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# Introdution

We live in a connected world. Think about your everyday experiences shopping online, interactions on various social media platforms, the banking system. The fundamental mathematical idea that capture world's relational nature is a graph, or network. Social networks help friends to communicate; financial networks help businesses, road or rail networks. As these networks are indispensable part of our lives, there is a field called Network Science to specifically study these network [Garey and Johnson, 1979]. Study of these networks help us to understand and improve our interconnected world.

Real-world networks have distinctive properties such as small-world, scale-free degree distribution, and community structure. The small-world phenomenon is characterized by having high clustering coefficient and small average path lengths, showing how most of the nodes can be reached from any other by a few number of steps [Watts and Strogatz, 1998]. The scale-free property shows a power law degree distribution which implies many nodes have a few connections while there are a few nodes with many connections, that impact network robustness and vulnerability [Barabási and Albert, 1999]. Erdős-Rényi [Erdős and Rényi, 1960] and Gilbert [Gilbert, 1959] provide models for random network generation where connections are formed at random. The real-world networks are then compared against these models to understand these properties.

There are graph products based network generation methods (such as Cartesian product, direct product, Kronecker product, corona product) that captures more complex network structures [Leskovec *et al.*, 2010a]. These models provide deep insights into the real-world networks form and evolve. The authors [Sharma *et al.*, 2015, 2017] proposed a network generation framework based on the corona product of graphs. They introduced a framework for generating corona graphs by iteratively using corona product on the seed graph. Apart from studying the structural properties of corona graphs they also investigated diameter, degree and betweeness distribution.

Several signed network generative models are proposed in literature with a goal to preserve structural or statistical properties of real world signed networks. For example, in [Hsieh *et al.*, 2012] a low-rank model based on matrix completion technique is proposed to generate signed networks which can inherit structural balance property of real networks. In [Derr *et al.*, 2018], the authors propose a parametric model to preserve degree distribution, sign distribution, and balance/unbalanced triangle distribution of signed networks. This model is inspired by the popular Chung-Lu model for generation of unsigned networks. Recently,

a model is proposed based on local preferential attachment to generate signed networks that have community structure and high positive clustering coefficient [Li *et al.*, 2018]. This thesis first porposes corona product of signed two signed graphs based on the corona product of two graphs introduced by [Frucht and Harary, 1970].

Let $G^s$ be asigned graph on $n$ nodes, and $H^s$ an signed graph. Then the corona product graph $G^s \circ H^s$ of $G^s$ and $H^s$ is defined as the graph obtained by taking one copy of $G^s$ and $n$ copies of $H^s$, and then joining the $i^{th}$ vertex of $G$ to every vertex in the $i^{th}$ copy of $H^s$. I studied structural and spectral properties of the corona product of signed graphs. Spectral properties of the existing network generative models are completely unexplored. Thus how the spectral properties relate to the structural properties of the networks generated by these models is unclear. The thesis also defines signed corona graphs and propose them as signed graph generation model. The formulas to calculate number of signed edges, triangles, and positive and negatve degree are provided. Spectral property such as algebraic conflict [Aref, 2019; Kunegis *et al.*, 2010], the smallest signed Laplacian eigenvalue of the network can estimate the degree of balance. The study investigated the algebraic conflict for some sample seed graphs. It is shown that we can identify a seed graph to preserve certain statistical properties of real-world signed graphs.

Finding analytical expression of eigenvalues and Laplacian eigenvalues of product graphs, such as corona product graphs, is one of the fundamental problems in spectral graph theory [Sharma *et al.*, 2015; McLeman and McNicholas, 2011]. Another contribution of thesis is to introduce the generalized corona product of signed graphs and study their structural balance and spectral properties. I introduced the concepts of signed coronal, Laplacian signed coronal (L-signed coronal), and Signless Laplacian signed coronal (Q-signed coronal) for signed graphs by building upon the idea of 'coronal' introduced by McLeman and McNicholas [McLeman and McNicholas, 2011] as well as 'Laplacian Coronal' and 'Signless Laplacian coronal' introduced by Laali et al. [Laali *et al.*, 2016]. Furthermore, we studied the spectral properties of generalized corona products applied to signed graphs, including the derivation of the signed characteristic polynomial, signed Laplacian polynomial, and signed signless Laplacian polynomial. Additionally, I introduced the concept of 'signed marked graph isomorphism,' enabling us to establish sufficient conditions for the generalized product of distinct collections of signed graphs to exhibit co-spectral behavior.

Real-world systems such as transportation systems, social networks, collaboration networks, communication networks can be modeled as complex networks. Controlling these networks is one of the challenging problem in modern network science. In large-scale networks, due to heaving computational complexity, fundamental controllability criteria like Kallman rank criteria and PBH test can not be applied directly. Since communities or motifs are the building blocks of large networks, graphs products are natural choice to generate such networks by using two or more smaller ones. In [Wang *et al.*, 2020], the authors laid down necessary and sufficient condition on controllability of corona product graphs. This is a low dimensional condition and hence can reduce the computational overhead for large-scale networks.

## 1.1 BASIC DEFINITIONS

In this section, I have explained various concepts, definitions, and terminology that helps in understanding the subsequent chapters. I have also describe different matrices that will help in analyzing the structural and spectral properties of signed networks. A brief summary of literature survey and introduction of related work is explained. In this section, the definitions, basic concepts, and notations used in the thesis are listed.

### 1.1.1 Graph-theoretic concepts

The notion of 'Graphs' was first introduced in the eighteenth century by Swiss mathematician Leonard Euler in solving, The Königsberg Bridge problem, but he did not explicitly used the term 'Graph'. One hundred and fifty years later, an English mathematician Sylvester introduced this term.

A graph $G = (V, E)$ is defined as a collections of nodes and edges, where $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of nodes and $E \subseteq V \times V$ is a finite set of edges. The relationship between two nodes $v_i$ and $v_j$ is symmetric if $(v_i, v_j) \in E$ implies $(v_j, v_i) \in E$. $|V| = n$ and $|E| = m$ represents the number of nodes and number of edges in graph $G$, respectively. A graph $G$ can be represented in the form a square matrix of order $n \times n$, called adjacency matrix $A$. The entry in $A$, $a_{ij} = 1$ if $(v_i, v_j) \in E$, otherwise $a_{ij} = 0$.

**Definition 1.1.1.** ***Undirected and directed Graph*** *If we consider the direction of the links or edges between the nodes then graphs can be directed or undirected. For an undirected graph, A is a symmetric matrix and* $(v_i, v_j) = (v_j, v_i)$*, for* $1 \leq i, j \leq n$.

**Definition 1.1.2.** ***Path -*** *A path of length l is a walk in which all the nodes and edges are distinct. A path consisting of n nodes is denoted by* $P_n$.

**Definition 1.1.3.** ***Cycle -*** *A cycle is a closed walk in which all the edges and nodes are distinct except the first and last. A cycle with n nodes is denoted by* $C_n$.

**Definition 1.1.4.** ***Degree (d) -*** *The degree of a node is the number of edges incident on it.*

**Definition 1.1.5.** ***Complete graph -*** *A simple graph with each pair of nodes having an edge between them,* $E = V \times V$*. The complete graph with n nodes is denoted by* $K_n$.

**Definition 1.1.6.** ***Sub-graph -*** *A graph* $G' = (V', E')$ *is a subgraph of* $G = (V, E)$ *if and only if* $V' \subseteq V$ *and* $E' \subseteq E$.

**Definition 1.1.7.** ***r-regular graph -*** *A graph* $G = (V, E)$ *is r-regular if all the nodes have degree r and consequently* $rn/2$ *edges.*

**Definition 1.1.8.** ***Triangle or triad -*** *A triangle is a 3-node clique where each node is adjacent to one another.*

In this thesis, we use the term graph and network interchangeably.

**Matrices associated with graphs**
The matrices associated with graphs are used in the analysis of networks. The most common matrices used are adjacency matrix, Laplacian matrix.

**Definition 1.1.9.** ***Adjacency matrix -*** *The adjacency matrix, A, of a graph $G = (V,E)$ of n nodes is a $n \times n$. The i, jth element of the adjacency matrix A is given by:*

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise} \end{cases}$$

*where $a_{ij}$ is the $i^{th}$ element of A and $V = \{v_1, v_2, \ldots, v_n\}$ be the set of vertices and $1 \leq i, j \leq n$.*

*The adjacency matrix of an undirected graph is symmetric and hence its eigenvalues are real. The set of eigenvalues of a matrix are called as the spectrum of the graph. The largest eigenvalue of a matrix is called spectral radius. It signifies the robustness of the underlying graph [Wang* et al.*, 2003].*

**Definition 1.1.10.** ***Degree matrix -*** *The degree matrix of a graph G pf order n is $n \times n$ diagonal matrix $i^{th}$ diagonal element ($a_{ii}$) is equal to the number of degree of node i.*

**Definition 1.1.11.** ***Laplacian matrix -*** *The laplacian matrix L of a simple graph G of order n is a $n \times n$ matrix and is given by$L = D - A$. If $d_i$ is the degree of $i^{th}$ node of G then*

$$l_{ij} = \begin{cases} -1 & \text{if } a_{ij} = 1, \\ d_i & \text{if } i = j, \\ 0 & otherwise, \end{cases}$$

*where $l_{ij}$ is $ij^{th}$ element of L and $V = \{v_1, v_2, \ldots, v_n\}$ and $1 \leq i, j \leq n$.*

Laplacian matrix is a symmetric positive semi-definite matrix and therefore it has non-negative eigenvalues.

### 1.1.2 Signed graph-theoretic concepts

A signed graph is a graph in which some edges are designated with positive sign reflecting positive relationship between the constituent pair of nodes and other edges are assigned negative sign which represents negative relationship. Thus signed graph is an ordered tuple $G^s = (V, E^s, \sigma)$ where $V$ denotes the set of nodes, $E^s \subseteq V \times V$ the edge set, and $\sigma : E^s \rightarrow \{+, -\}$ is called the signature function.

**Definition 1.1.12.** ***Structural balance -*** *A signed network is called balanced or in balanced state if and only if all the cycles (closed walks) in the signed network are balanced [Harary, 1953; Cartwright and Harary, 1956]. A signed cycle is called balanced if the number of negative edges in it is even otherwise it is called unbalanced.*

**Definition 1.1.13.** ***Positive and negative degree -*** *The positive and negative degree of a node u in a signed graph are defined by number of positive and negative edges incidental to the node u denoted by $d^+(u)$ and $d^-(u)$, respectively. The degree of u are given by $d^{\pm}(u) = d^+(u) + d^-(u)$.*

**Definition 1.1.14.** ***Signed Triangle or triad -*** *Let $T_i$ denote a triangle, a signed subgraph on three nodes in a signed network with i negative edges in it, $i = 0, 1, 2, 3$, Fig. 1.1.*



**(a)** $T_0$ **(b)** $T_1$ **(c)** $T_2$ **(d)** $T_3$

**Figure 1.1 :** Possible triangles in an (undirected) signed network. triangles (a) and (c) are balanced, and triangles (b) and (d) are unbalanced. Solid and dashed edges represent positive and negative edges, respectively.

**Definition 1.1.15.** ***Net-degree -*** *The net-degree of a node u of a singed graph is given by $d(u) = d^+(u) - d^-(u)$.*

**Definition 1.1.16.** ***Net-regular signed graph*** *A signed graph is called net-regular if every node of it has the same net-degree d. Note that net-regular signed graphs are signed counterpart of unsigned regular graphs.*

## 1.2 NETWORK MODELS

A network model explains the evolution of the network and helps to understand its functionality. Here, we discuss some widely studied models.

Complex networks is a young and active area of scientific research inspired largely by the empirical study of real-world networks such as computer networks, social networks, biological networks, ecological networks, and telecommunication networks. A lot of interesting and important properties of complex networks have been discovered, including sparsity, the small world phenomenon (also called six degree separation, as exemplified by studies such as those of Harvard sociologist Stanley [Milgram, 1967], the play of [Guare, 1990], and Watt's observation about handshakes [Watts, 2004]), and power law distribution (also known as long tail or heavy tail distribution or scale-free, as noted by Barabási and Albert [Barabási and Albert, 1999] in their observations of actor collaboration networks and the World Wide Web). The field continues to develop at a brisk pace, bringing together researchers from mathematics, physics, biology, telecommunications, computer science, sociology, epidemiology, and other areas [Motter and Albert, 2012]. Commonly used metrics in complex networks include average degree, diameter, entropy, average distance, centrality, betweenness, density, clustering coefficient, transitivity, and diffusion time.

### *Unsigned Network Models*

Modeling complex networks is a significant challenge in this area. Traditional models such as the Erdős-Rényi random network model often do not fit real-world networks well. For instance, while the Erdős-Rényi model has a small diameter, it lacks the high number of triangles typical in social networks, where friends of one person often know each other. To address these discrepancies, scientists have proposed various models:

The Watts-Strogatz model [Watts and Strogatz, 1998] begins with a ring lattice and rewires edges with a probability $p$, which can drastically reduce the diameter. The BC small world model [Bollobás and Chung, 1988] adds a random matching to a ring of $n$ vertices, showing a diameter of approximately $\log n$. Newman and Watts [Watts and Strogatz, 1998] introduced the NW small world model, adding a Poisson number of shortcuts to create long-distance edges. Barabási and Albert [Barabási and Albert, 1999] proposed the preferential attachment model to explain power law distribution, showing that the diameter is approximately $\log n/(\log\log n)$. Chung and Lu [Chung and Lu, 2002, 2004] developed the Chung-Lu model with distances of $O(\log n/(\log\log n))$. Callaway et al. [Callaway *et al.*, 2001] proposed the CHKNS model, inspired by Barabási and Albert's work. See also [Chung and Lu, 2006; Durrett, 2007] for more details.

### *Signed Network Models*

Signed networks include edges that carry a sign, indicating positive or negative relationships. Research on signed network reconstruction has produced several models:

Ludwig and Abell [Ludwig and Abell, 2007] proposed an evolutionary model (Evo) for generating signed networks, focusing on controlling the sign of edges and bounding the maximum number of unbalanced triangles. However, this model is limited by its computational complexity due to the need to account for various triad types. Vukasinovic et al. [Vukasinovic *et al.*, 2014] introduced an interaction-based (IB) model that preserves the proportion of balanced triangles while generating synthetic networks corresponding to given input signed networks. Derr et al. [Derr *et al.*, 2018] developed the BSCL model, which integrates the transitive Chung-Lu (TCL) model [Pfeiffer *et al.*, 2012] with balance theory [Heider, 1946] to ensure that the distribution of signed triangles in synthetic networks resembles that of the input networks.

The research in growing networks aims to model the dynamics of network growth. The signed growing network (SGN) model [Li *et al.*, 2018] incorporates local preferential attachment and requires three parameters: $n$ (total number of nodes), $l$ (total links per node), and $q$ (ratio of negative edges). Additionally, the two-layered generative architecture (2L-SNM) [Pandey *et al.*, 2022] allows network growth through probabilistic node insertion and structural balance preservation or through internal growth processes that produce unbalanced triangles.

Signed network modeling is crucial for understanding complex relationships where interactions can be positive or negative. These models help in various applications, such as predicting social dynamics, analyzing sentiment in social media, and studying conflicts and alliances in social or political contexts. The ability to model and simulate signed networks

provides insights into the structural balance and dynamics of real-world systems, making it an important area of research in network theory and applications.

### 1.2.1 Graph Product-Based Models

Graph product-based models are important because they allow the construction of large and complex networks from simpler, well-understood components. These models combine two or more graphs to create a new graph that inherits properties from the original graphs. This approach is particularly useful in modular network design, where complex networks can be broken down into smaller, manageable sub-networks, or when modeling multi-layered or multi-dimensional systems. For instance, Cartesian, tensor, and strong graph products can be used to model networks with different types of interactions or multi-modal data, providing a more comprehensive understanding of the system's overall behavior. By using product-based models, researchers can more easily explore the interplay between different network layers, study the emergent properties of the combined networks, and apply these models in fields like biology, communication systems, and social network analysis. This modeling strategy is essential for scaling up to real-world applications where networks are too large or complex to be studied as a single entity.

**Product graph approach** There are a few graph product operations that are used for network modeling [Hammack *et al.*, 2011].

**Kronecker Graphs -** Kronecker graphs are introduced by [Leskovec *et al.*, 2010a] and are based on the Kronecker product of graphs. Kronecker graph models are generates networks using minimal number of parameters. The synthetic graphs generated using Kronecker products are fitted with real-world networks [Leskovec *et al.*, 2010a]. The authors defined two types of recursive Kronecker graphs as Deterministic Kronecker graphs and Stochastic Kronecker graphs.

for a given graph $G$ having two matrices matrix $A = [a_{i,j}]$ and $B$ of sizes $nm$ and $n \times m$ respectively, the Kronecker product matrix $C$ of dimensions $(nn) \times (mm)$ is given by

$$C = A \otimes B = \begin{bmatrix} a_{1,1}B & a_{1,2}B & \cdots & a_{1,m}B \\ a_{2,1}B & a_{2,2}B & \cdots & a_{2,m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1}B & a_{n,2}B & \cdots & a_{n,m}B \end{bmatrix}$$

Kronecker product of graphs is defined based on the Kronecker product of matrices. If $G$ and $H$ are the graphs with adjacency matrices $A(G)$ and $A(H)$ respectively, then their Kronecker product is given by $A(G) \otimes A(H)$.

We can also generate a sequence of Kronecker graphs by iteratively using Kronecker product (Kronecker power) on the seed graph with adjacency matrix $(K_1)$. The $K^{th}$ power of $K_1$ is defined as the matrix $K_1^{[k]}$ (abbreviated as $K_k$), and is given by

$$K_1^{[k]} = K_k = \underbrace{K_1 \otimes K_1 \otimes \cdots \otimes K_1}_{k \text{ times}} = K_{k-1} \otimes K_1$$

For a given seed graph $K_1$ with $n$ nodes and $m$ edges, the total number of nodes and edge in $K^k$ are $n^{2^k}$ and $m^{2^k}$. If the eigenvalues of $K_1$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$ then the eigenvalues of $K_k$ are given by $(\lambda_1, \lambda_2, \ldots, \lambda_n)^{[k]}$. Similarly if $v_1, v_2, \ldots, v_n$ are the eigenvectors of $K_1$, then the eigenvectors of $K_k$ are given by $k^{th}$ Kronecker power of the vector $(a_n)^{[k]}$. Similarly if $v_1, v_2, \ldots, v_n)$, which gives a multinomial distribution for the components of each eigenvectors in $K_k$.

**Generalized graph products**

Generalized graph product based approaches are proposed for designing data communication network in terms of small sub-graphs [Parsonage *et al.*, 2011]. Give two graphs $G$ and $H$, they studied Cartesian product ($G \square H$), Tensor product ($G \otimes H$), Strong product ($G \boxtimes H$), Lexicographic product ($G \bullet H$), Rooted product ($G \circ H$), and Corona product ($G \odot H$). (Note that, in this thesis we have used $\circ$ to denote Corona product.) The above graph products can be expressed in terms of Kronecker product of matrices.

Let $A(G)$ and $A(H)$ be the adjacency matrices of graphs $G$ and $H$, of order $n$ and $m$), respectively. Then $A(G \square H)$, $A(G \boxtimes)$, $A(G \bullet H)$,and $A(G \circ H)$ represents the adjacency matrices of the Cartesian product, Strong product, Lexicographic product, and Rooted product. These products in terms of Kronecker product are given as

$$\begin{aligned}
G \square H &= A_G \otimes I_m + I_n \otimes A_H, \\
G \boxtimes H &= A_G \otimes I_m + I_n \otimes A_H + A_G \otimes A_H, \\
G \bullet H &= A_G \otimes J_m + I_n \otimes A_H, \\
G \circ H &= A_G \otimes D_{m,k} + I_n \otimes A_H,
\end{aligned}$$

where $I_n$ is the identity matrix, $J_n$ is the $n \times n$ matrix with 1 in every row element, and $D_{m,k}$ is $m \times m$ matrix with 1s in $k^{th}$ diagonal and 0's in the left row elements. The authors also provided a generalized notation to represent the product graphs as given

$$A(G \otimes_{f,g} H) = A_G \otimes f(A_H) + I_n \otimes g(A_H) \tag{1.1}$$

$$= \begin{bmatrix} a_{1,1} f(A_H) & \cdots & a_{1,n} f(A_H) \\ \vdots & \ddots & \vdots \\ a_{n,1} f(A_H) & \cdots & a_{n,n} f(A_H) \end{bmatrix} + \begin{bmatrix} g(A_H) & 0 & \cdots & 0 \\ 0 & g(A_H) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & g(A_H) \end{bmatrix} \tag{1.2}$$

where $a_{i,j} = (A_G)_{ij}$, and functions $f$ and $g$ depends on the specific graphs products defined above.

Let $\{H_i\}_{i=1}^n$ represents $n$ different graphs, then the more generalized structure of the equation 1.2 is given by

$$A(G \otimes_{f,g} (H_1, H_2, \ldots, H_n)) = \begin{bmatrix} a_{1,1}(A_{H_1}, A_{H_1}) & \cdots & a_{1,n} f(A_{H_1}, A_{H_n}) \\ \vdots & \ddots & \vdots \\ a_{n,1} f(A_{H_n}, A_{H_1}) & \cdots & a_{n,n} f(A_{H_n}, A_{H_n}) \end{bmatrix} + \begin{bmatrix} g(A_{H_1}, A_{H_1}) & 0 & \\ & \ddots & \\ 0 & \cdots & g(A_{H_n}, A_{H_n}) \end{bmatrix}$$

(1.3)

When each $H_i$ is equal to each other then we obtain the standard graph operations.

**Corona product graph**

Corona product of graphs was introduced by Frucht and Harary [Frucht and Harary, 1970]. Given two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ of orders $n$ and $m$ respectively. The corona product of $G$ and $H$ is a graph, $G \circ H$, obtained by taking $n$ instances of $H$ and each such instance of $H$ gets connected to each node of $G$. The corona product graph $G \circ H$ has $n(1+m)$ number of nodes and $e_1 + n(e_2 + m))$ edges, where $e_1$ and $e_2$ are the number of edges of $G$ and $H$ respectively.

Starting with a simple graph $G$, corona graphs $G^(m) = G^{(m-1)} \circ G$, for $m \geq 1$, are obtained by taking corona product of $G$ with itself iteratively. In this case, $G$ is called the seed graph for the corona graphs. Thus, given a seed graph, as the number of iterations increases during the formation of corona graphs, this process provides a mathematical model for a growing network, where the number of nodes expands exponentially with each iteration.

**Generalized corona product graphs** The generalized corona product of unsigned graph is studied by [McLeman and McNicholas, 2011]. The pectrum of such graph is obtained by using the coronal. Assume that $H$ is a graph on $n$ vertices with the adjacency matrix $B$, when considered as a matrix over the field of a rational function $\mathscr{C}(\lambda)$, the characteristic matrix $\lambda I - B$ has determinant $\det(\lambda I - B) = f_H(\lambda) \neq 0$, indicating it is inevitable. The coronal $\chi_H(\lambda) \in \mathscr{C}(\lambda)$ of $H$ is defined to be the sum of the entries of the matrix $(\lambda I - B)^{-1}$. Therefore, this can be obtained as

$$\chi_H(\lambda) = \mathbf{1}^T (\lambda I - B)^{-1} \mathbf{1}$$

where $\mathbf{1}$ denotes the all-one vector of compatible dimension, that is $n$.

By utilizing the definition of coronal, we can define the generalized corona product of signed graphs. Given simple graphs $G, H_1, \ldots, H_n$ of order $n$, then the generalized corona, denoted $G \circ \bigwedge_{l=1}^{n} H^s_{\mu_l}$, is the graph obtained by taking one copy of graphs $G, H_1, \ldots, H_n$ and joining the $i^{th}$ vertex of $G$ to every vertex of $H_i$. In this paper, we determine and study the characteristic, Laplacian and signless Laplacian polynomial of $G, H_1, \ldots, H_n$. This leads us to construct new pairs of cospectral, L-cospectral and Q-cospectral graphs using the corresponding matrices.

## 1.3 CONTROLLABILITY OF NETWORKS

A multi-agent networked system can be represented by a graph $G = (V, E)$ where the sets $V = \{v_1, v_2, \ldots, v_n\}$ and $E \subset V \times V$ represent the agents and the interactions between them, respectively. If the networked system follows the leader-follower framework, the set $V$ can be partitioned into leaders ($V^l$) and followers ($V^f$) sets. The interactions among the agents are represented by the adjacency matrix $A$, and an entry in $i^{th}$ represents the presence of negative/positive relation between $v_i$ and $v_j$ in $V$. The Laplacian matrix of $G^s$ is given by $L = D - A$. Let the laplacian spectrum of $G^s$ is $\Gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$.

An Linear time invarient (LTI) control system [Williams and Lawrence, 2007] denoted by

a pair of matrices $(A,B)$ is uncontrollable if and only if there exists a left eigenvalue-eigenvector pair $(\lambda,v)$ of $A$ such that $v^TB=0$.

$$\dot{X}=AX+BU \tag{1.4}$$

is uncontrollable if and only if for every left eigenvector $y$ of $A$, $y^TB\neq 0$, where $A\in\mathbb{R}^{n\times n}$ is a matrix of system, $B\in\mathbb{R}^{n\times m}$ is the input matrix, $X\in\mathbb{R}^n$ is a state vector of the system, and $U\in\mathbb{R}^m$ is the control input vector, respectively.

## 1.4 OVERVIEW OF THE THESIS

In this thesis, I introduce corona product of signed graph using the notion of corona product of unsigned graphs. Then I explore the structural and spectral properties of the those graphs. I extend this definition to define a class of signed networks and termed them signed corona graphs, which were obtained by using the corona product on a seed graph. I have also studied graph dynamics and explored various characteristics of signed corona graphs.

In Chapter 2, I introduce the corona product of signed graphs, extending the classical notion of corona products from unsigned graphs. This involves by introducing the marking scheme for assigning edge signs and examines how these modifications affect the balance and spectral properties of the resulting graphs. I focus on the structural and spectral properties of signed corona graphs. The structural properties include formulas to calculate positive and negative edges and count of type of signed triangles. Later I give detailed derivations of eigenvalues and eigenvectors for matrices associated with the corona product of signed graphs. I also examine the algebraic conflict within these structures. This section connects the theoretical findings with practical applications, demonstrating how signed corona graphs can be used to model and analyze real-world networks with signed interactions. The implications for various domains, including social and biological systems, are discussed, emphasizing the relevance and potential of this approach in understanding complex systems. The corona product is presented as a powerful tool for constructing larger, complex signed networks from smaller seed graphs, called signed corona graphs, offering a method to maintain and control certain properties (degree distribution, positive and negative edges distribution, and type of triangles distribution) in the resulting network. The chapter sets the foundation for understanding the complexities of signed networks and their applications in various fields.

In Chapter 3, I have broadened the application of the generalized corona product from its original definition for unsigned graphs to now include signed graphs. This expansion has enabled an in-depth exploration of the structural and spectral properties unique to signed graphs. I have developed statistical measures, including positive and negative edge counts and triangle classifications, specifically for the generalized corona product of signed graphs. These measures have facilitated to identify sufficient conditions under which the generalized corona product of signed graphs is unbalanced.

Additionally, the research has produced explicit formulations for the signed coronal, L-signed coronal, and Q-signed coronal, with a special emphasis on certain structured signed graphs. I have also derived the characteristic polynomial, Laplacian polynomial, and Signless

Laplacian polynomial for the generalized corona product of signed graphs, all expressed in terms of their corresponding signed coronal, L-signed coronal, and Q-signed coronal. Furthermore, I have established key criteria for detecting instances where generalized corona product signed graphs might exhibit cospectral properties, including L-cospectrality and Q-cospectrality.

In Chapter 4, I extend the definition of corona product of signed graph introduced in Chapter 2 into generalized corona product of signed graphs. Then I derived formulas for structural balance and spectral properties of these graphs. Later I define signed coronal based on the notion of coronal of a graph [Laali *et al.*, 2016], I determine computable formulae of characteristic, Laplacian, and signless Laplacian polynomials of generalized corona product of signed graphs. Finally, I provide sufficient conditions for the generalized corona product of some distinct collections of signed graphs to be co-spectral.