6
Construction of Cospectral Graphs

Reciprocation of Mathematics and Physics is a well studied topic in the Philosophy of Science. Theoretical physics has generated a platform for the application of abstract mathematics. This interrelation has significantly evolved different branches of Mathematics. Partial transpose is a well known tool for entanglement detection. In this chapter, we utilize it towards a well known problem of graph theory, that is, construction of co-spectral graphs. It is based on our work [**?**].

6.1 AN INTRODUCTION TO GRAPH SPECTRA

The graph spectrum reveals fundamental properties of a graph, links the discrete universe to the continuous one through algebraic, analytic and geometric techniques via eigenvalues. Spectral graph theory is a relatively new branch of Mathematics that has been evolving rapidly.

Given any graph $G = (V(G), E(G))$ the adjacency matrix is denoted by $A(G)$, which is defined in the equation (2.9). Spectra of a matrix M is the multi-set $\Lambda(M)$ consisting of the eigenvalues of the matrix M. Spectra of a graph G is denoted by $\Lambda(G)$ and is defined by the spectra of its adjacency matrix. It provides significant information about the structure of the graph G [Bapat, 2010].

Recall that, two graphs G and H are isomorphic if there is a bijective mapping $f : V(G) \to V(H)$, such that, $(u, v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(H)$. We have mentioned in Chapter 2 that given two isomorphic graphs G and H, there exists a permutation matrix P of order $|V(G)|$, such that, $A(G) = PA(H)P^{\dagger}$. Isomorphic graphs have equal spectra and other graph theoretic properties. The graph isomorphism problem is to determine the existence of a graph isomorphism between two graphs G and H.

Two isomorphic graphs has same degree sequence and spectra. But there are non-isomorphic graphs with equal degree sequence and spectra [Cvetkovic et al., 1980, Chapter 6]. The graphs G and H are called cospectral if they have equal spectra. Therefore, finding new classes of non-isomorphic cospectral graphs is an interesting problem in graph theory. Number of non-isomorphic co-spectral graphs has been calculated for graphs with order, $|V(G)| \leq 12$ [Brouwer and Spence, 2009]. Note that, there are 165 billion non-isomorphic graphs with 12 vertices. Hence, the computation is not trivial. There are a few algorithmic procedures for constructing non-isomorphic cospectral graphs. Seidel switching [Seidel, 1974] and the Godsil-McKay switching [Godsil and McKay, 1982] are two well-known techniques to generate families of co-spectral graphs. Current developments in this direction include [Abiad and Haemers, 2012; Butler, 2010; Dalfó and Fiol, 2016]. Another interesting question is to find out which graphs are determined by their spectra [van Dam and Haemers, 2003].

In this chapter, we shall show how partial transpose helps us to construct non-isomorphic cospectral graphs. In this thesis, we introduced the idea of partial transpose, in Chapter 4. We partition the vertex set into clusters in the equation (2.14). We have described the clustering on the vertex set in the Subsection 2.2.6. Based on it we have defined Graph Theoretical Partial Transpose

(GTPT) in the definition 4.3. Two clustered and labelled graphs are GTPT equivalent if one is obtained from another using GTPT. A number of graph theoretic properties of partial transpose were discussed in Section 4.2.

Recall the equation 2.15 which shows that the clustering on the vertex set partitions the adjacency matrix, $A(G)$, into blocks as follows,

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}_{N\times N}, \tag{6.1}$$

where, $A_{\nu,\mu} = A^t_{\mu,\nu}$ are block matrices of order n. When confusion occurs, we mention the block matrix $A_{\mu,\nu}$ as $A_{\mu,\nu}|_G$ to specify it as a part of $A(G)$. If G^τ be the GTPT equivalent of G then the adjacency matrix of G^τ is given by,

$$A(G^\tau) = \begin{bmatrix} A^t_{11} & A^t_{12} & \dots & A^t_{1m} \\ A^t_{21} & A^t_{22} & \dots & A^t_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A^t_{m1} & A^t_{m2} & \dots & A^t_{mm} \end{bmatrix}_{N\times N}. \tag{6.2}$$

Here, G and G^τ may be isomorphic or non-isomorphic. They may be cospectral or non-cospectral. We are interested in those graphs G for which G^τ is non-isomorphic and cospectral. For this investigation we need a relation between $A(G)$ and $A(G^\tau)$ which we shall derive in the following sections.

6.2 A SYSTEM OF LYAPUNOV EQUATIONS AND ITS SOLUTION

The Lyapunov equation is named after the Russian mathematician Aleksandr Lyapunov. It mainly appears in control theory and stability analysis. Let $A^\dagger$ be the conjugate transpose of a matrix A of order n. Given matrices A and Q, the Lyapunov equation is,

$$AX + XA^\dagger + Q = 0, \tag{6.3}$$

where X is an unknown matrix of order n. In our case, the matrix A is a real matrix. Therefore, we consider $A^\dagger$ as A^t. Also, in this chapter we explicitly mention $A^\dagger$ as $\overline{A}^t$ whenever required.

Any complex square matrix, A and its transpose, A^t have equal spectra, and Jordon canonical forms. It can be proved that given a complex square matrix A there is an invertible matrix P s.t. $A^t = P^{-1}AP$ [Kaplansky, 1974, Theorem 66]. But, computing P in this expression is a tedious task. The following property of P is immediate.

Theorem 6.1. *If $A^t = P^{-1}AP$, and $A = P^{-1}A^tP$ hold simultaneously then P is a unitary matrix.*

Proof. We have, $A = P^{-1}A^tP = P^{-1}P^{-1}APP = (P^2)^{-1}AP^2$. Thus, $P^2 = I$, that is, P is self-invertible. Also, $A = P^{-1}A^tP \Rightarrow A^t = (P^{-1}A^tP)^t = P^t(A^t)^t(P^{-1})^t = P^tA(P^t)^{-1}$. Again, $A = P^{-1}A^tP \Rightarrow A^t = PAP^{-1}$. Comparing them we get, $P = P^t$, that is, P is a symmetric matrix. Combining all these properties, we conclude that P is a unitary matrix. □

When a matrix is unitary equivalent to its transpose was studied in [Garcia and Tener, 2009]. Here, we consider those binary matrices which are unitary equivalent to its transpose. We

are interested in finding P for any two matrices A and B such that the relations $A^t = P^{-1}AP$ and $B^t = P^{-1}BP$ holds simultaneously, which will be applied in the next section. We write the relation $A^t = P^{-1}AP$ as $PA^t = AP$, that is $PA^t - AP = 0$. Thus, for $A^t = P^{-1}AP$, and $B^t = P^{-1}BP$ to hold simultaneously, a common solution with non-singular matrix X of the following system of matrix equations is essential:

$$\begin{aligned} AX - XA^t &= 0, \\ BX - XB^t &= 0. \end{aligned} \tag{6.4}$$

A standard technique for solving the Lyapunov equation $AX - XA^t = 0$ is to convert it to a system of linear equations with vectorization and Kronecker product. But, there is no standard procedure to solve a system of Lyapunov equations in the literature till date.

Definition 6.1. Vectorization of a matrix: *For any complex matrix $A = (a_{ij})_{m\times n}$, there exists a vector* $\text{vec}(A) \in \mathbb{C}^{mn}$*, defined by [Horn and Johnson, 1991, Definition 4.2.9],*

$$\text{vec}(A) = [a_{11}, a_{21}, \dots a_{n1}, a_{12}, a_{22}, \dots a_{mn}]^t.$$

Also, given a vector $x = [x_1, x_2, \dots x_{mn}]^t$ we may construct a matrix A such that $\text{vec}(A) = x$*. We denote,* $A = \text{vec}^{-1}(x)$*.*

We express these matrix equations (6.4) into two systems of linear equations [Horn and Johnson, 1991, Lemma 4.3.1]

$$\begin{aligned} (I \otimes A - A^t \otimes I)\,\text{vec}(X) &= \text{vec}(0) = 0, \\ (I \otimes B - B^t \otimes I)\,\text{vec}(X) &= \text{vec}(0) = 0. \end{aligned} \tag{6.5}$$

Given any two matrices A and B we call the expression $I \otimes A + B \otimes I$ as the Kronecker sum of the matrices A and B. Therefore in the above system of equations we are considering the Kronecker sum of A and $-A^t$ as well as B and $-B^t$. The expression $(I \otimes A - A^t \otimes I)$ plays a significant role in this chapter. We call it the compatibility indicator and denote it by $S(A)$. Hence, solution of the above equations depend on $S(A)$ and $S(B)$. They have non-trivial solutions if $S(A)$ and $S(B)$ are not full rank matrices. Also, rank of a matrix is determined by its eigenvalues. We mention the following lemma describing the eigenvalues of a Kronecker sum.

Lemma 6.1. *Given complex square matrices A and B, eigenvalues of their Kronecker sum, $I \otimes A + B \otimes I$ are given by $\gamma + \mu$ where $\gamma \in \Gamma(A)$ and $\mu \in \Gamma(B)$ [Horn and Johnson, 1991, Theorem 4.4.5].*

As A, and A^t have same spectra, the spectra of $S(A)$ contains all possible values of $\lambda_i - \lambda_j$, where $\lambda_i, \lambda_j \in \Lambda(A)$. Thus, $S(A)$ has zero eigenvalues. It is always a singular matrix having a non-trivial null space, determined by its row space [Horn and Johnson, 1991, Exercise, Page 270]. The system (6.4) has a solution if and only if the null spaces of $S(A)$ and $S(B)$ contain a vector x in common. Another useful property of $S(A)$ is:

Lemma 6.2. *If A be a normal matrix, then $S(A)$ is also a normal matrix.*

Proof. This proof follows from the simple calculation.

$$S(A)^t = (I \otimes A - A^t \otimes I)^t = I \otimes A^t - A \otimes I. \tag{6.6}$$

Therefore,

$$\begin{aligned} S(A)S(A)^t =& (I \otimes A - A^t \otimes I)(I \otimes A^t - A \otimes I) \\ =& (I \otimes A)(I \otimes A^t) - (I \otimes A)(A \otimes I) - (A^t \otimes I)(I \otimes A^t) + (A^t \otimes I)(A \otimes I) \\ =& (I \otimes AA^t) - (A \otimes A) - (A^t \otimes A^t) + (A^tA \otimes I) \end{aligned} \tag{6.7}$$

$$
\begin{aligned}
S(A)S(A)^t =& (I \otimes A^t A) - (A \otimes A) - (A^t \otimes A^t) + (AA^t \otimes I) \\
& [\text{ as } A \text{ is a normal matrix, } AA^t = A^t A;] \\
=& (I \otimes A^t)(I \otimes A) - (A \otimes I)(I \otimes A) - (I \otimes A^t)(A^t \otimes I) + (A \otimes I)(A^t \otimes I) \\
=& (I \otimes A^t - A \otimes I)(I \otimes A - A^t \otimes I) = S(A)^t S(A).
\end{aligned} \tag{6.8}
$$

Hence, $S(A)$ is a normal matrix. □

Theorem 6.2. *Let A and B be two commuting normal matrix, then there exists a matrix P such that $A^t = P^{-1}AP$, and $B^t = P^{-1}BP$, hold simultaneously.*

Proof. As the normal matrices A, and B commute there exist a unitary operator U, and diagonal matrices D_a, and D_b such that, $A = \overline{U}^t D_a U$, and $B = \overline{U}^t D_b U$. Transposing them we get $A^t = U^t D_a \overline{U}$, and $B^t = U^t D_b \overline{U}$. As U is a unitary matrix, we have $U\overline{U}^t = \overline{U}^t U = I = \overline{U}U^t = U^t\overline{U} = I$. Now,

$$
\begin{aligned}
S(A) &= I \otimes A - A^t \otimes I = I \otimes \overline{U}^t D_a U - U^t D_a \overline{U} \otimes I \\
&= (U^t I \overline{U}) \otimes (\overline{U}^t D_a U) - (U^t D_a \overline{U}) \otimes (\overline{U}^t I U) \\
&= (U^t \otimes \overline{U}^t)(I \otimes D_a - D_a \otimes I)(\overline{U} \otimes U).
\end{aligned} \tag{6.9}
$$

Let $\mathcal{U} = (\overline{U} \otimes U)$. Then, $\overline{\mathcal{U}}^t = \overline{(\overline{U} \otimes U)}^t = (U^t \otimes \overline{U}^t)$ and $\mathcal{U}\overline{\mathcal{U}}^t = (\overline{U} \otimes U)(U^t \otimes \overline{U}^t) = (\overline{U}U^t) \otimes (U\overline{U}^t) = I$.

Therefore, $S(A) = \overline{\mathcal{U}}^t (I \otimes D_a - D_a \otimes I)\mathcal{U}$. We denote $S(D_a) = I \otimes D_a - D_a \otimes I$. Thus, the equation $S(A)\,\mathrm{vec}(x) = 0$ takes the form, $S(D_a)\mathcal{U}\,\mathrm{vec}(x) = 0$. Denoting $y = \mathcal{U}\,\mathrm{vec}(x)$ we get, $S(D_a)y = 0$. Similarly, from the equation $S(B) = 0$ we get $S(D_b)y = 0$.

In the above equations, y is a vector of order m^2. Let $\hat{y}_i = (y_{(i-1)m+1}, y_{(i-1)m+2}, \ldots, y_{im})^t$. Thus, we may split the vector y into vectors $\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m$, such that, $y = (\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m)^t$. Note that, column vectors of $\mathrm{vec}^{-1}(y)$ are $\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m$.

Let $D_a = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$. Thus, $I \otimes D_a = \mathrm{diag}\{D_a, D_a, \ldots D_a(m\text{-times})\}$. Also, $D_a \otimes I = \mathrm{diag}\{\lambda_1 I, \lambda_2 I, \ldots \lambda_m I\}$. It indicates that, in each of the diagonal matrices, $S(D_a)$, and $S(D_b)$ at least m diagonal entries are zero. They are $1, (m+2), (2m+3), \ldots m^2$-th terms in the diagonal. Thus, $1, (m+2), (2m+3), \ldots m^2$-th term of the vector y may be an arbitrary complex number. It indicates that $\{\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m\}$ forms a set of linearly independent vectors. In particular, we may chose $\hat{y}_i = \iota_i$, the i-th column vector of the identity matrix I. It belongs to null spaces of both $S(D_a)$, and $S(D_b)$. Recall that, $\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m$ are column vectors of $\mathrm{vec}^{-1}(y)$. Thus, $\mathrm{vec}^{-1}(y)$ is a full rank matrix.

Now, $y = \mathcal{U}\,\mathrm{vec}(x)$, that is, $\mathrm{vec}(x) = (\overline{U} \otimes U)y$, which is a unitary evolution on the vectors $\hat{y}_1, \hat{y}_2, \ldots \hat{y}_m$. A unitary evolution does not change the linear independence. Thus, the non-singular matrix $P = \mathrm{vec}^{-1}(\mathrm{vec}(x))$ will satisfy the system of Lyapunov equations (6.4). Therefore, there is a matrix P, such that, $A^t = P^{-1}AP$, and $B^t = P^{-1}BP$ holds simultaneously. □

The above theorem is sufficient for $A^t = P^{-1}AP$, and $B^t = P^{-1}BP$. There may be two matrices A and B such that these hold but do not satisfy the above conditions.

6.3 CONSTRUCTION OF COSPECTRAL GRAPHS

Now we are in position to construct cospectral graphs. The following theorem shows that the GTPT is applicable for generating cospectral mates for a very big class of graphs.

Theorem 6.3. *Any bipartite graph G is co-spectral to its GTPT equivalence G^τ.*

Proof. Let G be a bipartite graph with equal bipartitions on the vertex set. We choose a vertex labelling on G, such that, partitions of the vertex set $V(G)$ are

$$C_1 = \{v_{1,1}, v_{1,2}, \dots v_{1,m}\}, \text{ and } C_2 = \{v_{2,1}, v_{2,2}, \dots v_{2,m}\}. \tag{6.10}$$

We define a mapping $f : G \to G^\tau$ by,

$$f(v_{i,j}) = \begin{cases} v_{2,j} & \text{for } i = 1, \\ v_{1,j} & \text{for } i = 2. \end{cases} \tag{6.11}$$

As G is a bipartite graph, there is no edge $(v_{i,k}, v_{i,l})$ for $i = 1$, and 2. Note that, if $(v_{1,k}, v_{2,l}) \in E(G)$, then $(v_{1,l}, v_{2,k}) \in E(G^\tau)$, that is, $(f(v_{2,l}), f(v_{1,k})) = (f(v_{1,k}), f(v_{2,l})) \in E(G^\tau)$, and conversely. Hence, f acts as an isomorphism between G, and G^τ. Thus, they are co-spectral.

Let G is a bipartite graph with different size of bipartitions, say $C_1 = \{v_{1,1}, v_{1,2}, \dots v_{1,m_1}\}$, and $C_2 = \{v_{2,1}, v_{2,2}, \dots v_{2,m_2}\}$. For simplicity, let $m_1 < m_2$. We may add $(m_2 - m_1)$ isolated vertices at the partition C_1 to construct a new graph G_1. In this case, we have proved that G_1 and G_1^τ are co-spectral. Note that, G_1^τ has $(m_2 - m_1)$ isolated vertices at its C_2, corresponding to newly added $(m_2 - m_1)$ isolated vertices in G. By removing them we get an induced sub-graph of G_1^τ, that is co-spectral to G. □

Bipartite graphs do not have any edge between two vertices belonging to same bipartition. GTPT is also applicable when G has a subgraph with equal bipartition under some additional conditions.

Corollary 6.1. *Let G be a graph with two clusters C_1 and C_2 such that $\langle C_1 \rangle$ and $\langle C_2 \rangle$ are identical, then G is co-spectral to G^τ.*

Proof. Note that, $\langle C_1, C_2 \rangle$ is a bipartite subgraph of G with equal bipartitions. Now we can apply the theorem 6.3 on $\langle C_1, C_2 \rangle$. Consider the following isomorphism,

$$f(v_{\alpha i}) = \begin{cases} v_{2i} & \text{if } \alpha = 1; \\ v_{1i} & \text{if } \alpha = 2. \end{cases} \tag{6.12}$$

It projects the cluster C_1 on C_2, and C_2 on C_1. The induced sugbraphs $\langle C_1 \rangle$, and $\langle C_2 \rangle$ are identical by assumption. Thus, G and G^τ are isomorphic and therefore co-spectral. □

Therefore GTPT preserves spectrum for a very large class of graphs. But, in all the above cases G and G^τ are isomorphic. Non-isomorphic cospectral graphs generated by GTPT are rare. A well-known conjecture in graph theory says that almost all co-spectral graphs are isomorphic [van Dam and Haemers, 2003]. The next theorem may be considered as our first step, where non-isomorphic cospectral graphs are generated.

We have observed that the GTPT modifies the edges in the subgraphs $\langle C_\mu, C_\nu \rangle$. It leads us to an intuition that generation of co-spectral graphs depends on the selection of these subgraphs. The subgraph $\langle C_\mu, C_\nu \rangle$ contributes to the block matrices $A_{\mu,\nu}$ and $A_{\nu,\mu}$ of $A(G)$. In the adjacency

matrix $A(G^\tau)$ these blocks are transposed to $A^t_{\mu,\nu}$ and $A^t_{\nu,\mu}$. At this point we need the simultaneous solution of systems of Lyapunov equations derived in the last section. Also, in the last chapter we have derived graph theoretic conditions for constructing a family of normal commuting matrices. Combining them we get the following.

Theorem 6.4. *Let G be a graph such that all the subgraphs $\langle C_\mu \rangle$, and $\langle C_\mu, C_\nu \rangle$ satisfy conditions for commutativity and normality mentioned in the subsection 5.2.1, then, G and G^τ are cospectral.*

Proof. According to the assumption, the blocks $A_{\mu,\nu}$ of the adjacency matrix generate a family of commuting normal matrices. The theorem 6.2 justifies that there is a matrix P such that $A^t_{\mu\nu} = P^{-1}A_{\mu\nu}P$, for all μ and ν. Hence, combining all these we get,

$$A(G^\tau) = \begin{bmatrix} A^t_{1,1} & A^t_{1,2} & \dots & A^t_{1,n} \\ A^t_{2,1} & A^t_{2,2} & \dots & A^t_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A^t_{n,1} & A^t_{n,2} & \dots & A^t_{n,n} \end{bmatrix} = \begin{bmatrix} P^{-1}A_{1,1}P & P^{-1}A_{1,2}P & \dots & P^{-1}A_{1,n}P \\ P^{-1}A_{2,1}P & P^{-1}A_{2,2}P & \dots & P^{-1}A_{2,n}P \\ \vdots & \vdots & \vdots & \vdots \\ P^{-1}A_{n,1}P & P^{-1}A_{n,2}P & \dots & P^{-1}A_{n,n}P \end{bmatrix}$$

$$\text{or } A(G^\tau) = \begin{bmatrix} P^{-1} & 0 & \dots & 0 \\ 0 & P^{-1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & P^{-1} \end{bmatrix} \begin{bmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,n} \\ A_{2,1} & A_{2,2} & \dots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n,1} & A_{n,2} & \dots & A_{n,n} \end{bmatrix} \begin{bmatrix} P & 0 & \dots & 0 \\ 0 & P & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & P \end{bmatrix}$$
$$= \mathcal{P}^{-1}A(G)\mathcal{P},$$

where, $\mathcal{P} = \text{diag}\{P, P, \dots P\}$. Hence, $A(G)$ and $A(G^\tau)$ are similar. It means G and G^τ are cospectral. □

The above theorem provides a sufficient condition for co-spectrality, which may be further generalized. We have applied the theorem 6.2, which provides a sufficient condition for existing P, such that $A^t = P^{-1}AP$ and $B^t = P^{-1}BP$ hold simultaneously. But the theorem is applicable even when the normality and commutativity conditions are not satisfied. Formally we state it as follows.

Theorem 6.5. *Let G be a graph, such that for any two blocks $A_{\mu\nu}$ and $A_{\alpha\beta}$ of the adjacency matrix there is a matrix P, such that, $A^t_{\mu\nu} = P^{-1}A_{\mu\nu}P$, and $A^t_{\alpha\beta} = P^{-1}A_{\alpha\beta}P$ holds simultaneously. Then G and G^τ are co-spectral.*

Proof. The proof is similar to the last theorem. □

Example 6.1. *Consider the GTPT equivalent graphs G and G^τ depicted in the figure 6.1. The adjacency matrix*

$$A(G) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \tag{6.13}$$



(a) G **(b)** G^τ

Figure 6.1 : GTPT equivalent non-isomorphic cospectral graphs

and

$$A(G^\tau) = \begin{bmatrix} A^t_{11} & A^t_{12} & A^t_{13} \\ A^t_{21} & A^t_{22} & A^t_{23} \\ A^t_{31} & A^t_{32} & A^t_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{6.14}$$

We may calculate that the eigenvalues of $A(G)$, and $A(G^\tau)$ are $\{-1.7321, 0, 1.7321, 1, -1, 0\}$. Therefore G and G^τ are co-spectral. There is a matrix P such that $A_{21} = A^t_{12} = P^{-1}A_{12}P$. Also $A_{23} = A_{32} = I$ and $A_{11} = A_{22} = \cdots = 0$. Thus, $A^t_{\mu,\nu} = P^{-1}A_{\mu,\nu}P$ for all blocks $A_{\mu,\nu}$.

A multipartite graph has more than two clusters. Also, the induced subgraphs $\langle C_\mu \rangle$ are empty graphs. The following corollary shows that GTPT is applicable for generating cospectral mates for multipartite graphs. It may also be considered as another generalization of our statements on bipartite graphs.

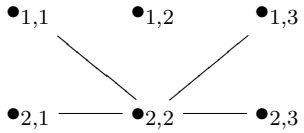
Corollary 6.2. *Let G be a multipartite graph with equal size of clusters and any two non-empty induced sub-graphs, $\langle C_\mu, C_\nu \rangle$, and $\langle C_\alpha, C_\beta \rangle$, are identical. Then, the graphs G and G^τ are co-spectral.*

Proof. According to assumptions, the induced sub-graph $\langle C_\mu \rangle : i = 1, 2, \ldots n$ is an empty graph. Hence, $A_{\mu,\mu} = 0$ for all $i = 1, 2, \ldots m$. Also, for any two non-empty induced sub-graphs $\langle C_\mu, C_\nu \rangle_G = \langle C_\alpha, C_\beta \rangle_G$. Thus, given any two non-zero blocks, there is an invertible matrix P, such that, $A^t_{ij} = P^{-1}A_{ij}P$. Therefore, G and G^τ are co-spectral, by the above theorem. □

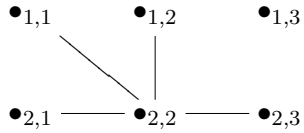
Now we like to state an interesting characteristic of GTPT. Let G and G_1 be two isomorphic graphs with different vertex labellings. Let G^τ, and G^τ_1 be GTPT equivalent to G, and G_1, respectively. Then G^τ, and G^τ_1 may be neither isomorphic nor co-spectral. The following example illuminates the fact.

Example 6.2. *Consider two isomorphic graphs G and G_1 depicted in the figure 6.2. Their GTPT equivalents are the graphs G^τ and G^τ_1. Note that, G^τ is a circle of four vertices and G^τ_1 is a circle of three vertices with a degree one vertex. Therefore, they are neither isomorphic nor cospectral.*

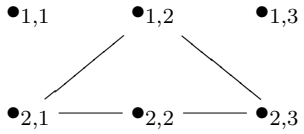
This characteristic allows us to generate an isomorphic copy of G, say G_1, such that their GTPT equivalent graphs G^τ of G^τ_1 are also co-spectral. We have already mentioned that the two



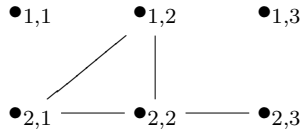
(a) Graph G



(b) Graph G_1 isomorphic to G



(c) G^τ



(d) G_1^τ

Figure 6.2 : GTPT equivalents of two isomorphic graphs may not be isomorphic or cospectral

isomorphic graphs are different in vertex labellings. Till now we have been arranging the vertices into clusters in the horizontal direction. Alternatively, we may take the clusters in the vertical directions. We call it as alternative clustering, which will be applicable for further constructions.

Definition 6.2. ***Alternative clusters (AL):*** *Let $V(G)$ be partitioned into clusters $C_\mu : \mu = 1, 2, \dots n$, where, $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \dots v_{\mu,m}\}$. We define alternative clusters on $V(G)$ as $C'_\mu : \mu = 1, 2, \dots m$, where, $C'_\mu = \{v_{\nu,\mu} : \nu = 1, 2, \dots n\}$.*

Following table will make the definition of alternative clusters clearer:

	C'_1	C'_2	$\dots$	C'_m
C_1	$v_{1,1}$	$v_{1,2}$	$\dots$	$v_{1,m}$
C_2	$v_{2,1}$	$v_{2,2}$	$\dots$	$v_{2,m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
C_n	$v_{n,1}$	$v_{n,2}$	$\dots$	$v_{n,m}$

Theorem 6.6. *Let G_1 be generated from the labelled clustered graph G with the alternative clustering. Also, G^τ and G_1^τ are GTPT equivalent graphs of G and G_1, respectively. Then we have the followings:*

1. *G and G_1 are isomorphic.*
2. *G^τ and G_1^τ are isomorphic.*
3. *If G and G^τ are co-spectral then G_1 and G_1^τ are also co-spectral.*

Proof. We prove the three statements one by one.

***G* and *G_1* are isomorphic:** This isomorphism is determined by the bijective mapping $f : V(G) \to V(G_1)$ with $f(v_{ij}) = v_{ji}$ for $i = 1, 2, \dots n$ and $j = 1, 2, \dots m$. Proof is simple enough as only the vertex labelling is changed.

***G^τ* and *G_1^τ* are isomorphic:** This isomorphism is again determined with the bijective mapping $f : V(G^\tau) \to V(G_1^\tau)$ with $f(v_{ij}) = v_{ji}$ for $i = 1, 2, \dots n$ and $j = 1, 2, \dots m$. Note that, G, G_1, G^τ,



(a) G_1 **(b)** G^τ

Figure 6.3 : Graphs of 6.1 after alternative clustering

and G_1^τ have equal number of vertices and edges. There are three types of edges in $E(G)$. We discuss them in the following three cases.

1. **Case 1:** Let there be an edge $(v_{ij}, v_{ik}) \in E(G)$ for some $j \neq k$. Then $(v_{ji}, v_{ki}) \in E(G_1)$ after the alternative clustering. As GTPT does not change this edge, $(v_{ij}, v_{ik}) \in E(G^\tau)$ and $(v_{ji}, v_{ki}) \in E(G_1^\tau)$. That is $(f(v_{ij}), f(v_{ik})) \in E(G_1^\tau)$. Again for any edge of the form $(v_{ji}, v_{ki}) \in E(G_1^\tau)$, we have $(f^{-1}(v_{ji}), f^{-1}(v_{ki})) = (v_{ij}, v_{ik}) \in E(G_1^\tau)$.

2. **Case 2:** Let there be an edge $(v_{ij}, v_{kj}) \in E(G)$ for some $j \neq k$. Then proof is similar to the case 1.

3. **Case 3:** Let there be an edge $(v_{ij}, v_{kl}) \in E(G)$ for some $i \neq k$ and $j \neq l$. Then after alternative clustering on G, $(v_{ji}, v_{lk}) \in E(G_1)$. After GTPT on G, and G_1 we have $(v_{il}, v_{kj}) \in E(G^\tau)$, and $(v_{jk}, v_{li}) \in E(G_1^\tau)$, respectively. As all the graphs are simple, $(v_{jk}, v_{li}) = (v_{li}, v_{jk}) = (f(v_{il}), f(v_{kj}))$. Again for any edge of the form $(v_{jk}, v_{li}) \in E(G_1^\tau)$ there is $(f^{-1}(v_{jk}), f^{-1}(v_{li})) = (v_{il}, v_{kj}) \in E(G^\tau)$.

Combining all these cases, we may say G^τ and G_1^τ are isomorphic.

If G and G^τ are co-spectral then G_1 and G_1^τ are also co-spectral: It follows trivially from above two results. □

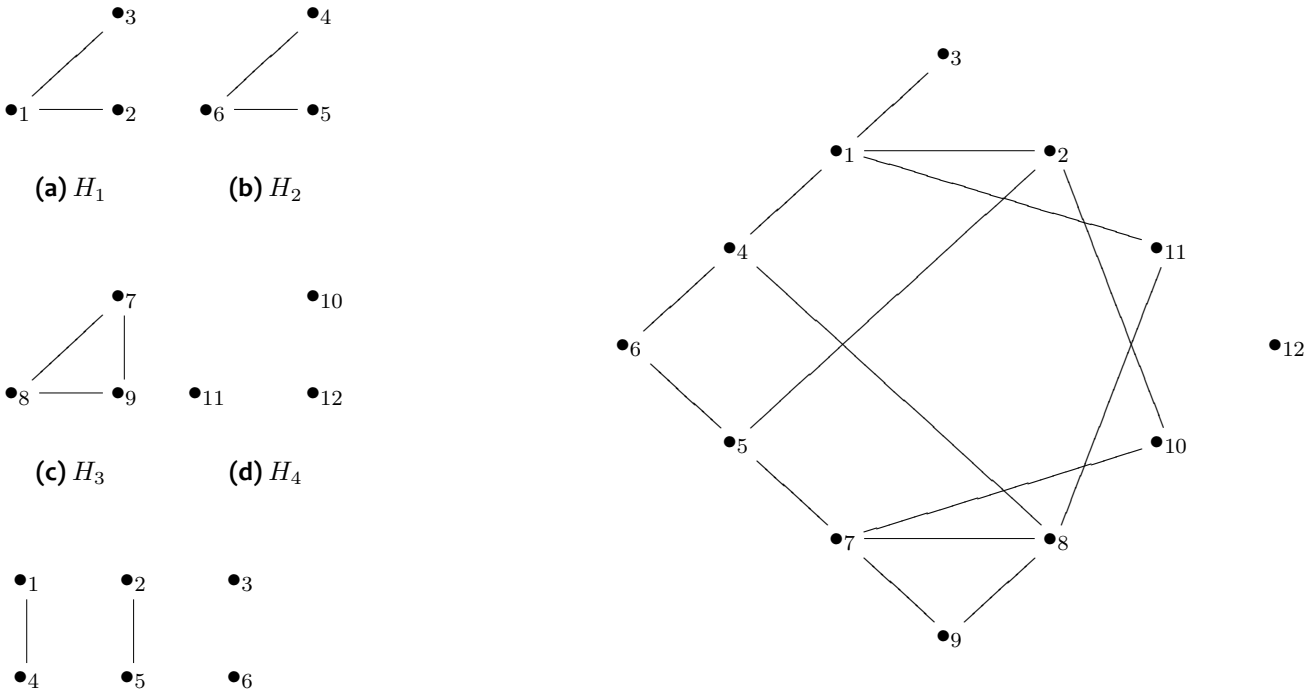
Symbolically, $G \cong G_1$ and $G^\tau \cong G_1^\tau$. But there are graph G such that, $G \ncong G^\tau$ and $G_1 \ncong G_1^\tau$. This relationship may be expressed by the figure below.

$$\begin{array}{ccc} G & \xrightarrow{GTPT} & G^\tau \\ {\scriptstyle AL}\downarrow & & \downarrow{\scriptstyle AL} \\ G_1 & \xrightarrow[GTPT]{} & G_1^\tau \end{array}$$

Example 6.3. *We apply the alternative clustering on the graph depicted in the example 6.1. We depict the new graphs G_1 and G_1^τ in the figure 6.3. As, G_1 and G_1^τ are isomorphic copies of G and G^τ, respectively, it follows that G_1 and G_1^τ are co-spectral.*

Now, we will develop a number of procedures for generating families of big co-spectral graphs from small GTPT equivalent co-spectral graphs based on the theorem 6.5. Before stating the first procedure we recall the definition 4.4 of partially symmetric graphs stated in Chapter 4. Partial symmetric graphs remain invariant under GTPT.

Procedure 6.1. *Consider m graphs of order n, $H_1, H_2, \ldots H_m$. Construct a graph G of order $n \times m$ with m clusters $C_1, C_2, \ldots C_m$, each containing n vertices. We construct $E(G)$, such that,*



(a) H_1 **(b)** H_2 **(c)** H_3 **(d)** H_4

(e) Partial symmetric graph

(f) Graph G

Figure 6.4 : Example of Procedure 6.1

1. *The induced subgraph generated by each cluster* $\langle C_\mu \rangle_G = H_\mu$.
2. *The subgraphs* $\langle C_\mu, C_\nu \rangle_G$ *are partially symmetric graphs.*

Theorem 6.7. *Let G be constructed with the above procedure. Then, G and* G^τ *are cospectral.*

Proof. Note that, $A(G)$ is a block diagonal matrix. The diagonal blocks are $A(H_i)$, which are symmetric matrices. Off diagonal blocks are either zero or symmetric, as we have considered the subgraph $\langle C_\mu, C_\nu \rangle_G$ as a partially symmetric graph. Hence, transpose on individual blocks keeps them unchanged. Thus, $A(G) = A(G^\tau)$. Therefore, G and G^τ are isomorphic and co-spectral. □

Example 6.4. *Consider the graphs depicted in the figure 6.4. We take four graphs* H_1, H_2, H_3, *and* H_4 *each having three vertices. The graph G consists of four clusters* $C_1 = \{1, 2, 3\}, C_2 = \{4, 5, 6\}, C_3 = \{7, 8, 9\}$ *and* $C_4 = \{10, 11, 12\}$. *The induced subgraphs* $\langle C_1 \rangle, \langle C_2 \rangle, \langle C_3 \rangle$, *and* $\langle C_4 \rangle$ *are* H_1, H_2, H_3, *and* H_4, *respectively. We consider copies of a partial symmetric graph for the induced subgraphs* $\langle C_1, C_2 \rangle$, $\langle C_2, C_3 \rangle$, $\langle C_3, C_4 \rangle$, *and* $\langle C_1, C_4 \rangle$. *Now the GTPT equivalent* G^τ *is identical to G. Hence, G and* G^τ *are cospectral.*

For the next procedures our aim is to generate cospectral GTPT equivalent graphs from smaller graphs. It is discussed in literature that the ratio of non-isomorphic cospectral graphs to the corresponding non-isomorphic graphs decrease with order. Hence, finding non-isomorphic cospectral graphs in higher order is important. Here, we call small graphs as seed. From a seed we construct bigger graphs. We define seed graphs as follows. For simplicity we consider the graphs in the figure 6.3 as seed in the corresponding examples.

Definition 6.3. ***Seed graph:*** *The graph G is called seed of a graph H if the co-spectral graphs G and G^τ are induced sub-graph of the co-spectral graphs H and H^τ, respectively.*

Procedure 6.2. *Let $V(G)$ consist of two clusters C_1 and C_2 which are co-spectral to G^τ. Construct graph H with three clusters C_1, C_2 and C_3 such that $\langle C_1 \cup C_2\rangle_H$ is identical to G, $\langle C_1, C_3\rangle_H$ is a null graph, and graph G is isomorphic to the induced sub-graph $\langle C_2 \cup C_3\rangle_H$ by the mapping $f : G \to \langle C_2 \cup C_3\rangle_H$ defined by,*

$$f(v_{\mu,k}) = \begin{cases} v_{2,k} & \text{if } \mu = 2 \\ v_{3,k} & \text{if } \mu = 1 \end{cases}. \tag{6.15}$$

Theorem 6.8. *Let H be produced from G with the above procedure. If G is co-spectral to G^τ then H is co-spectral to H^τ. Also, G^τ is identical to the induced sub-graph, $\langle C_1 \cup C_2\rangle_{H^\tau}$.*

Proof. To construct H we keep all the vertices and edges of clusters C_1 and C_2 unchanged. As G is an induced sub-graph of H, $\langle C_1 \cup C_2\rangle_H$ is identical to G, $A_{\mu,\nu}|_H = A_{\mu,\nu}|_G$, for $\mu, \nu = 1, 2$. Also, the induced sub-graph, $\langle C_1 \cup C_3\rangle_H$ is a null graph. Hence, $A_{1,3} = A_{3,1} =$ the zero matrix. The mapping $f : G \to \langle C_2 \cup C_3\rangle_H$ indicates $A_{2,3} = A_{1,2}^t$. We say there exists a matrix P, such that, $P^{-1}A_{1,2}P = A_{1,2}^t$ and $P^{-1}A_{2,3}P = A_{2,3}^t$ holds simultaneously. Now theorem 6.5 confirms that H and H^τ has same spectra. Also, G^τ is also an induced sub-graph of H^τ. □

Corollary 6.3. *The graph H can be further extended by adding a number of clusters with edges one by one provided for any two clusters C_i and C_j, either $\langle C_i \cup C_j\rangle_H$ is isomorphic to G or an empty graph.*

Proof. This result holds immediately as, in every step of extension any new block matrix A is either a zero matrix or satisfies compatibility condition with the existing blocks. □

Example 6.5. *We consider the GTPT equivalent non-isomorphic cospectral graphs depicted in the figure 6.3 as seed. New graphs H and H^τ are depicted in the figure 6.5. We have applied the procedure two times to get two pairs of non-isomorphic cospectral graphs. Many more non-isomorphic cospectral graphs of bigger size can be generated by repeated application of the procedure.*

In the last procedure, we consider the seed graph with two clusters. We can easily generalize if for a seed graph having more than two clusters. We restate it in the next procedure.

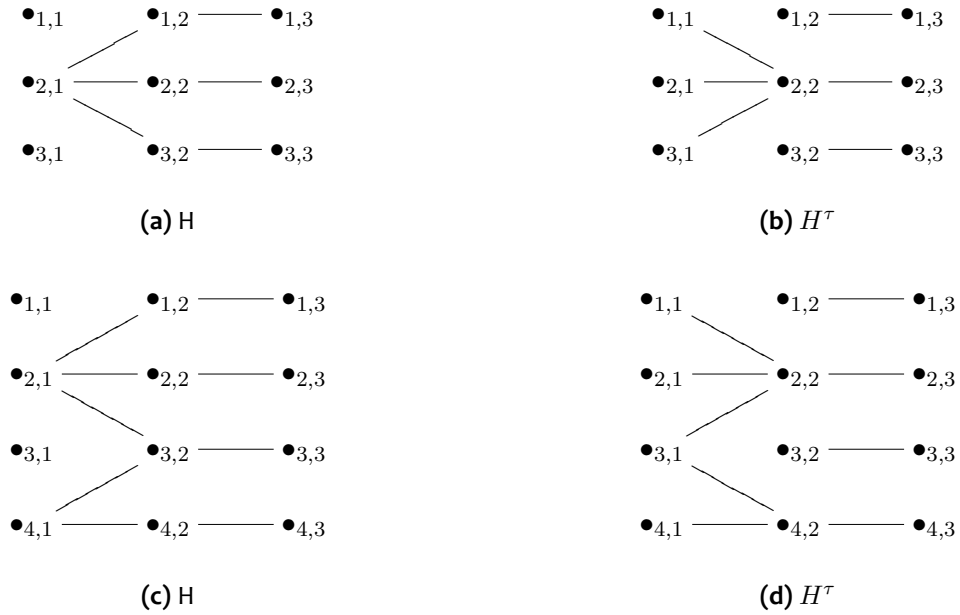
Procedure 6.3. *Let G be a graph cospectral to G^τ such that $V(G)$ consists of clusters $C_\mu : \mu = 1, 2, \ldots m$. Construct a graph H with clusters $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \ldots v_{\mu,n}\}, \mu = 1, 2, 3, \ldots N$ for $N \geq m$ such that:*

1. *Any subgraph $\langle C_\mu\rangle_H, \mu = 1, 2, 3, \ldots N$ is identical to a subgraph $\langle C_\nu\rangle_G, \nu = 1, 2, 3, \ldots m$.*
2. *The induced sub-graph, $\langle C_\mu, C_\nu\rangle_H$, for $\mu, \nu = 1, 2, 3, \ldots N$ is either a null graph; or identical to $\langle C_\alpha, C_\beta\rangle_G$, where $\alpha, \beta = 1, 2, 3, \ldots n$; or $A(\langle C_\mu C_\nu\rangle_H) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, where I is the identity matrix.*

Theorem 6.9. *Let H be a graph generated from G by the above procedure, then H is cospectral to H^τ.*

Proof. Note that, blocks in $A(H)$ are either zeros, or identity matrix, or is included in $A(G)$. We have assumed that G is cospectral to G^τ. Hence there is an invertible matrix P such that $A^t = P^{-1}AP$ holds for any block of $A(H)$.Therefore, H and H^τ are co-spectral by the theorem 6.5. □

Without extending number of clusters in the graph we may increase number of vertices in existing clusters. This is achieved in the next procedure.



(a) H **(b)** H^τ

(c) H **(d)** H^τ

Figure 6.5 : Example of Procedure 6.2

Procedure 6.4. *Let G be cospectral to G^τ with clusters $C_\mu : \mu = 1, 2, \ldots m$, where, $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \ldots, v_{\mu,n}\}$. Construct the graph H as follows:*

1. *$V(H)$ consists of clusters $C_i : i = 1, 2, \ldots n$, such that, $C_i = C_{i,1} \cup C_{i,2}$, where, $C_{i,1} = \{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$, and $C_{i,2} = \{v_{i,n}, v_{i,(n+1)}, \ldots v_{i,(2n-1)}\}$. Note that, $C_{i,1} \cap C_{i,2} = v_{i,n}$.*
2. *The induced sub-graph $\langle \cup_{i=1}^{m} C_{i,1} \rangle_H$ is identical to G.*
3. *The induced sub-graph $\langle \cup_{i=1}^{m} C_{i,2} \rangle_H$ is isomorphic to G by the mapping $f : G \to \langle \cup_{i=1}^{m} C_{i,2} \rangle_H$ defined by, $f(v_{i,k}) = v_{i,(2n-k)}$.*

Theorem 6.10. *Let H be generated from G by the above procedure. Then, H is co-spectral to H^τ. Also, G^τ is identical to the induced sub-graph, $\langle \cup_{i=1}^{n} C_{i,1} \rangle_{H^\tau}$.*

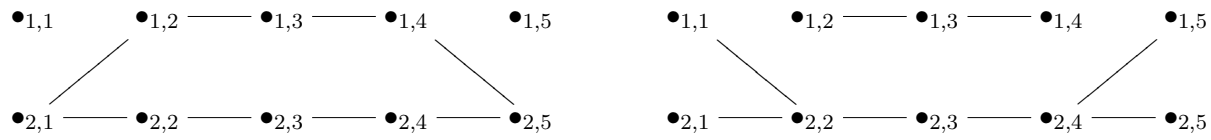
Proof. Let G_1, and H_1 be the alternative clustering of G, and H, respectively. As clusters on H are $D_1, D_2, \ldots D_n$, clusters on $V(H_1)$ would be given by $D'_1, D'_2 \ldots D'_{2m-1}$, where $D'_j = \{v_{1j}, v_{2j}, \ldots v_{nj}\}$. Clearly,

$$V(H) = \bigsqcup_{i=1}^{n} D_i = \bigsqcup_{j=1}^{2m-1} D'_j = V(H_1). \tag{6.16}$$

Also $D_i = C_i \cup \widehat{C}_i$ where $C_i \cap \widehat{C}_i = \{v_{im}\}, i = 1, 2, \ldots n$. Note that, $\bigsqcup_{j=1}^{m} D'_j = \bigsqcup_{i=1}^{n} C_i$ indicates,

$$G_1 = \langle \bigsqcup_{j=1}^{m} D'_j \rangle \approx \langle \bigsqcup_{i=1}^{n} C_i \rangle = G, \tag{6.17}$$

and $D'_m = \{v_{im} : i = 1, 2, \ldots n\}$. Consider the map $f : \bigsqcup_{i=1}^{n} C_i \to \bigsqcup_{i=1}^{n} \widehat{C}_i$ defined by $f(v_{i,l}) = v_{i,2m-l}, l = 1, \ldots, m, i = 1, \ldots, n$. After alternative clustering it acts as an identity isomorphism



(a) H **(b)** H^τ

Figure 6.6 : Example of procedure 6.4

as $f(D_l) = D_{2m-l}$. Also, $f(D_m) = D_m$. Therefore, f induces a graph isomorphism between all the subgraphs generated by the clusters $\{D'_m, D'_{m+1}, \dots D'_{2m-1}\}$ and subgraphs generated by the cluster $\{D'_m, D'_{m-1}, \dots D'_1\}$. As an example f acts as an identity isomorphism as follows,

$$\begin{aligned} &f : \langle D'_m, D'_{m-1}\rangle \to \langle D'_m, D'_{m+1}\rangle, \\ &f : \langle D'_{m-1}, D'_{m-2}\rangle \to \langle D'_{m+1}, D'_{m+1}\rangle, \\ &\vdots \\ &f : \langle D'_1, D'_2\rangle \to \langle D'_{2m-1}, D'_{2m-2}\rangle. \end{aligned} \tag{6.18}$$

Let $A(G_1) = [A_{ij}]_{m\times m}$, and $A(H_1) = [B_{ij}]_{(2m-1)\times(2m-1)}$ where A_{ij} and B_{ij} are block matrices of order $n \times n$. The mapping f provides the following relationship between blocks of $A(G_1)$ and $A(H_1)$.

$$A_{ij} = \begin{cases} B_{ij} & \text{for all } i, j = 1, 2, \dots m, \\ B_{2m-i,2m-j} & \text{for all } i, j = (m+1), (m+2), \dots (2m-1). \end{cases} \tag{6.19}$$

Also $B_{ij} = 0$ for all other combinations of i and j.

We have shown that if G and G^τ are cospectral, then G_1 and G_1^τ are also cospectral (Theorem 6.6). Thus, there is a matrix P such that $A^t_{ij} = P^{-1}A_{ij}P$ for all i, j. It indicates $B^t_{ij} = P^{-1}B_{ij}P$ for all i, j. Thus, H_1 and H_1^τ are cospectral. Again, applying the alternative clustering on H_1 we say H and H^τ are cospectral. □

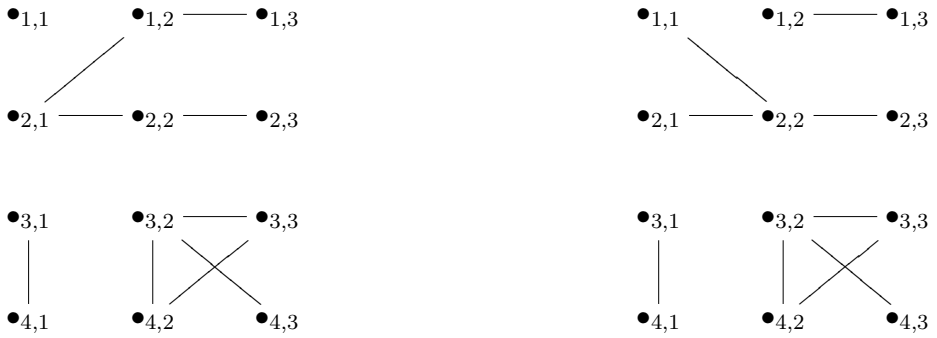
Example 6.6. *We consider the graphs depicted in the figure 6.3 as G and G^τ. In each cluster we add vertices and edges as mentioned in the above procedure. New graphs H and H^τ are depicted in the figure 6.6.*

Procedure 6.5. *Let G be cospectral to G^τ having clusters $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \dots, v_{\mu,n}\}$ with $\mu = 1, 2, \dots m$. Construct the graph H by extending G with new clusters $C_{m+i} : i = 1, 2, \dots$ each consisting of n vertices such that $\langle C_{m+i}, C_{m+j}\rangle$ are partially symmetric graphs. Then, H is co-spectral to H^τ.*

Proof. As there is no edge $(v_{i,l}, v_{j,k}) \in E(H)$ such that $i = 1, 2, \dots m$ and $j = (m+1), (m+2), \dots$, we can divide the graph H into disjoint parts. They are induced sub-graphs $\langle \cup_{i=1}^m C_\mu\rangle_H$ and $\langle \cup_{\mu>n} C_\mu\rangle_H$. This leads to the adjacency matrix becoming a block diagonal matrix,

$$A(H) = A(\langle \cup_{i=1}^n C_i\rangle_H) \oplus A(\langle \cup_{i=(n+1)}^\infty C_i\rangle_H). \tag{6.20}$$

Here $\oplus$ denotes direct sum of matrices. As, $\langle \cup_{i=(n+1)}^\infty C_i\rangle_{H^\tau}$ is a partially symmetric graph GTPT does not change it. Hence the induced sub-graphs $\langle \cup_{i=(n+1)}^\infty C_i\rangle_H$, and $\langle \cup_{i=(n+1)}^\infty C_i\rangle_{H^\tau}$ is identical.



(a) H **(b)** H^τ

Figure 6.7 : Example of procedure 6.5

Also, the induced sub-graphs $\langle \cup_{i=1}^{n} C_i \rangle_H$ and $\langle \cup_{i=1}^{n} C_i \rangle_{H^\tau}$ are co-spectral, according to our assumptions. Eigenvalues of the block diagonal matrices are eigenvalues of the individual blocks [Horn and Johnson, 2012, Page 48, Problem 1.1.P4]. Combining all these, we conclude H and H^τ are co-spectral. □

Example 6.7. *Consider the graphs shown in figure 6.1 as G and G^τ. They have two clusters. According to the above procedure we have added two additional clusters such that the induced subgraphs generated by the new clusters form partial symmetric graphs. Note that new graph H is cospectral to H^τ depicted in figure 6.7.*

We can easily generalize the above procedure. Instead of partially symmetric graphs we may consider a graph G which is cospectral to G^τ. Also instead of adding additional clusters we may add vertices in the existing clusters. In other words, we may increase the number of clusters in the alternative clustering. The last procedure is very effective in generating non-isomorphic cospectral graphs.

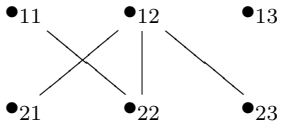
Procedure 6.6. *Let G be any bipartite graph with equal bipartition considered as cluster $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \ldots, v_{\mu,n}\} : \mu = 1, 2$. Construct a graph H by adding an additional cluster C_3 with new edges $\{(v_{2i}, v_{3i}) : i = 1, 2, \ldots n\}$. Then H and H^τ are cospectral.*

Proof. The edge set $\{(v_{2i}, v_{3i}) : i = 1, 2, \ldots n\}$ contributes an identity block in $A(H)$. For any P, $I^t = I = P^{-1}IP$ holds. Now applying theorem 6.5 we can conclude that H and H^τ are cospectral. □

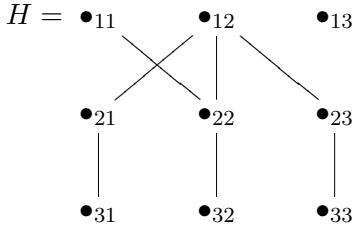
Instead of adding a single cluster in C_3 with G we may add any number of clusters such that the edge set between two clusters contribute either a zero block or an identity block in the adjacency matrix. By repeated application we may find classes of bigger colpectral graphs.

Example 6.8. *Consider the bipartite graph G depicted in the figure 6.8a. After adding a cluster of vertices and edges equivalent to the identity matrix we get a new graph H in the figure 6.8b. Its partial transpose H^τ is given in the figure 6.8c. After calculating eigenvalues we can conclude that H and H^τ are cospectral. Note that H and H^τ are non-isomorphic.*

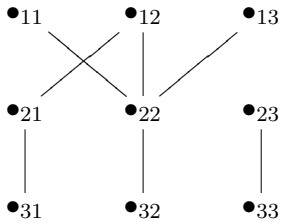
In the next section, we shall find further examples of classes of GTPT equivalent non-isomorphic cospectral graphs. Also we shall compute the number of graphs in these classes.



(a) A bipartite graph G



(b) Graph H after adding a cluster and edges equivalant to identity



(c) H^{τ}

Figure 6.8 : Example of procedure 6.6

6.4 EXAMPLES OF NON-ISOMORPHIC COSPECTRAL GTPT EQUIVALENT GRAPHS

In this section, we construct different classes of graphs and find non-isomorphic GTPT equivalent non-isomorphic cospectral graphs in this class. For simplicity, we begin with a bipartite graph $H = (V(H), E(H))$ with equal bipartitions $C_\mu = \{v_{\mu,1}, v_{\mu,2}, \dots v_{\mu,m}\}, \mu = 1, 2$ for all the examples. We generate a new graph G from H such that G and G^τ are GTPT cospectral. We denote the edge set of a path and a circle in C_i as $\text{Path}_i = \{(v_{i,k}, v_{i,(k+1)}) : k = 1, 2, \dots (m-1)\}$, and $\text{Circle}_i = \{(v_{i,k}, v_{i,(k+1)}) : k = 1, 2, \dots (m-1)\} \cup \{(v_{i,1}, v_{i,m})\}$, respectively.

Example 1. We add new edges and vertices with H such that the new graph $V(G) = V(H) \cup C_3$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,i}) : i = 1, 2, \dots m\}$. Note that, new vertices and edges introduce identity blocks in the new adjacency matrices. Recall that G and G^τ are always cospectral.

Example 2. Here $V(G) = V(H)$ and $E(G) = E(H) \cup \text{Path}_2$. Here G and G^τ are cospectral in some cases depending on choice of the graph H.

Example 3. Here, $V(G) = V(H) \cup C_3$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,j}) : (v_{1,j}, v_{2,i}) \in E(H)\}$. The graphs G and G^τ are cospectral depending on H.

Example 4. Here, $V(G) = V(H) \cup C_3$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,j}) : (v_{1,j}, v_{2,i}) \in E(H)\} \cup \text{Path}_2$. The graphs G and G^τ are cospectral depending on H.

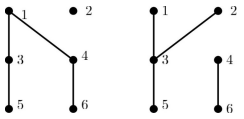
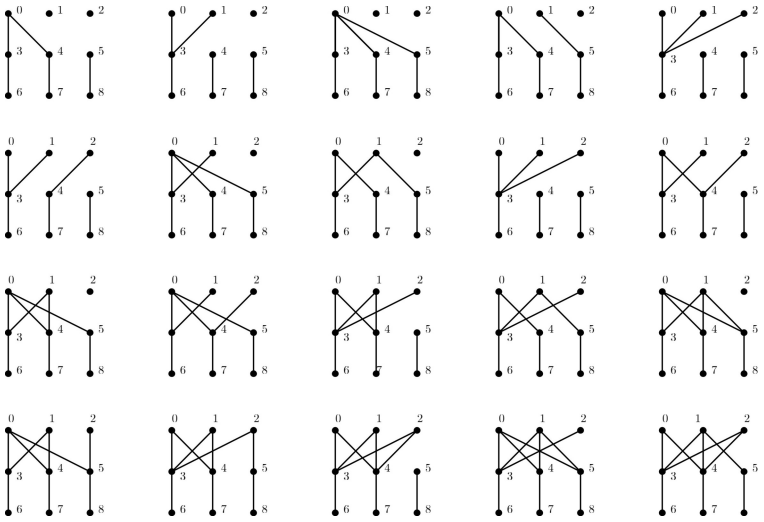
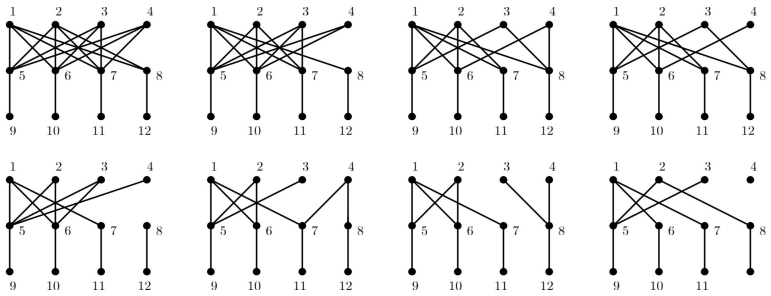
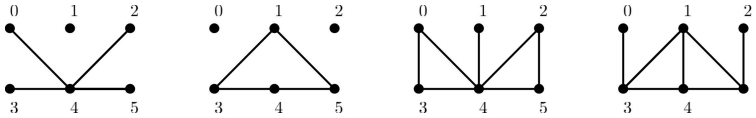
Example 5. Here, $V(G) = V(H)$ and $E(G) = E(H) \cup \text{Circle}_2$. In this case also, G and G^τ are cospectral depending on particular H.

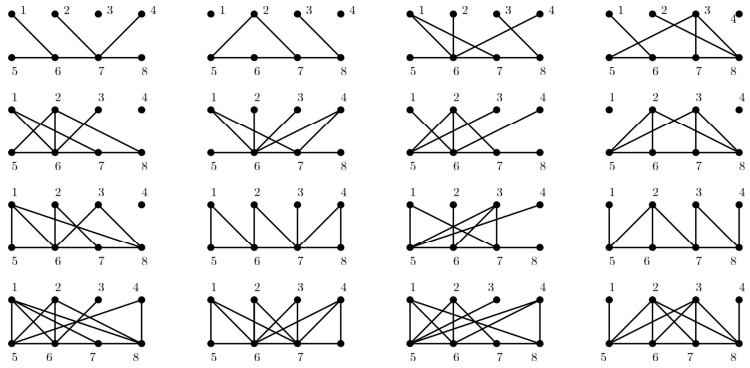
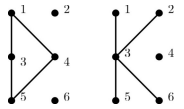
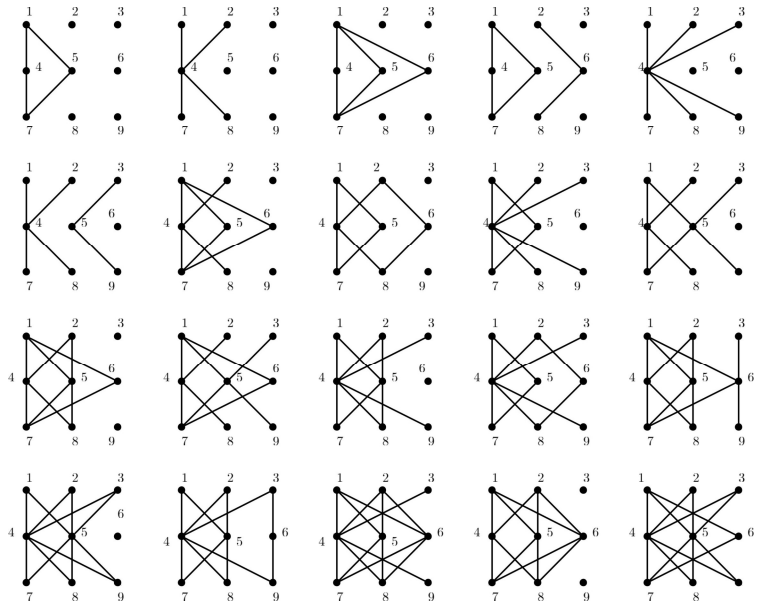
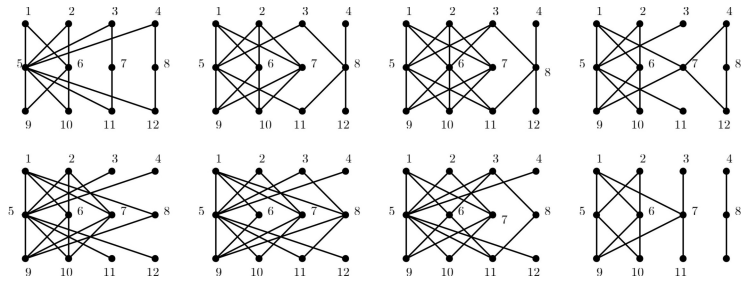
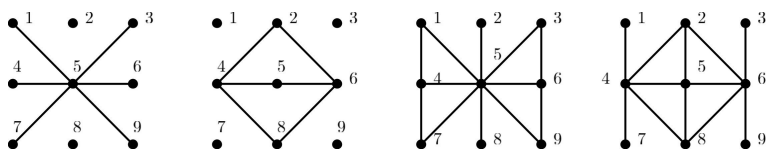
Example 6. Here, $V(G) = V(H) \cup C_3$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,j}) : (v_{1,j}, v_{2,i}) \in E(H)\} \cup \text{Path}_2 \cup \text{Path}_3$. The graphs G and G^τ are cospectral depending on H.

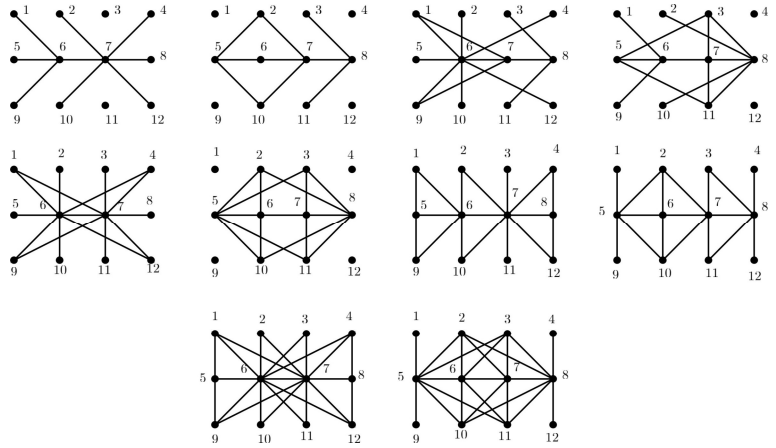
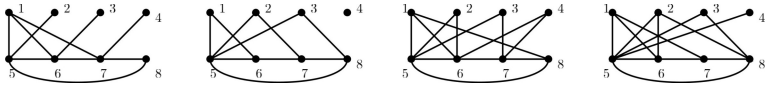
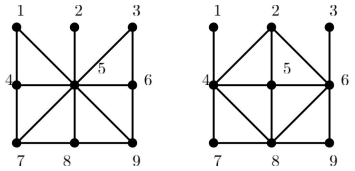
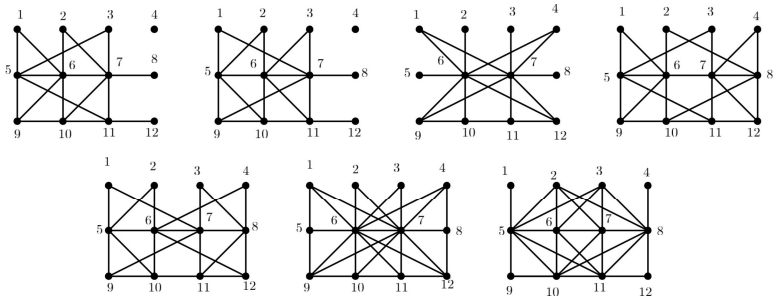
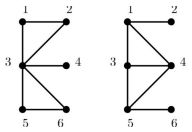
Example 7. Here, $V(G) = V(H) \cup C_3$ and $E(G) = E(H) \cup \text{Path}_1 \cup \text{Path}_2 \cup \text{Path}_3$. The graphs G and G^τ are cospectral depending on H.

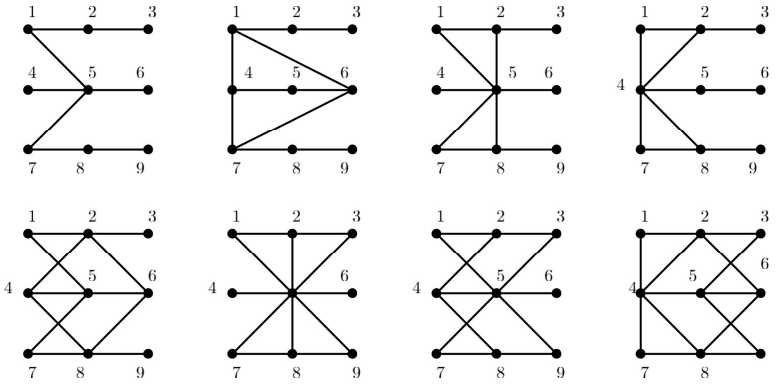
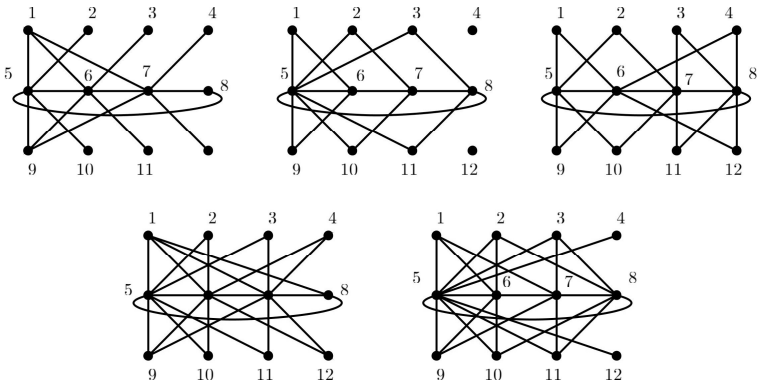
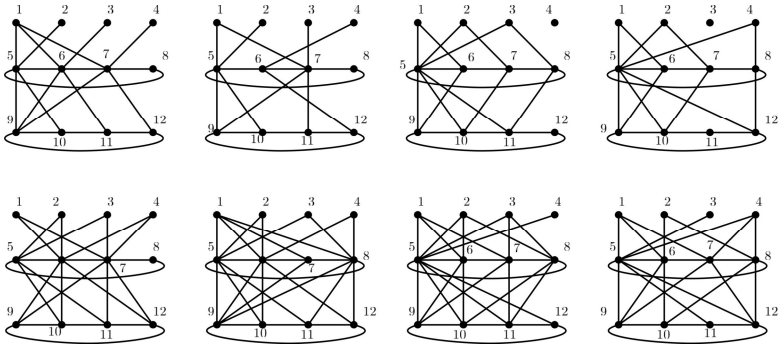
Example 8. Here, $V(G) = V(H)$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,j}) : (v_{1,j}, v_{2,i}) \in E(H)\} \cup \text{Circle}_2$. In this case also, G and G^τ are cospectral depending on particular H.

Example 9. Here, $V(G) = V(H)$ and $E(G) = E(H) \cup \{(v_{2,i}, v_{3,j}) : (v_{1,j}, v_{2,i}) \in E(H)\} \cup \text{Circle}_2 \cup \text{Circle}_3$. In this case also, G and G^τ are cospectral depending on H.

Example number	m	Number of graphs	Examples (we provide some examples when number of graphs are large)
1 (n = 3)	2	2	
	3	20	
	4	250	
2 (n = 2)	2	0	No GTPT equivalent non-isomorphic cospectral graph is available
	3	4	
	4	16	

			
3 (n = 3)	2	2	
	3	20	
	4	250	
4 (n = 3)	2	0	No GTPT equivalent non-isomorphic cospectral graph is available
	3	4	
	4	10	

			
5 (n = 2)	2	0	No GTPT equivalent non-isomorphic cospectral graph is available
	3	0	No GTPT equivalent non-isomorphic cospectral graph is available
	4	4	
6 (n = 3)	2	0	No GTPT equivalent non-isomorphic cospectral graph is available
	3	2	
	4	7	
7 (n = 3)	2	2	
	3	113	

<table>
<tr><td></td><td></td><td></td><td></td></tr>
<tr><td rowspan="3">(n = 3)</td><td>8</td><td>2</td><td>0</td><td>No GTPT equivalent non-isomorphic cospectral graph is available</td></tr>
<tr><td></td><td>3</td><td>0</td><td>No GTPT equivalent non-isomorphic cospectral graph is available</td></tr>
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<tr><td rowspan="3">(n = 3)</td><td>9</td><td>2</td><td>0</td><td>No GTPT equivalent non-isomorphic cospectral graph is available</td></tr>
<tr><td></td><td>3</td><td>0</td><td>No GTPT equivalent non-isomorphic cospectral graph is available</td></tr>
<tr><td></td><td>4</td><td>8</td><td></td></tr>
</table>

We used networkx library [Schult and Swart, 2008] for computer generation of the above graphs. For isomorphism checking, this library use was made of the algorithm stated in [Cordella et al., 2001].

6.5 WHAT NEXT?

In this work we have first shown that partial transpose in applicable in graph theoretical problems. Its use was earlier restricted to entanglement detection. It is a tool of linear algebra but not studied extensively by mathematicians, till now. Therefore, GTPT opens up a new topic of mathematical exploration. In this light, a number of problems can be attempted in future:

1. GTPT makes a well defined change in a graph. It keeps the eigenvalues fixed under some conditions. Which other graph theoretic properties are invariant under GTPT?

2. As a counter question it may be asked which properties of a graph are not invariant under GTPT. A detailed study on GTPT is required for getting a complete picture.

3. There are well known methods for constructing cospectral graphs. Laplacian and signless Laplacian cospectral graphs are not studied well. A method for constructing Laplacian cospectral graphs using GTPT would be welcome.