

$$I_m = \underbrace{\int_0^{\gamma}\int_0^{\gamma}\cdots\int_0^{\gamma}}_{m-1 \text{ times}} \left\{\omega\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)+\omega\sigma_2 y_{m-1} \right.$$

$$\left. +\omega\sigma_2(1+\sigma_1)y_{m-2}+\cdots+\omega\sigma_2(1+\sigma_1)^{m-2}y_1+b\right\}^{-\frac{\beta}{\omega}} dy_1dy_2\ldots dy_{m-1}.$$

Now, we solve I_m by iterative process. Consider

$$I_2 = \int_0^{\gamma}\{\omega\delta_0(1+(1+\sigma_1))+\omega\sigma_2 y_1+b\}^{\frac{-\beta}{\omega}}dy_1$$

$$= \left[\frac{\{\omega\delta_0(1+(1+\sigma_1))+\omega\sigma_2\gamma+b\}^{\left(-\frac{\beta}{\omega}+1\right)}-\{\omega\delta_0(1+(1+\sigma_1))+b\}^{\left(-\frac{\beta}{\omega}+1\right)}}{\sigma_2(\omega-\beta)}\right]$$

$$= \frac{1}{\sigma_2(\omega-\beta)}\left[(-1)\left\{\omega\delta_0(1+(1+\sigma_1))+\sum_{s\in S^1_{A_1,0}}s+b\right\}^{\left(-\frac{\beta}{\omega}+1\right)}\right.$$

$$\left.+(-1)^0\left\{\omega\delta_0(1+(1+\sigma_1))+\sum_{s\in S^1_{A_1,1}}s+b\right\}^{\left(-\frac{\beta}{\omega}+1\right)}\right],$$

where $A_1=\{\omega\sigma_2\gamma\}$. Similarly,

$$I_3 = \int_0^{\gamma}\int_0^{\gamma}\{\omega\delta_0(1+(1+\sigma_1)+(1+\sigma)^2)+\omega\sigma_2 y_2+\omega\sigma_2(1+\sigma_1)y_1+b\}^{-\frac{\beta}{\omega}}dy_1dy_2$$

$$= \frac{1}{\sigma_2^2(1+\sigma_1)(2\omega-\beta)(\omega-\beta)}\left[\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_2,2}}s+b\right\}^{\left(-\frac{\beta}{\omega}+2\right)}\right.$$

$$-\sum_{l=1}^{2}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right)+\sum_{s\in S^l_{A_2,1}}s+b\right\}^{\left(-\frac{\beta}{\omega}+2\right)}$$

$$\left.+\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^3-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_2,0}}s+b\right\}^{\left(-\frac{\beta}{\omega}+2\right)}\right]$$

and

$$I_4 = \int_0^{\gamma}\int_0^{\gamma}\int_0^{\gamma}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\omega\sigma_2 y_3+\omega\sigma_2(1+\sigma_1)y_2+\omega\sigma_2(1+\sigma_1)^2y_1\right\}^{-\frac{\beta}{\omega}}dy_3dy_2dy_1$$

$$= \frac{1}{\sigma_2^3(1+\sigma_1)^{1+2}(3\omega-\beta)(2\omega-\beta)(\omega-\beta)}\left[\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_3,3}}s+b\right\}^{\left(-\frac{\beta}{\omega}+3\right)}\right.$$

$$-\sum_{l=1}^{3}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^l_{A_3,2}} s+b\right\}^{\left(-\frac{\beta}{\omega}+3\right)}$$
$$+\sum_{l=1}^{3}\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^l_{A_3,1}} s+b\right\}^{\left(-\frac{\beta}{\omega}+3\right)}$$
$$-\left\{\omega\delta_0\left(\frac{(1+\sigma_1)^4-1}{\sigma_1}\right)+\sum_{s\in S^1_{A_3,0}} s+b\right\}^{\left(-\frac{\beta}{\omega}+3\right)}\Bigg],$$

where $A_2 := \{\omega\sigma_2\gamma, \omega\sigma_2(1+\sigma_1)\gamma\}$ and $A_3 = \{\omega\sigma_2\gamma, \omega\sigma_2\gamma(1+\sigma_1), \omega\sigma_2\gamma(1+\sigma_1)^2\}$. By proceeding in the same line, we get

$$I_m = \frac{1}{\sigma_2^{m-1}(1+\sigma_1)^{\frac{(m-1)(m-2)}{2}}\left(\prod_{i=1}^{m-1}\{i\omega-\beta\}\right)} \times\left[\sum_{i=0}^{m-1}(-1)^{m-1-i}\sum_{l=1}^{\binom{m-1}{i}}\left(\omega\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)+\left(\sum_{s\in S^l_{A_{m-1},i}} s\right)+b\right)^{-\frac{\beta}{\omega}+m-1}\right],$$

where $A_{m-1} = \{\omega\sigma_2\gamma, \omega\sigma_2\gamma(1+\sigma_1), \ldots, \omega\sigma_2\gamma(1+\sigma_1)^{m-2}\}$. On using the above value of I_m in Eq. (3.4.15), we get $\zeta_m = \Upsilon_m$ and hence, the required result follows from Eq. (3.4.7), Eq. (3.4.8) and Eq. (3.4.9). □

Below we give three consecutive corollaries which immediately follow from Theorem 3.4.2.

Corollary 3.4.4 *Assume that $\beta = 1$, $\omega \to 0$ and $b = 1/\lambda$ (i.e., the shocks occur according to the HPP with intensity $\lambda > 0$) in Theorem 3.4.2. Then the pmf and the mean of M are given by*

$$P(M=m) = \Omega_{m-1} - \Omega_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty}\Omega_m,$$

respectively, where $\Omega_0 = 1$ and, for $m = 1, 2, \ldots,$

$$\Omega_m = \left(\frac{\gamma\exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^m-1}{\sigma_1}\right)\right\}}{\xi^m(1+\sigma_1)^{(m-1)^2}\lambda^{m-1}\sigma_2^{m-1}}\right)\left[\prod_{i=1}^{m-1}\left(1-\exp\{-\lambda\sigma_2(1+\sigma_1)^{i-1}\gamma\}\right)\right].$$

Corollary 3.4.5 *Assume that $\sigma_1 \to 0$ in Theorem 3.4.2. Then the pmf and the mean of M are given by*

$$P(M=m) = \Psi_{m-1} - \Psi_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty}\Psi_m,$$

respectively, where $\Psi_0 = 1$ *and, for* $m = 1, 2, \ldots,$

$$\Psi_m = \frac{\gamma b^{\frac{\beta}{\omega}}}{\xi^m \sigma_2^{m-1} \left(\prod_{i=1}^{m-1}(i\omega - \beta)\right)} \sum_{i=0}^{m-1} (-1)^{m-1-i} \binom{m-1}{i} (m\omega\delta_0 + i\omega\gamma\sigma_2 + b)^{\left(-\frac{\beta}{\omega}+m-1\right)}.$$

Corollary 3.4.6 *Assume that* $\sigma_2 \to 0$ *in Theorem 3.4.2. Then the pmf and the mean of* M *are given by*

$$P(M = m) = \Phi_{m-1} - \Phi_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \Phi_m,$$

respectively, where

$$\Phi_m = \frac{\gamma^m}{\xi^m (1+\sigma_1)^{\frac{m(m-1)}{2}}} \left[\frac{b}{b + \omega\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right)}\right]^{\frac{\beta}{\omega}}, \quad m = 0, 1, 2, \ldots. \qquad \square$$

In the following theorem we derive the the same result, as in Theorem 3.4.2, under the assumption that shocks' magnitudes follow the exponential distribution. The proof of the second part immediately follows from the first part and is therefore omitted.

Theorem 3.4.3 *Let shocks occur according to the HPP with intensity* $\lambda > 0$*. Assume that* $Y_i \sim \exp(\theta)$*,* $i \in \mathbb{N}$*, for some* $\theta > 0$*. Then the pmf and the mean of* M*, for the general* δ*-shock model with the recovery function* $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$*, are given by*

$$P(M = m) = \Delta_{m-1} - \Delta_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{i=0}^{\infty} \Delta_m,$$

respectively, where $\Delta_0 = 1$ *and*

$$\begin{aligned} \Delta_m &= \theta^{m-1} \left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}}\right) \exp\left\{-\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1}\right)\right\} \\ &\quad \times \prod_{i=1}^{m-1} \left(\frac{1 - \exp\left\{-\gamma(\lambda\sigma_2(1+\sigma_1)^{m-1-i} + \theta)\right\}}{\lambda\sigma_2(1+\sigma_1)^{m-1-i} + \theta}\right), \quad m = 1, 2, \ldots. \end{aligned}$$

Proof: Since $Y_i \sim \exp(\theta)$, for all $i \in \mathbb{N}$, for some $\theta > 0$, we have

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_m) = \prod_{i=1}^{m} f_{Y_i}(y_i) = \prod_{i=1}^{m} \theta \exp\{-\theta y_i\}, \quad 0 < y_1, \ldots, y_n < \infty, \qquad (3.4.16)$$

and $F_{Y_m}(\gamma) = (1 - \exp\{-\theta\gamma\})$ for all $m \in \mathbb{N}$. Then, from Eq. (3.4.6), we have

$$P(M = 1) = 1 - (1 - \exp\{-\theta\gamma\}) \exp\{-\lambda\delta_0\} = \Delta_0 - \Delta_1. \tag{3.4.17}$$

By proceeding in the same line as done in the proof of Theorem 3.4.2, we get, from Eq. (3.4.10) and Eq. (3.4.16), that

$$\zeta_m = \left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right) \right\} \theta^{m-1} J_m,$$

where

$$\begin{aligned} J_m &= \underbrace{\int_0^\gamma \int_0^\gamma \cdots \int_0^\gamma}_{m-1 \text{ times}} \prod_{i=1}^{m-1} \exp\left\{ -y_i \left(\lambda\sigma_2 (1+\sigma_1)^{m-1-i} + \theta \right) \right\} dy_1 dy_2 \ldots dy_{m-1} \\ &= \prod_{i=1}^{m-1} \left(\int_0^\gamma \exp\left\{ -y_i \left(\lambda\sigma_2 (1+\sigma_1)^{m-1-i} + \theta \right) \right\} dy_i \right) \\ &= \prod_{i=1}^{m-1} \left(\frac{1 - \exp\left\{ -\gamma(\lambda\sigma_2 (1+\sigma_1)^{m-1-i} + \theta) \right\}}{\lambda\sigma_2 (1+\sigma_1)^{m-1-i} + \theta} \right). \end{aligned}$$

Upon using the above value of ζ_m in Eq. (3.4.8), we get

$$\begin{aligned} P(M = m) &= \theta^{m-2} \left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{(m-1)(m-2)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^{m-1} - 1}{\sigma_1} \right) \right\} J_{m-1} \\ &\quad - \theta^{m-1} \left(\frac{(1 - \exp\{-\theta\gamma\})}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right) \right\} J_m \\ &= \Delta_{m-1} - \Delta_m, \quad m = 2, 3, \ldots. \end{aligned} \tag{3.4.18}$$

Finally, the result follows from Eq. (3.4.17) and Eq. (3.4.18). □

The following two corollaries follow from Theorem 3.4.3.

Corollary 3.4.7 *Assume that $\sigma_1 \to 0$ in Theorem 3.4.3. Then the pmf and the mean of M are given by*

$$P(M = m) = \Theta_{m-1} - \Theta_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = 1 + \left[\frac{(\lambda\sigma_2 + \theta) \exp\{-\lambda\delta_0\}(1 - \exp\{-\theta\gamma\})}{\lambda\sigma_2 + \theta(1 - \exp\{-\lambda\delta_0\}) + \theta \exp\{-(\lambda\delta_0 + \lambda\sigma_2\gamma + \theta\gamma)\}} \right],$$

respectively, where $\Theta_0 = 1$ and

$$\Theta_m = \theta^{m-1}(1 - \exp\{-\theta\gamma\}) \exp\{-m\lambda\delta_0\} \left(\frac{1 - \exp\{-(\lambda\sigma_2 + \theta)\gamma\}}{\lambda\sigma_2 + \theta} \right)^{m-1}, \quad m = 1, 2, \ldots.$$

Corollary 3.4.8 *Assume that $\sigma_2 \to 0$ in Theorem 3.4.3. Then the pmf and the mean of M are given by*

$$P(M = m) = \Xi_{m-1} - \Xi_m, \quad m = 1, 2, \ldots,$$

and

$$E(M) = \sum_{m=0}^{\infty} \Xi_m,$$

respectively, where

$$\Xi_m = \left(\frac{(1 - \exp\{-\theta\gamma\})^m}{(1+\sigma_1)^{\frac{m(m-1)}{2}}} \right) \exp\left\{ -\lambda\delta_0 \left(\frac{(1+\sigma_1)^m - 1}{\sigma_1} \right) \right\}, \quad m = 0, 1, 2, \ldots.$$

3.4.3 Mean lifetime

In the following theorem, we derive the mean lifetime of a system for the defined model. The proof is similar to Theorem 3.3.2 and is therefore omitted.

Theorem 3.4.4 *Let shocks occur according to the HPGGP with the set of parameters $\{\lambda, \nu, k, \alpha, l\}$, $\nu \geq 0$, $k, \alpha, l > 0$. Assume that $Y_i \sim U[0, \xi]$, $i \in \mathbb{N}$, for some $\xi > 0$. Then the mean lifetime of a system for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$ is given by*

$$\begin{aligned} E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \int_0^\infty \frac{\Gamma_\nu(k, (\alpha + \lambda t) l)}{(\alpha + \lambda t)^{k-\nu}} dt \\ &\quad + \frac{\gamma \alpha^{k-\nu}}{\Gamma_\nu(k, \alpha l)} \sum_{n=1}^{\infty} \int_{g^{n-1}(\delta_0, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}})}^{\infty} \left(\frac{\Gamma_\nu(k+n, (\alpha + \lambda t) l)}{n! (\alpha + \lambda t)^{k+n-\nu}} \right) \left(\frac{\lambda}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} v(t, n) dt, \end{aligned}$$

provided the expectation exists; here $v(t, n)$ is the same as in Theorem 3.4.1. □

Below we give three corollaries. In the first corollary, we give a nice relation between the mean lifetime of the system and the expected number of shocks before the failure of the system. The proof of this corollary is straightforward and is therefore omitted.

Corollary 3.4.9 *Assume that $\nu = 0$, $\alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ) in Theorem 3.4.4. Then, we have*

$$E(L) = \frac{1}{\lambda} E(M) = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{m=1}^{\infty} \Omega_m,$$

where Ω_m is the same as in Corollary 3.4.4.

Corollary 3.4.10 *Assume that $\sigma_2 \to 0$ in Theorem 3.4.4. Then, for $k > 1$,*

$$E(L) = \frac{\alpha^{k-\nu}}{\lambda \Gamma_\nu(k, \alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi} \right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \frac{\Gamma_\nu\left(k-1, \left(\alpha + \lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\right) l\right)}{\left(\alpha + \lambda\delta_0 \left(\frac{(1+\sigma_1)^n - 1}{\sigma_1}\right)\right)^{k-\nu-1}}.$$

Proof: From Theorem 3.4.4, we have

$$
\begin{aligned}
E(L) &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty} \frac{\lambda^n}{n!} \frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left(\frac{\gamma}{\xi}\right)^n \\
&\quad \times \left[t - \left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{\lambda^n}{n!} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty} \frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{(\alpha+\lambda t)^{k+n-\nu}} \\
&\quad \times \left[t - \left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{\lambda^n}{n!} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty} \int_0^{\infty} \frac{y^{n+k-1}\exp\{-(\alpha+\lambda t)y\}}{(y+l)^\nu} \\
&\quad \times \left[t - \left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dy dt \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{\lambda^n}{n!} \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \int_0^{\infty} \frac{y^{n+k-1}\exp\{-\alpha y\}}{(y+l)^\nu} \left\{ \int_{\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0}^{\infty} \exp\{-\lambda t y\} \right. \\
&\quad \left. \times \left[t - \left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\delta_0\right]^n dt \right\} dy \\
&= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{\lambda^n}{n!} \frac{\Gamma(n+1)}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \int_0^{\infty} \frac{y^{n+k-1}\exp\{-\alpha y\}}{(y+l)^\nu} \frac{\exp\left\{-\lambda y\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right\}}{(\lambda y)^{n+1}} dy \\
&= \frac{\alpha^{k-\nu}}{\lambda\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \int_0^{\infty} \frac{y^{k-2}\exp\left\{-y\left[\alpha+\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right]\right\}}{(y+l)^\nu} dy \\
&= \frac{\alpha^{k-\nu}}{\lambda\Gamma_\nu(k,\alpha l)} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \frac{\Gamma_\nu\left(k-1,\alpha l+\lambda\delta_0 l\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right)}{\left(\alpha+\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right)^{k-\nu-1}},
\end{aligned}
$$

where the third and the last equalities follow from Eq. (1.2.2). Thus the result is proved.□

Two special cases of the above corollary are as follows. When $\lambda = 1/b$, $\nu = 0$, $k = \beta$ (> 1) and $\alpha = 1$ (i.e., the shocks occur according to the Pólya process with the set of parameters $\{\beta, b\}$), we have

$$
E(L) = \left(\frac{b}{\beta-1}\right) \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \left[\frac{b}{b+\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)}\right]^{\beta-1}.
$$

Further, when $\nu = 0$, $\alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ), we have

$$
E(L) = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{\gamma}{\xi}\right)^n \frac{1}{(1+\sigma_1)^{\frac{n(n-1)}{2}}} \exp\left\{-\lambda\delta_0\left(\frac{(1+\sigma_1)^n-1}{\sigma_1}\right)\right\}.
$$

Corollary 3.4.11 *Assume that $\sigma_1 \to 0$ in Theorem 3.4.4. Then, for $k \to \infty$, we have*

$$\begin{aligned} E(L) &= \left(\frac{\alpha}{\lambda}\right)\left(\lim_{k\to\infty}\frac{\Gamma_\nu(k-1,\alpha l)}{\Gamma_\nu(k,\alpha l)}\right) \\ &\quad + \lim_{k\to\infty}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(\frac{1}{\lambda\xi}\right)^n \frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}}\frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}, \end{aligned}$$

provided the above expression has a finite value.

Proof: From Theorem 3.4.4, we have

$$\begin{aligned} E(L) &= \int_0^\infty \frac{\alpha^{k-\nu}}{(\alpha+\lambda t)^{k-\nu}}\frac{\Gamma_\nu(k,(\alpha+\lambda t)l)}{\Gamma_\nu(k,\alpha l)}dt \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)(\alpha+\lambda t)^{k-\nu}}\sum_{n=1}^{\infty}\int_{n\delta_0}^{\infty}\left(\frac{\lambda}{\xi}\right)^n\frac{\Gamma_\nu(k+n,(\alpha+\lambda t)l)}{n!(\alpha+\lambda t)^n}v(t,n)dt \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\int_0^\infty\int_0^\infty \frac{\exp\{-(\alpha+\lambda t)y\}y^{k-1}}{(y+l)^\nu}dydt \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)}\sum_{n=1}^{\infty}\int_{n\delta_0}^{\infty}\int_0^\infty\left(\frac{\lambda}{\xi}\right)^n\frac{\exp\{-(\alpha+\lambda t)y\}y^{k+n-1}}{(y+l)^\nu}\frac{v(t,n)}{n!}dydt \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\int_0^\infty\frac{\exp\{-\alpha y\}y^{k-1}}{(y+l)^\nu}\left(\int_0^\infty\exp\{-\lambda ty\}dt\right)dy \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)}\sum_{n=1}^{\infty}\int_0^\infty\left(\frac{\lambda}{\xi}\right)^n\frac{\exp\{-\alpha y\}y^{k+n-1}}{n!(y+l)^\nu}\left(\int_{n\delta_0}^\infty\exp\{-\lambda ty\}v(t,n)dt\right)dy \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\int_0^\infty\frac{\exp\{-\alpha y\}y^{k-1}}{(y+l)^\nu}\left(\int_0^\infty\exp\{-\lambda ty\}dt\right)dy \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_0^\infty(-1)^i\binom{n-1}{i}\left(\frac{\lambda}{\xi}\right)^n\frac{\exp\{-\alpha y\}y^{k+n-1}}{n!(y+l)^\nu} \\ &\quad \times\left(\int_{n\delta_0+i\gamma\sigma_2}^\infty\frac{\exp\{-\lambda yt\}\left(t-n\delta_0-i\gamma\sigma_2\right)^{2n-1}}{(2n-1)!}dt\right)dy \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\int_0^\infty\frac{\exp\{-\alpha y\}y^{k-1}}{(y+l)^\nu}\left(\int_0^\infty\exp\{-\lambda ty\}dt\right)dy \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_0^\infty(-1)^i\binom{n-1}{i}\left(\frac{\lambda}{\xi}\right)^n\frac{\exp\{-(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)y\}y^{k+n-1}}{n!(y+l)^\nu} \\ &\quad \times\left(\int_0^\infty\frac{\exp\{-\lambda yx\}x^{2n-1}}{(2n-1)!}dx\right)dy \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\int_0^\infty\frac{\exp\{-\alpha y\}y^{k-2}}{\lambda(y+l)^\nu}dy \\ &\quad + \frac{\alpha^{k-\nu}\gamma}{\Gamma_\nu(k,\alpha l)}\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}\int_0^\infty(-1)^i\binom{n-1}{i}\left(\frac{1}{\lambda\xi}\right)^n\frac{\exp\{-(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)y\}y^{k-n-1}}{n!(y+l)^\nu}dy \\ &= \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)}\left[\frac{\Gamma_\nu(k-1,\alpha l)}{\lambda\alpha^{k-\nu-1}}+\sum_{n=1}^{\infty}\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left(\frac{1}{\lambda\xi}\right)^n\frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}}\right], \end{aligned}$$

where the second and the last equalities follows from Eq. (1.2.2), for $k > n$, for all $n \in \mathbb{N}$. Therefore,

$$E(L) = \lim_{k\to\infty} \frac{\alpha^{k-\nu}}{\Gamma_\nu(k,\alpha l)} \left[\frac{\Gamma_\nu(k-1,\alpha l)}{\lambda \alpha^{k-\nu-1}} + \sum_{n=1}^{\infty} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left(\frac{1}{\lambda\xi}\right)^n \frac{\Gamma_\nu(k-n,(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)l)}{(\alpha+n\lambda\delta_0+i\gamma\sigma_2\lambda)^{k-\nu-n}} \right]$$

and hence, the result is proved. □

An important special case of Corollary 3.4.11 is as follows. When $\nu = 0$, $\alpha = k$ and $k \to \infty$ (i.e., the shocks occur according to the HPP with intensity λ), we have

$$E(L) = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{n=1}^{\infty} \left(\frac{\gamma \exp\{-\lambda n \delta_0\}}{\xi^n \lambda^{n-1} \sigma_2^{n-1}} \right) (1 - \exp\{-\lambda\sigma_2\gamma\})^{n-1}.$$

In the next theorem, we prove the same result, as in Corollary 3.4.9, under the assumption that shocks' magnitudes follow the exponential distribution. Here, we also establish an useful relationship between the mean lifetime of the system and the expected number of shocks that the system may experience before its failure.

Theorem 3.4.5 *Let shocks occur according to the HPP with intensity $\lambda > 0$. Assume that $Y_i \sim \exp(\theta)$, $i \in \mathbb{N}$, for some $\theta > 0$. Then the mean lifetime of a system for the general δ-shock model with the recovery function $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$ is given by*

$$E(L) = \frac{1}{\lambda} E(M) = \frac{1}{\lambda} + \frac{1}{\lambda} \sum_{m=1}^{\infty} \Delta_m,$$

where Δ_m is the same as in Theorem 3.4.3.

Proof: We can write

$$L = X_1 + X_2 + \cdots + X_M,$$

where M is the random variable representing the fatal shock that causes the system's failure. As the shocks occur according to the HPP with intensity $\lambda > 0$, $\{X_1, X_2, \ldots, X_n\}$ are iid random variables with the common mean $1/\lambda$. Thus, from the Wald's equation, we get

$$E(L) = E(X_1)E(M) = \frac{1}{\lambda} E(M),$$

where $E(M)$ is the same as in Theorem 3.4.3. Hence, the result is proved. □

Now, we illustrate the result given in Corollary 3.4.9. We assume the relevant parameters as follows: $\lambda = 0.5$ and $\xi = 9$. In Figure 3.1, we plot $E(L)$ against $\sigma_1 \in [0, 1]$, for fixed $\sigma_2 = 0.01$ and $\gamma = 8$. In Figure 3.2, we plot $E(L)$ against $\sigma_2 \in [0, 1]$, for fixed $\sigma_1 = 0.01$ and $\gamma = 8$. In Figure 3.3, we plot $E(L)$ against $\gamma \in (0, \xi]$, for fixed $\sigma_1 = 0.1$ and $\sigma_2 = 0.01$. Figures 3.1 and 3.2 show that $E(L)$ decreases as σ_1 or σ_2 increases, whereas the reverse scenario is observed in Figure 3.3.

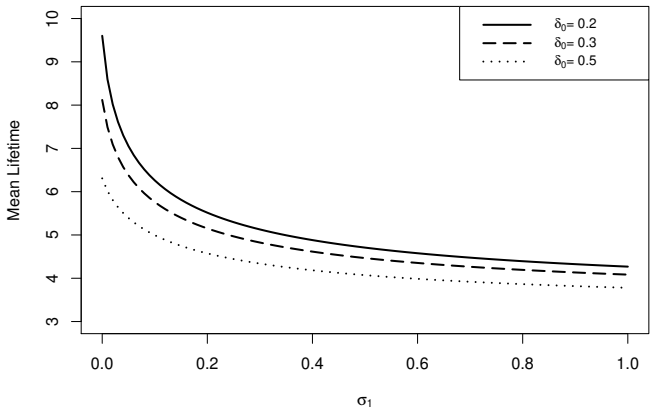


Figure 3.1: Mean lifetime $w.r.t.$ σ_1

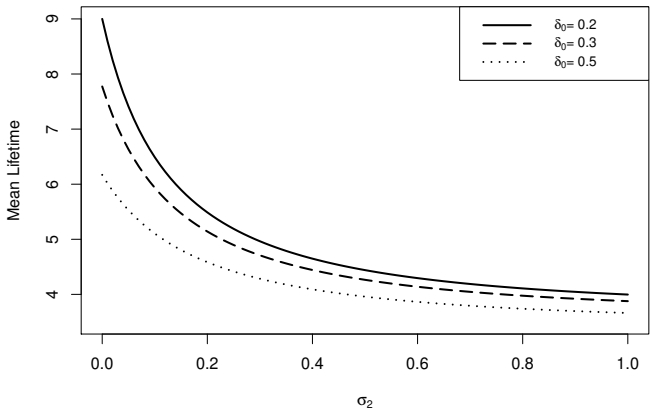


Figure 3.2: Mean lifetime $w.r.t.$ σ_2

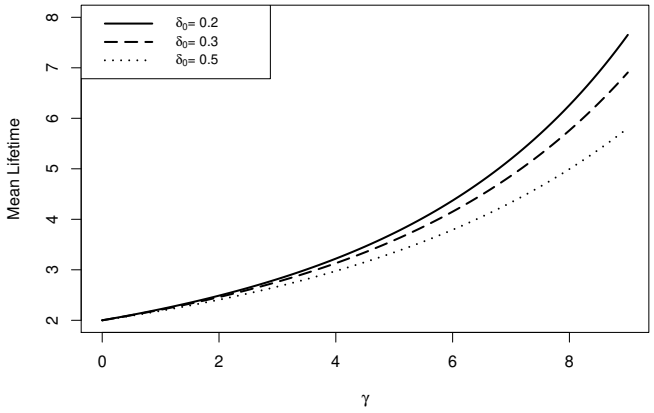


Figure 3.3: Mean lifetime $w.r.t.$ γ

3.5 Application: Optimal replacement policy

In this section, we consider the optimal replacement policy N^* for a system with the lifetime described by the general δ-shock model (the linear recovery function). The system is replaced by a new one when it experiences N^* failures. Imperfect repairs are performed between replacements. In what follows, as an example, we study the optimal replacement policy for the developed general δ-shock model with the linear recovery time and the HPP process of shocks. We consider two different distributions of shocks' magnitudes, namely, the exponential and uniform distributions.

Below we give a list of assumptions that are similar to those considered in [Lam and Zhang, 2004; Tang and Lam, 2006; Eryilmaz, 2017b].

Assumptions:

1. A new system is put into operation at time $t = 0$ and is repaired instantly once it is failed. The system is replaced by a new identical one after the N-th failure is observed.

2. The system is subject to external shocks that occur according to the HPP with the intensity λ (> 0).

3. After the n-th repair, the recovery function is given by $\delta_n : [0,\infty)\times[0,\infty) \to [\kappa^n\delta_0, \infty)$ such that $\delta_n(t, y) = \kappa^n\delta(t, y)$, $n = 1, 2, \ldots$; here $\kappa \geq 1$ and $\delta(t, y) = \delta_0 + \sigma_1 t + \sigma_2 y$, $\delta_0 \geq 0$, $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$. Moreover, the system is not affected by any external shocks during the repair time.

4. Let R_i be the repair time of the system after the i-th failure, $i = 1, 2, \ldots$. Then the sequence $\{R_1, R_2, \ldots\}$ forms an increasing geometric process such that $E(R_n) = \frac{\mu}{r^{n-1}}$, $n = 1, 2, \ldots$.

5. The repair cost is c; the reward rate is c_r when the system is operating. The replacement cost has two parts: the basic replacement cost is B_c, whereas the other one is proportional to the replacement time Z with rate c_p. Further, we assume that $E(Z) = \rho$.

6. The HPP, the geometric process and the replacement time Z are independent.

Now, we discuss the replacement policy for the following two cases.
Case-1: $Y_j \sim U[0, \xi]$, for all $j \in \mathbb{N}$, where $\xi > 0$;
Case-2: $Y_j \sim \exp(\theta)$, for all $j \in \mathbb{N}$, where $\theta > 0$.
Let $L_1^{(i)}$ denote the random operating time of the system to the first failure for Case-i, $i = 1, 2$. Further, let $L_n^{(i)}$ be the random time elapsed since the $(n-1)$th repair to n-th failure for Case-i, $i = 1, 2$, $n = 2, 3, \ldots$. Let W_i be a random length of a cycle under replacement policy N for Case-i, $i = 1, 2$. Then

$$W_i = \sum_{n=1}^{N} L_n^{(i)} + \sum_{n=1}^{N-1} R_n + Z.$$

From Corollary 3.4.9, we have

$$E(L_n^{(1)}) = \frac{1}{\lambda}\left[1+\sum_{m=1}^{\infty}\left(\frac{\gamma\exp\left\{-\lambda\delta_0\left(\frac{(1+\kappa^{n-1}\sigma_1)^m-1}{\sigma_1}\right)\right\}}{\xi^m(1+\kappa^{n-1}\sigma_1)^{(m-1)^2}\lambda^{m-1}\left(\kappa^{n-1}\sigma_2\right)^{m-1}}\right)\right.$$
$$\left.\times\left\{\prod_{i=1}^{m-1}\left(1-\exp\{-\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{i-1}\gamma\}\right)\right\}\right] \tag{3.5.1}$$

and, from Theorem 3.4.5, we have

$$E(L_n^{(2)}) = \frac{1}{\lambda}\left[1+\sum_{m=1}^{\infty}\theta^{m-1}\left(\frac{(1-\exp\{-\theta\gamma\})}{(1+\kappa^{n-1}\sigma_1)^{\frac{m(m-1)}{2}}}\right)\exp\left\{-\lambda\delta_0\left(\frac{(1+\kappa^{n-1}\sigma_1)^m-1}{\sigma_1}\right)\right\}\epsilon_m\right], \tag{3.5.2}$$

where

$$\epsilon_m = \prod_{i=1}^{m-1}\left(\frac{1-\exp\left\{-\gamma(\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{m-1-i}+\theta)\right\}}{\lambda\kappa^{n-1}\sigma_2(1+\kappa^{n-1}\sigma_1)^{m-1-i}+\theta}\right),\ m=1,2,\ldots.$$

Therefore,

$$E(W_i) = \sum_{n=1}^{N}E(L_n^{(i)}) + \sum_{n=1}^{N-1}\frac{\mu}{r^{n-1}} + \rho, \quad i=1,2.$$

Further, the expected cost of a cycle for Case-i, $i=1,2$, is given by

$$E\left\{c\sum_{n=1}^{N-1}R_n - c_r\sum_{n=1}^{N}L_n^{(i)} + B_c + c_pZ\right\} = c\sum_{n=1}^{N-1}\frac{\mu}{r^{n-1}} - c_r\sum_{n=1}^{N}E(L_n^{(i)}) + B_c + c_p\rho.$$

Then the average cost, denoted by $C_i(N)$, for Case-i, $i=1,2$, can be calculated as

$$C_i(N) = \frac{\text{Expected cost incurred in a cycle}}{\text{Expected length of a cycle}}$$
$$= \frac{c\sum_{n=1}^{N-1}\frac{\mu}{r^{n-1}} - c_r\sum_{n=1}^{N}E(L_n^{(i)}) + B_c + c_p\rho}{\sum_{n=1}^{N}E(L_n^{(i)}) + \sum_{n=1}^{N-1}\frac{\mu}{r^{n-1}} + \rho},$$

where $E(L_n^{(1)})$ and $E(L_n^{(2)})$ are the same as in Eq. (3.5.1) and Eq. (3.5.2), respectively.

In Tables 3.1 and 3.2, we tabulated the values of $C_1(N)$ and $C_2(N)$, for different values of N. We assume the relevant parameters as follows: $\kappa = 1.06, \mu = 10, r = 0.80, c = 6, c_r = 3, B_c = 100, \delta_0 = 0.5, \rho = 30, \sigma_1 = 0.2, \sigma_2 = 0.1, c_p = 4, \lambda = 0.40, \gamma = 1$ and $\xi = 5$, $\theta = 0.4$. Note that, for both cases, the average magnitudes of shocks are the same (which is equal to 2.5). Tables 3.1 and 3.2 indicate that the optimal average replacement costs are $C_1(6) = 5.081326$ and $C_2(6) = 4.948404$. Although these are different, the optimal replacement policies, for both cases, are the same and are given by $N^* = 6$. Hence, for both cases, the systems should be replaced immediately after the 6-th failure.

Table 3.1: Values of $C_1(N)$

N	$C_1(N)$	N	$C_1(N)$	N	$C_1(N)$	N	$C_1(N)$
1	6.403121	11	5.367308	21	5.856849	34	5.987039
2	5.709264	12	5.438418	22	5.879559	38	5.994079
3	5.349527	13	5.506007	23	5.898941	42	5.997328
4	5.169131	14	5.568817	24	5.915413	46	5.998806
5	5.093363	15	5.626171	25	5.929358	50	5.999470
6	**5.081326**	16	5.677809	26	5.941125	60	5.999932
7	5.108563	17	5.723764	27	5.951022	70	5.999992
8	5.159521	18	5.764263	28	5.959326	80	5.999999
9	5.223903	19	5.799656	29	5.966274	90	6.000000
10	5.294744	20	5.830367	30	5.972076	100	6.000000

Table 3.2: Values of $C_2(N)$

N	$C_2(N)$	N	$C_2(N)$	N	$C_2(N)$	N	$C_2(N)$
1	6.294425	11	5.289444	21	5.841683	34	5.985941
2	5.567181	12	5.370442	22	5.867000	38	5.993613
3	5.198814	13	5.447119	23	5.888572	42	5.997133
4	5.020134	14	5.518161	24	5.906876	46	5.998725
5	4.951095	15	5.582876	25	5.922347	50	5.999437
6	**4.948404**	16	5.641023	26	5.935379	60	5.999929
7	4.986342	17	5.692676	27	5.946324	70	5.999991
8	5.048593	18	5.738118	28	5.955491	80	5.999999
9	5.124352	19	5.777767	29	5.963150	90	6.000000
10	5.206297	20	5.812114	30	5.969535	100	6.000000