Chapter 1

Introduction and a Brief Review of Literature

1.1 Introduction

Reliability theory is a field of study that originated from the need to comprehend and enhance the reliability of a complex system. During World War II, the development and the usage of complex electrical and mechanical systems (such as aircrafts, radar systems, communication equipments, etc.) highlighted the importance of reliability theory. The field of machine maintenance is one of the areas where reliability theory was first used with detailed mathematical understanding [cf. Barlow and Proschan, 1965]. Various applications of reliability theory could be found in different branches of science and engineering, namely, manufacturing, finance, mechanical, electrical, industrial, civil, communications, etc. This theory is helpful in describing different aspects of a system, e.g., reliability, mean time to failure, failure rate, optimal maintenance strategy, etc.

Most of the systems used in real-life have two fundamental characteristics. First, each component of a system has some contribution in functioning the system. Second, the system lifetime cannot decrease as and when a failed component is replaced by a good one. These two primary considerations serve as the basis for defining the so-called coherent systems. In our day-to-day lives, we use a number of different systems, including television sets, refrigerators, cars, etc. In most cases, the basic structures of these systems match with coherent systems. Thus, the study of different reliability aspects of coherent systems is considered an important research area in reliability theory. In a coherent system, it is assumed that each component shares the same fraction of contributions in functioning the system. On the contrary, there are many real-life scenarios where the aforementioned assumption regarding the equality of contributions (weights) of all the components may not hold. In practice, we sometimes deal with systems wherein some specific components are more important (critical) than others. The basic structures of these systems are mathematically equivalent with the weighted coherent systems. Thus, the study of weighted coherent systems is another thrust area of research in reliability theory.

In the literature, there are many useful tools available to study different reliability aspects of a system. For example, stochastic orders are frequently used, as effective tools, to compare the lifetimes of different systems. In the last seven decades, stochastic orders have been employed in several fields, including statistics, mathematics, economics, physics, biology, etc. Like stochastic orders, the notions of different stochastic ageings are extensively used in survival analysis and reliability theory. Stochastic ageings broadly describe how a system deteriorates as time progresses. Thus, one of the important problems in reliability theory is the study of different closure properties of various ageing classes. This study helps us in understanding how the reliability of a system changes with the changes of the reliabilities of its components.

Different stochastic dependence notions are frequently used in reliability theory to describe how the components of a system depend on each other. Technical systems are often formed by several components that are operated simultaneously in the same environment and share the same load. In such a system, the failure of a component affects the lifetimes of the remaining components. Consequently, the reliability of such a system depends not only on the lifetimes of its components but also on the underlying dependence structure. Thus, the study of various reliability aspects of systems with dependent components is considered as an important research problem in reliability theory.

Most of the systems used in real-life are not only costly but also complex in nature. The breakdown of such a system leads to a significant financial loss along with other associated damages. Thus, one of the long-lasting questions in reliability theory is how we can enhance the reliability of a system. This problem is often taken care of by allocating one or more redundant component(s) into a system. Due to financial and other constraints, it is not always possible to assign redundant components to each original component of a system. Different reliability importance measures play a significant role in identifying the key components of a system to which the redundant components may be assigned. A reliability importance measure is used to evaluate the impact of a component or a subsystem on the reliability (or performance) of its corresponding system. Moreover, different importance measures are used in various decision-making processes related to maintenance, optimization, and risk management. Thus, the study of different importance measures and their applications is one of the frontier areas of research in reliability theory.

1.2 Review of Literature

This section provides a thorough literature review of the problems addressed in this thesis. We divide this section into thirteen subsections. The Subsection 1.2.1 consists of some important univariate and multivariate reliability measures. In Subsection 1.2.2, we give definitions, motivations and usefulness of different univariate and multivariate stochastic orders. In Subsection 1.2.3, we discuss different majorization orders. Different stochastic ageing classes are discussed in Subsection 1.2.4. In Subsection 1.2.5, we discuss the notion of copula and its connection with the CDF. A brief discussion on coherent systems and their applications is given in Subsection 1.2.6. In Subsections 1.2.7, 1.2.8 and 1.2.9, we discuss order statistics, sequential order statistics and random weighted k-out-of-n systems, respectively. Different kinds of redundancy notions are discussed in Subsection 1.2.10. In Subsection 1.2.11, we discuss various importance measures and their applications in reliability theory. In Subsection 1.2.12, we discuss the notion of system signature. Lastly, some useful definitions are given in Subsection 1.2.13.

1.2.1 Some important reliability measures

In this subsection, we discuss some useful measures, namely, the reliability function, the quantile function, the hazard rate function, the reversed hazard rate function and the mean residual life function. We begin this subsection with the following definition.

Definition 1.2.1 *Let X be a non-negative random variable defined on a probability space. Then, the cumulative distribution function (CDF) of X is given by*

$$F_X(t) = P(X \leq t), \quad t > 0. \qquad \square$$

Like the CDF, the reliability function and the quantile function are two important measures. The reliability function largely describes the probability of surviving a system or an organism longer than x units of time, for a given $x > 0$. In survival analysis, the reliability function is known by the name of survival function. On the other hand, the quantile function is defined as the lowest possible value for which the distribution function is greater than p, for a given $p \in (0, 1]$. We below provide the definitions of these two important measures.

Definition 1.2.2 *Let X be a non-negative random variable defined on a probability space. Then, the reliability or survival function (RF) of X is given by*

$$\bar{F}_X(t) = P(X > t), \quad t > 0.$$

Definition 1.2.3 *Let X be a non-negative random variable defined on a probability space. Then, the quantile function of X is given by*

$$F_X^{-1}(p) = \inf\{t > 0 : F_X(t) \geq p\}, \quad p \in (0, 1]. \qquad \square$$

Multivariate distribution functions extend the notion of univariate distribution functions. A multivariate distribution function depicts the joint behaviour of multiple random variables. It offers a mathematical framework for analyzing and modeling the relationships between two or more random variables, enabling us to comprehend their collective behaviour and interdependencies. We below provide the definitions of multivariate distribution and survival functions.

Definition 1.2.4 *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a vector of n non-negative random variables defined on the same probability space. Then, the CDF of $\boldsymbol{X}$ is given by*

$$F_{\boldsymbol{X}}(\boldsymbol{t}) = P(X_1 \leq t_1, \ldots, X_n \leq t_n), \quad (t_1, \ldots, t_n) \in \mathbb{R}_+^n.$$

Definition 1.2.5 *Let $\boldsymbol{X} = (X_1, \ldots, X_n)$ be a vector of n non-negative random variables defined on the same probability space. Then, the RF of $\boldsymbol{X}$ is given by*

$$\bar{F}_{\boldsymbol{X}}(\boldsymbol{t}) = P(X_1 > t_1, \ldots, X_n > t_n), \quad (t_1, \ldots, t_n) \in \mathbb{R}_+^n. \qquad \square$$

The development of an appropriate multivariate extension of the univariate quantile function has a deep and storied history. Next, we recall the definition of the multivariate quantile transform, which is also referred to as the standard structure of multivariate quantile function [Rosenblatt, 1952; Óbrien, 1975; Arjas and Lehton, 1978].

Definition 1.2.6 *Let F_1 be the marginal cumulative distribution function of X_1, and let $F_{i|1,\ldots,i-1}(\cdot|t_1, \ldots, t_{i-1})$ be the conditional cumulative distribution function of X_i, given $X_1 = t_1, \ldots, X_{i-1} = t_{i-1}$, for $i = 2, \ldots, n$. For each $\boldsymbol{u} = (u_1, \ldots, u_n) \in (0, 1)^n$, the multivariate quantile function is recursively defined as*

$$t_1(\boldsymbol{u}) = F_1^{-1}(u_1),$$

and

$$t_i(\boldsymbol{u}) = F_{i|1,\ldots,i-1}^{-1}(u_i|t_1, \ldots, t_{i-1}), \quad i = 2, \ldots, n,$$

where F_1^{-1} is the quantile function of X_1 and $F_{i|1,\ldots,i-1}^{-1}$ is the quantile function of the univariate conditional random variable $X_i|X_1 = t_1, \ldots, X_{i-1} = t_{i-1}$, $i = 2, \ldots, n$. $\square$

Similar to the univariate case, if the joint cumulative distribution function of $\boldsymbol{X}$ is

$$F_{\boldsymbol{X}}(\boldsymbol{t}) = \int_{\prod_{i=1}^{n}(0,t_i]} f(\boldsymbol{u})\, d\boldsymbol{u},$$

then $\boldsymbol{X}$ is said to be a continuous random vector with the joint probability density function $f(\cdot)$. From the joint cumulative distribution function, one may calculate the marginal cumulative distribution function of X_i as

$$F_{X_i}(t) = P(X_i \leq t) = \lim_{\substack{t_j \to \infty \\ j \neq i}} F_{\boldsymbol{X}}(t_1, \ldots, t_{i-1}, t, t_{i+1}, \ldots, t_n).$$

In reliability theory, there are different measures that are established based on the notions of residual lifetime and inactivity time. The study of system reliability involves in understanding the behaviours of several measures such as hazard rate or failure rate function, mean residual life function, reversed hazard rate function, etc. One may note that the hazard rate function and the mean residual life function are useful when we analyze a system that has reached the age t (residual lifetime) after being put into operation. On the other hand, the reversed hazard rate function is used in the study of the lifetime of a system whose failure is known to occur at or before a certain time t (inactivity time).

- **Hazard (failure) rate function**

The realization of the lifetime of a system can be seen in the form of failure, death, or any other similar event. Thus, in reliability analysis, it is crucial to have information regarding the probability of a failure occurring in the next (usually very small) time frame. The hazard rate (often referred to as the failure rate) function is used to quantify this probability. If the random variable X represents the lifetime of an individual or unit, then the corresponding hazard rate function provides a measure of the 'probability of an instantaneous failure at a given time $t > 0$'. In reliability theory, survival analysis, medical research, industrial life testing, and in other studies, the lifetime distributions are widely established by selecting an appropriate failure rate/hazard rate function. In different domains, the hazard rate is known by a number of names. For example, in extreme value theory, it is referred to as the 'intensity function' [cf. Gumbel, 1958], and in actuarial work, it is referred to as the 'force of mortality'. There are other instances where it is referred to as the 'age-specific force of mortality' and the 'intensity of mortality' [cf. Steffensen, 1930]. Its reciprocal for the normal distribution is referred to as the 'Mills ratio' in statistics [cf. Barlow *et al.*, 1963]. In epidemiology, it is referred to as the 'age-specific failure rate'. In reliability theory, we refer to it as either the 'failure rate' or the 'hazard rate' [cf. Barlow and Proschan, 1975]. The hazard rate function of a non-negative absolutely continuous random variable X is defined as

$$r_X(t) = \frac{f_X(t)}{\bar{F}_X(t)},\ t > 0,$$

which can also be represented as

$$r_X(t) = \lim_{\Delta \to 0+} \frac{1}{\Delta} \frac{\bar{F}_X(t) - \bar{F}_X(t+\Delta)}{\bar{F}_X(t)}.$$

Based on the above representation, we can say that this function represents the intensity of the failure of an unit. Another important characterization of the hazard rate function is that it uniquely determines its corresponding distribution function via the realation

$$\bar{F}_X(t) = \exp\left\{ -\int_0^t r_X(u)du \right\}.$$

Based on different monotonic behaviours of the hazard rate function, various ageing classes (namely, IFR, DFR, IFRA, etc.) were introduced in the literature [Lai and Xie, 2006]. For more information about other aspects of the hazard rate function, one may look into [Barlow and Proschan, 1975; Marshall *et al.*, 2011; Finkelstein, 2008], and the references therein.

- **Reversed hazard rate function**

Similar to the hazard rate function, the reversed hazard rate function is also an important measure introduced by von Mises [1936]. This measure is useful to analyze left censored or right truncated data sets [see Andersen *et al.*, 2018; Sengupta and Nanda, 2010]. In forensic science, it is used to estimate the exact time of failure of a system. It is also used in actuarial science, most notably by insurance companies when determining the premiums for new policyholders. The reversed hazard rate function of a non-negative absolutely continuous random variable X is defined as

$$\tilde{r}_X(t) = \frac{f_X(t)}{F_X(t)}, \quad t > 0.$$

This can equivalently be written as

$$\tilde{r}_X(t) = \lim_{\Delta \to 0+} \frac{1}{\Delta} \frac{F_X(t) - F_X(t-\Delta)}{F_X(t)}.$$

Consequently, $\Delta \tilde{r}_X(t)$ approximately represents the conditional probability of failure of a unit in the interval $(t-\Delta, t]$ given the condition that it has already failed at or before t. Like the hazard rate function, the reversed hazard rate function also uniquely determines the distribution via the relation

$$F_X(t) = \exp\left\{ -\int_t^\infty \tilde{r}_X(u)du \right\}.$$

Different properties of the reversed hazard rate function were discussed in [Block *et al.*, 1998; Chandra and Roy, 2001; Sengupta and Nanda, 1999], and the references therein.

- **Mean residual life function**

The mean residual life function is another important measure like the hazard and the reversed hazard rate functions. Demographers are often interested in life expectancy which can be determined by using the mean residual life function. This measure is used to determine the optimal burn-in time in reliability theory. Apart from these, this measure has huge applications in many other disciplines such as economics, biomedical science, actuarial science, renewal theory, dynamic programming, branching processes, etc. The hazard rate function represents an immediate failure rate at any point of time, whereas the mean residual life function gives a summary of the whole residual life. Thus, the mean residual life function is more intuitively appealing than the hazard rate function when it comes to the modeling and interpretation of failure data. The mean residual life function of a random variable X is defined as

$$m_X(t) = \begin{cases} E[X - t|X > t], & \text{for } t < t_0 \\ 0, & \text{otherwise,} \end{cases}$$

where $t_0 = \sup\{t : \bar{F}_X(t) > 0\}$. If X is a non negative random variable, then $m_X(0) = E(X)$. By the finiteness of $E(X)$ we have that $m_X(t) < \infty$, for all $0 < t < \infty$. However, it is possible that $\lim_{t\to\infty} m_X(t) = \infty$. In this context, it is important to note that

$$m_X(t) = \frac{1}{\bar{F}_X(t)} \int_t^\infty \bar{F}_X(u)du, \quad \text{when } t_0 = \infty.$$

Similar to other measures mentioned before, the mean residual life function also uniquely determines the distribution in a one-to-one fashion via the relation

$$\bar{F}_X(t) = \frac{E(X)}{m_X(t)} \exp\left\{ -\int_0^t \frac{du}{m_X(u)} \right\} \quad \text{over } \{t : P(X > t) > 0\}.$$

Different properties of the mean residual life function were studied by [Shaked and Shanthikumar, 2007; Marshall *et al.*, 2011; Finkelstein, 2008; Barlow and Proschan, 1975; Lai and Xie, 2006], and the references therein.

- **Multivariate dynamic hazard rate function**

In survival analysis and reliability theory, the multivariate hazard rate function represents the conditional failure rate of an event given the survival or failure of other events. It extends the notion of the hazard rate function from the univariate random variable to multiple random variables. The multivariate hazard rate function is particularly useful when studying the interdependencies or interactions between multiple events or variables. Applications of the multivariate hazard rate function can be found in various fields. For example, in reliability theory, it helps us to assess the reliability and the failure behaviour of a complex system with multiple components or failure modes. In the literature, this function was defined in many different ways. Here, we particularly discuss the multivariate dynamic hazard rate function [Shaked and Shanthikumar, 2007].

Let $\boldsymbol{X} = (X_1, \ldots, X_m)$ be a nonnegative random vector with an absolutely continuous distribution function, where X_i represents the lifetime of the i-th component of a system, $i = 1, \ldots, m$. Let us pretend that an observer keeps track of when and what components of the system fail over a long period of time. Then, a typical 'history' after a given time $t > 0$ can be written in the form

$$h_t = \{\boldsymbol{X}_I = \boldsymbol{t}_I, \boldsymbol{X}_{\bar{I}} > t\boldsymbol{e}\}, 0\boldsymbol{e} \leq \boldsymbol{t}_I \leq t\boldsymbol{e}, I \subset \{1, \ldots, m\};$$

here, $\boldsymbol{t}_I = (t_{i_1}, \ldots, t_{i_k})$, $\bar{I}$ is the complement of $I = (i_1, \ldots, i_k)$ in $\{1, \ldots, m\}$ and $\boldsymbol{e} = (1, \ldots, 1)$. Given the history h_t, let $i \in \bar{I}$ be a component that is still alive at time t, the multivariate conditional hazard rate function of X_i, at time t, is defined as follows:

$$\lambda_{i|I}\left(t|\boldsymbol{t}_I\right) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} P\left(t < T_i \leq t + \Delta t | \boldsymbol{T}_I = \boldsymbol{t}_I, \boldsymbol{T}_{\bar{I}} > t\boldsymbol{e}\right),$$

where, $0\boldsymbol{e} \leq \boldsymbol{t}_I \leq t\boldsymbol{e}$, and $I \subset \{1, \ldots, m\}$ [see Shaked and Shanthikumar, 2007].

1.2.2 Stochastic orders

There are many instances where we compare two or more random variables/vectors. These comparisons are largely based on the comparison of certain measures associated with the underlying random variables. A comparison of the magnitudes of two random variables can be done by using their means or medians, which is both the simplest and the most common approach. This kind of comparison, however, relies on just two numbers (the means or medians) and hence, it can often be uninformative. Moreover, the means or medians sometimes do not exist. In many practical situations, one may have an access of more information than simply the two means or medians for the purpose of comparing two random variables. In most situations, one may obtain different kinds of information about the underlying random variables by considering some appropriate measures such as survival functions, quantile functions, hazard rate functions, mean residual functions, etc. The theory of stochastic orderings has expanded greatly during the last seven decades in response to the need for more precise comparisons of two random variables. There are many different kinds of stochastic orders introduced in the literature. Every stochastic order is important in its own way. For an encyclopaedic information on different stochastic orders, one may look into [Lehmann, 1955; Barlow and Proschan, 1975; Ross, 1983; Müller and Stoyan, 2002; Shaked and Shanthikumar, 2007].

Among the list of stochastic orders, the usual stochastic order is the most simplest one introduced by Mann and Whitney [1947]. This order compares the locations of two random variables and determines when one random variable is less likely to take large values compared to another random variable. We below provide the definition of this stochastic order.

Definition 1.2.7 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the usual stochastic order, denoted by $X \leq_{st} Y$ or $F_X \leq_{st} F_Y$, if*

$$\bar{F}_X(t) \leq \bar{F}_Y(t), \ \textit{for all}\ t > 0.$$

Another way to express this is by

$$F_X^{-1}(u) \leq F_Y^{-1}(u), \ \text{for all}\ u \in (0, 1).$$

Let X and Y be two random variables representing the lifetimes of two systems, and suppose that X is less than Y in the usual stochastic order. At a given time $t > 0$, if we observe that both systems are still working, then one may infer that their residual lifetimes are also likewise ordered, or in other words,

$$P(X > t + x|X > t) \leq P(Y > t + x|Y > t), \ \text{for all}\ x \ \text{and}\ t > 0.$$

Unfortunately, as shown by Marshall and Olkin [2007] in Example 1.A.8, this is not true. The hazard rate order is the one that confirms the aforementioned claim. This order compares the the failure rates of two random variables. We below give the definition of this order.

Definition 1.2.8 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the hazard rate order, denoted by $X \leq_{hr} Y$ or $F_X \leq_{hr} F_Y$, if*

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \ \textit{is increasing in}\ t > 0, \tag{1.2.1}$$

or equivalently,

$$r_X(t) \geq r_Y(t), \ \text{for all}\ t > 0. \tag{1.2.2}$$

Further, by writing $t = \bar{F}_X^{-1}(u)$ in (1.2.1), we have that $X \leq_{hr} Y$ if, and only if,

$$\frac{\bar{F}_Y \bar{F}_X^{-1}(u)}{u} \ \text{is decreasing in}\ u \in (0, 1). \tag{1.2.3}$$

Similar to the hazard rate order, the reversed hazard rate order is defined based on the reversed hazard rate functions. Moreover, it confirms that the following inequality holds true.

$$P(t-X>x|X\leq t)\geq P(t-Y>x|Y\leq t),\text{ for all } x \text{ and } t>0.$$

This order was introduced by Keilson and Sumita [1982]. In what follows, we give the definition of this order.

Definition 1.2.9 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the reversed hazard rate order, denoted by $X \leq_{rh} Y$ or $F_X \leq_{rh} F_Y$, if*

$$\frac{F_Y(t)}{F_X(t)} \textit{ is increasing in } t>0, \tag{1.2.4}$$

or equivalently,

$$\tilde{r}_X(t)\leq \tilde{r}_Y(t),\ \text{ for all } t>0. \tag{1.2.5}$$

Further, by writing $t=F_X^{-1}(u)$ in (1.2.4), we have that $X\leq_{rh} Y$ if, and only if,

$$\frac{F_YF_X^{-1}(u)}{u} \text{ is increasing in } u\in(0,1). \tag{1.2.6}$$

The likelihood ratio order is another useful stochastic order introduced by Ross [1983]. This order is often used as a sufficient condition for the aforementioned orders to hold. We below provide the definition of this order.

Definition 1.2.10 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the likelihood ratio order, denoted by $X\leq_{lr} Y$ or $F_X\leq_{lr} F_Y$, if*

$$\frac{f_Y(t)}{f_X(t)} \textit{ is increasing in } t>0. \tag{1.2.7}$$

This is equivalent to the fact that

$$P(X\in B)P(Y\in A)\leq P(X\in A)P(Y\in B)$$

for all measurable sets A and B such that $A\leq B$, where $A\leq B$ means that, for all $x\in A$ and $y\in B$, we have $x\leq y$ [cf. Müller, 1997]. Further, (1.2.7) can be expressed in an alternative form as [Lehmann and Rojo, 1992]

$$\bar{F}_X\bar{F}_Y^{-1}(u) \text{ is convex in } u\in(0,1). \tag{1.2.8}$$

Next, we discuss the mean residual life order. This order compares the remaining useful lifetimes of two systems or two individuals. We below give the definition of this order.

Definition 1.2.11 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the mean residual life order, denoted by $X \leq_{mrl} Y$ or $F_X \leq_{mrl} F_Y$, if*

$$m_X(t) \leq m_Y(t), \text{ for all } t > 0. \tag{1.2.9}$$

Further, this can equivalently be written as

$$\int_t^\infty \bar{F}_Y(u)du \Big/ \int_t^\infty \bar{F}_X(u)du \text{ is increasing in } t \text{ over } \{t : \int_t^\infty \bar{F}_X(u)du > 0\}.$$

In what follows, we discuss some stochastic orders based on relative ageings. The relative ageings largely describe how a system ages with respect to another system. Among the set of stochastic orders, there are four ageing faster orders that explain the relative ageings of two systems. We may refer the reader to [Kalashnikov and Rachev, 1986; Sengupta and Deshpande, 1994; Di Crescenzo, 2000; Finkelstein, 2006; Razaei *et al.*, 2015; Hazra and Nanda, 2016b; Mishra *et al.*, 2017; Misra and Francis, 2018] for more information about these orders. The definitions of the ageing faster orders are given below.

Definition 1.2.12 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be ageing faster than Y in the*

- (a) *hazard rate, denoted by $X \leq_c Y$ or $F_X \leq_c F_Y$, if $\Delta_X \circ \Delta_Y^{-1}(t)$ is convex on $[0, \infty)$, or equivalently, $r_X(t)/r_Y(t)$ is increasing in $t \in [0, \infty)$;*
- (a) *reversed hazard rate, denoted by $X \leq_b Y$ or $F_X \leq_b F_Y$, if $\tilde{r}_X(t)/\tilde{r}_Y(t)$ is decreasing in $t \in [0, \infty)$;*
- (b) *cumulative hazard rate, denoted by $X \leq_* Y$ or $F_X \leq_* F_Y$, if $\Delta_X \circ \Delta_Y^{-1}(t)$ is starshaped on $[0, \infty)$, or equivalently, $\Delta_X(t)/\Delta_Y(t)$ is increasing in $x \in [0, \infty)$;*
- (c) *quantile, denoted by $X \leq_{su} Y$ or $F_X \leq_{su} F_Y$, if $\Delta_X \circ \Delta_Y^{-1}(t)$ is superadditive on $[0, \infty)$.* □

Unlike the aforementioned stochastic orders, the dispersive order compares the variabilities between two random variables. This order is sometimes referred as the *tail order*. The increasing convex order is another such stochastic order that also compares the variabilities.

Definition 1.2.13 *Let X and Y be two absolutely continuous random variables with non-negative supports. Then X is said to be smaller than Y in the*

- (a) *dispersive order, denoted by $X \leq_{disp} Y$ or $F_X \leq_{disp} F_Y$, if $F_Y^{-1}(u) - F_X^{-1}(u)$ is increasing in $u \in (0, 1)$;*
- (b) *increasing convex order, denoted by $X \leq_{icx} Y$ or $F_X \leq_{icx} F_Y$, if $E(\phi(X)) \leq E(\phi(Y))$, for all increasing convex functions ϕ.*

The following diagram shows a chain of consequences resulting from the aforementioned list of stochastic orders [Shaked and Shanthikumar, 2007]. This diagram illustrates the hierarchy of orderings. It shows that the likelihood ratio order is the strongest one; the increasing convex order is the weakest one whereas other orders fall somewhere in between.

$$\begin{tikzcd}
 & X \leq_{hr} Y \arrow[dr] & & \\
X \leq_{lr} Y \arrow[ur] \arrow[dr] & & X \leq_{st} Y \arrow[r] & X \leq_{icx} Y \\
 & X \leq_{hr} Y \arrow[ur] & &
\end{tikzcd}$$

Analogous to the univariate orders, several multivariate stochastic orders, based on different multivariate reliability measures, were defined in the literature [Shaked and Shanthikumar, 2007]. These orders are useful to compare two or more random vectors. We below provide the definitions of some commonly used multivariate stochastic orders.

Definition 1.2.14 *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with non-negative supports. Further, let the multivariate probability density functions and the multivariate conditional hazard rate functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ be given by $f_{\boldsymbol{X}}(\cdot)$ and $f_{\boldsymbol{Y}}(\cdot)$, and $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$, respectively. Then, $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the*

(a) *usual multivariate stochastic order, denoted by $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$, if $E(\phi(\boldsymbol{X})) \leq E(\phi(\boldsymbol{Y}))$, for all increasing functions ϕ;*

(b) *dynamic multivariate hazard rate order, denoted by $\boldsymbol{X} \leq_{dyn-hr} \boldsymbol{Y}$, if*

$$\eta_{k|I\cup J}(u|\boldsymbol{s}_{I\cup J}) \quad \geq \quad \lambda_{k|I}(u|\boldsymbol{t}_I), \text{ for all } k \in \overline{I \cup J},$$

where $I \cap J = \emptyset$, $\boldsymbol{s}_I \leq \boldsymbol{t}_I \leq u\boldsymbol{e}$ and $\boldsymbol{s}_J \leq u\boldsymbol{e}$;

(c) *multivariate likelihood ratio order, denoted by $\boldsymbol{X} \leq_{lr} \boldsymbol{Y}$, if $f_{\boldsymbol{X}}(\boldsymbol{x}) f_{\boldsymbol{Y}}(\boldsymbol{y}) \leq f_{\boldsymbol{X}}(\boldsymbol{x} \wedge \boldsymbol{y}) f_{\boldsymbol{Y}}(\boldsymbol{x} \vee \boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;*

(d) *multivariate dispersive order, denoted by $\boldsymbol{X} \leq_{disp} \boldsymbol{Y}$, if $y_i(\boldsymbol{u}) - x_i(\boldsymbol{u})$ is increasing in $(u_1, \ldots, u_i) \in (0,1)^i$, for $i = 1, \ldots, n$, where $x_i(\boldsymbol{u})$ and $y_i(\boldsymbol{u})$ are multivariate quantile functions of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively.*

1.2.3 Majorization orders

Like stochastic orders, majorization orders are also quite useful for establishing various inequalities in different areas of mathematics, statistics, economics, etc. The notion of majorization orders was inherited from one of historic questions asked by Hardy *et al.*

[1929] as follows: What conditions are to be imposed on $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ so that

$$\sum_{i=1}^{n} g(x_i) \leq \sum_{i=1}^{n} g(y_i)$$

holds for all convex functions $g : \mathbb{R} \to \mathbb{R}$. The answer of this question is that $\boldsymbol{x}$ is to be majorized by $\boldsymbol{y}$ which was also provided by them. In short, the vector $\boldsymbol{x}$ majorizes $\boldsymbol{y}$ if the coordinates of $\boldsymbol{x}$ are more dispersed than the coordinates of $\boldsymbol{y}$, subject to the constraint that the sum of the coordinates of $\boldsymbol{x}$ and that of $\boldsymbol{y}$ are the same. A second origin of majorization orders is illustrated by the work of Schur [1923] on Hadamard's determinant inequality. To know more on this we refer the reader to Marshall *et al.* [2011].

Let $x_{(1)} \leq \cdots \leq x_{(n)}$ be the increasing arrangements and let $x_{[1]} \geq \cdots \geq x_{[n]}$ be the decreasing arrangements of the components of the vector $\boldsymbol{x}$=$(x_1, \ldots, x_n)$. The following definitions may be obtained in Marshall *et al.* [2011].

Definition 1.2.15 *Let I^n denote an n-dimensional Euclidean space where $I \subseteq \mathbb{R}^n$. Further, let $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in I^n$ be any two vectors.*

(i) The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]},\ j = 1, \ldots, n-1, \ \ \textit{and} \ \ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

or equivalently,

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)},\ j = 1, \ldots, n-1, \ \ \textit{and} \ \ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

(ii) The vector $\boldsymbol{x}$ is said to weakly supermajorize the vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\mathrm{w}}{\succeq} \boldsymbol{y}$) if

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

(iii) The vector $\boldsymbol{x}$ is said to weakly submajorize the vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \succeq_{\mathrm{w}} \boldsymbol{y}$) if

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

(iv) The vector $\boldsymbol{x}$ is said to be p-larger than the vector $\boldsymbol{y}$ (written as $\boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)} \quad \textit{for } j = 1, \ldots, n.$$

(v) *The vector* $\boldsymbol{x}$ *is said to reciprocally majorize the vector* $\boldsymbol{y}$ *(written as* $\boldsymbol{x} \overset{\text{rm}}{\succcurlyeq} \boldsymbol{y}$*) if*

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}} \quad \text{for } j = 1, \ldots, n. \qquad \square$$

It is worthwhile to mention here that

$$\boldsymbol{x} \overset{\text{m}}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\text{w}}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\text{p}}{\succeq} \boldsymbol{y} \Rightarrow \boldsymbol{x} \overset{\text{rm}}{\succeq} \boldsymbol{y}.$$

We below give the definition of the Schur-convex function.

Definition 1.2.16 *Let* I^n *denote an* n*-dimensional Euclidean space where* $I \subseteq \mathbb{R}^n$*. A function* $\psi : I^n \to \mathbb{R}$ *is said to be Schur-convex (resp. Schur-concave) on* I^n *if*

$$\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \text{ implies } \psi(\boldsymbol{x}) \geq (\text{resp. } \leq)\, \psi(\boldsymbol{y}) \text{ for all } \boldsymbol{x},\ \boldsymbol{y} \in I^n.$$

1.2.4 Stochastic ageings

Stochastic ageings largely describe how a system behaves as time progresses. There are three types of ageing notions, namely, no ageing, positive ageing and negative ageing. No ageing means that the system does not age over time. A system has the positive ageing property if its residual lifetime decreases in some stochastic sense as time progresses. On the other hand, negative ageing describes the scenario where the residual lifetime of a system increases in some stochastic sense as time progresses. A variety of positive and negative ageing classes (namely, IFR, IFRA, MIFRA, ILR, DFR, DRFR, DLR, NBU, NWU, etc.) were introduced in the literature to describe different ageing characteristics of a system [see Barlow and Proschan, 1975; Lai and Xie, 2006, and the references therein]. We below give the definitions of some ageing classes that are used in this thesis.

Definition 1.2.17 *Let* X *be an absolutely continuous random variable with non-negative support. Then* X *is said to have the*

(a) *increasing likelihood ratio (ILR) (resp. decreasing likelihood ratio (DLR)) property if* $f'_X(t)/f_X(t)$ *is decreasing (resp. increasing) in* $t \geq 0$;

(b) *increasing failure rate (IFR) (resp. decreasing failure rate (DFR)) property if* $r_X(t)$ *is increasing (resp. decreasing) in* $t \geq 0$;

(c) *decreasing reversed failure rate (DRFR) property if* $\tilde{r}_X(t)$ *is decreasing in* $t \geq 0$;

(d) *increasing failure rate in average (IFRA) (resp. decreasing failure rate in average (DFRA)) property if* $-\ln \bar{F}_X(t)/t$ *is increasing (resp. decreasing) in* $t \geq 0$;

(e) *multivariate increasing failure rate in average (MIFRA) property if* $E\left(\xi\left(X_1, \ldots, X_n\right)\right) \leq E^{1/\alpha}\left(\xi^\alpha\left(X_1/\alpha, \ldots, X_n/\alpha\right)\right)$*, for all continuous nonnegative increasing functions* ξ *and for all* $\alpha \in (0, 1)$;

(*f*) *new better than used (NBU) (resp. new worse than used (NWU)) property if* Δ_X *is superadditive (resp. subadditive) in* $t \geq 0$*, or equivalently,* $\bar{F}_X(x+t) \leq$ *(resp.* $\geq$*)* $\bar{F}_X(x)\bar{F}_X(t)$ *for all* $x, t \geq 0$. □

The interrelations among different ageing classes, for a nonnegative random variable, are given in the following diagram [cf. Franco *et al.*, 2003; Lai and Xie, 2006].

$$\text{ILR} \rightarrow \text{IFR} \rightarrow \text{IFRA} \rightarrow \text{NBU} \quad \text{and} \quad \text{DLR} \rightarrow \text{DFR} \rightarrow \text{DFRA} \rightarrow \text{NWU}.$$

1.2.5 Copula

From the joint cumulative distribution function, one can uniquely derive marginal cumulative distribution functions. However, in general, the joint cumulative distribution function cannot be derived from a given set of marginal cumulative distribution functions. In the specific scenario where the underlying random variables are independent, the joint distribution can be expressed through their marginals and consequently, the joint cumulative distribution function can be written as

$$F_{\boldsymbol{X}}(\boldsymbol{t}) = \prod_{i=1}^{n} F_{X_i}(t_i), \ (t_1, \ldots, t_n) \in \mathbb{R}_+^n.$$

Copula is a very useful notion in describing the dependence structure between components of a random vector. It builds a bridge between a multivariate cumulative distribution function and its corresponding one dimensional marginal cumulative distribution functions. In reliability theory, copulas are used to represent different dependence structures between components of a system. In actuarial science, copulas play an important role in the modeling of dependent losses and mortality. In finance, copulas are used in the areas of derivative pricing, asset allocation, risk management, credit scoring, etc. For modeling competing risks and correlated event timings, copulas are often used in biomedical research. In the field of engineering, several applications of copulas include hydrological modeling and multivariate process control. The joint CDF of a random vector $\boldsymbol{X}$=$(X_1, \ \ldots, X_n)$ can be written in terms of a copula as

$$\begin{aligned} F_{\boldsymbol{X}}(t_1, \ldots, t_n) &= P(X_1 \leq t_1, \ldots, X_n \leq t_n) \\ &= \mathcal{C}(F_{X_1}(t_1), \ldots, F_{X_n}(t_n)), \end{aligned}$$

where $\mathcal{C}(\cdot)$ is a copula. Similarly, the joint reliability function of $\boldsymbol{X}$ can be represented as

$$\begin{aligned} \bar{F}_{\boldsymbol{X}}(t_1, \ldots, t_n) &= P(X_1 > t_1, \ldots, X_n > t_n) \\ &= \bar{\mathcal{C}}(\bar{F}_{X_1}(t_1), \ldots, \bar{F}_{X_n}(t_n)), \end{aligned}$$

where $\bar{\mathcal{C}}(\cdot)$ is a survival copula. In the literature, different types of survival copulas were introduced to describe different dependence structures between components of a random vector. The commonly used copulas are the Farlie-Gumbel-Morgenstern (FGM) copula, the extreme-value copulas, the family of Archimedean copulas, the Clayton-Oakes (CO) copula, etc. Among all these copulas, the family of Archimedean copulas is the one that has paid more attention from the researchers due to its wide range of capturing the dependence structures. Moreover, these are mathematically tractable, and there is a large number of results available in the literature which can be used on a ready-made basis in different problems. For more information on this topic, we refer the reader to Nelsen [2006]. We below give the definition of the Archimedean copula [McNeil and Něslehová, 2009].

Definition 1.2.18 *Let* $\phi : [0, +\infty] \longrightarrow [0, 1]$ *be a decreasing continuous function such that* $\phi(0) = 1$ *and* $\phi(+\infty) = 0$, *and let* $\psi \equiv \phi^{-1}$ *be the pseudo-inverse of* ϕ. *Then*

$$\mathcal{C}(u_1, \ldots, u_n) = \phi\left(\psi(u_1) + \cdots + \psi(u_n)\right), \quad \textit{for } (u_1, \ldots, u_n) \in [0, 1]^n,$$

is called the Archimedean copula with generator ϕ *if* $(-1)^k \phi^{(k)}(x) \geq 0$, *for* $k = 0, 1, \ldots, n-2$, *and* $(-1)^{n-2}\phi^{(n-2)}(x)$ *is decreasing and convex in* $x \geq 0$. □

In what follows, we introduce some key notations. For an Archimedean copula with the generator ϕ, we write

$$H(u) = \frac{u\phi'(u)}{1-\phi(u)}, \quad R(u) = \frac{u\phi'(u)}{\phi(u)} \text{ and } G(u) = \frac{u\phi''(u)}{\phi'(u)}, \quad u > 0.$$

Note that $H(\cdot)$, $R(\cdot)$ and $G(\cdot)$ are all negative valued functions because $\phi(\cdot)$ is a decreasing and convex function.

1.2.6 Coherent system

In our day-to-day lives, we use many different kinds of systems, e.g., radio, car, TV, airplane, etc. A system is usually formed by various components that are interconnected in some way to form the whole. The survivability or failure of a system entirely depends on the performance of its components. The operational status of a system is usually classified as either completely functional or partly functional. However, we only take into account the systems that are either fully operational or entirely failed at any given point in time. Such systems are called binary-state systems which were first introduced and investigated by Birnbaum et al. [1961]. In what follows, we define the notion of structure function as a one-to-one connection between different states of a system and those of its components.

Consider a system with lifetime T. Let $\{X_1, \ldots, X_n\}$ be the set of random variables representing the lifetimes of its n components. Let us further define the state vector of $\boldsymbol{X}$ as $\boldsymbol{x} = (x_1, \ldots, x_n) \in \{0, 1\}^n$, where $x_i = 1$ if the ith component is working, and $x_i = 0$ if it has failed. Let us pretend for a moment that the components are in specific states. Then, the next obvious question is how we can determine the state of the system. A bridge between the states of a system and those of its components is determined by the mapping termed *structure function*, denoted by τ, and is defined as

$$\tau(\boldsymbol{x}) = \begin{cases} 1, & \text{if the system is functioning} \\ 0, & \text{if the system has failed.} \end{cases}$$

Then, the reliability function of T, denoted by $h_T(\cdot)$, is defined as the probability that it is in working state at any given point of time t. Consequently,

$$\bar{F}_T(t) = h_T(\boldsymbol{p}) = P(\tau(\boldsymbol{X}) = 1),$$

where $p_i = \bar{F}_{X_i}(t)$, $i = 1, \ldots, n$.

Most of the systems used in real-life satisy two basic requirements. First, each component of a system should have some role in functioning the system. Second, if a failed component of a system is replaced by a good one, then the lifetime of the system cannot decreased. Based on these two considerations, the notion of *coherent system* was introduced. Before giving the definition of a coherent system, we discuss several notions.

Definition 1.2.19 *The i-th component of a system having structure function $\tau(\cdot)$ is said to be irrelevant if $\tau(\cdot)$ is constant in x_i, i.e.,*

$$\tau(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = \tau(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n),$$

for all $x_j \in \{0,1\}$, $j = 1, \ldots, i-1, i+1, \ldots, n$. On the other hand, if this is not the case, then the component is said to be relevant.

Definition 1.2.20 *A structure function $\tau(\cdot)$ is said to be monotonically increasing if $\tau(\boldsymbol{x}) \leq \tau(\boldsymbol{y})$ whenever $\boldsymbol{x} \leq \boldsymbol{y}$; here the latter vector inequality is understood to be applied component-wise.* □

In what follows, we define coherent systems.

Definition 1.2.21 *A system is said to be coherent if each of its components is relevant and its structure function is monotonically increasing.* □

There are several approaches to represent the structure of a coherent system. One of the simplest and commonly used approaches is the representation given in terms of minimal path or cut sets [see Barlow and Proschan, 1975]. We first give the definitions of minimal cut and path sets, and then give the representation.

Definition 1.2.22 *A non-empty set $P \subseteq \{1, \ldots, n\}$ is said to be a path set of a coherent system with the structure function τ if $\tau(\boldsymbol{x}) = 1$ whenever $x_i = 1$ for all $i \in P$. Moreover, a path set P is said to be the minimal path set if it does not contain any other path sets.*

Definition 1.2.23 *A non-empty set $S \subseteq \{1, \ldots, n\}$ is said to be a cut set of a coherent system with the structure function τ if $\tau(\boldsymbol{x}) = 0$ whenever $x_i = 0$ for all $i \in S$. Moreover, a cut set S is said to be the minimal cut set if it does not contain any other cut sets.* □

Definition 1.2.24 *Consider a coherent system with the structure function* τ *and the lifetime* T*. Let* $\{X_1, \ldots, X_n\}$ *be the set of random variables representing the lifetimes of its* n *components, and let* $\mathcal{A} = \{P_1, \ldots, P_m\}$ *and* $\mathcal{S} = \{S_1, \ldots, S_p\}$ *be the sets of all minimal path sets and minimal cut sets of the coherent system, respectively. Then, the structure function of this coherent system can be represented as*

$$\tau(\boldsymbol{x}) = \max_{1 \le j \le m} \left\{ \min_{i \in P_j} x_i \right\} = \min_{1 \le j \le p} \left\{ \max_{i \in S_j} x_i \right\},$$

where $\boldsymbol{x} \in \{0,1\}^n$*. Moreover, the lifetime of the system can be written as*

$$T = \max_{1 \le j \le m} \left\{ \min_{i \in P_j} X_i \right\} = \min_{1 \le j \le p} \left\{ \max_{i \in S_j} X_i \right\}. \qquad \square$$

The well-known r-out-of-n systems are the special type of coherent systems. The r-out-of-n systems are frequently encountered in practice. An airplane that functions as long as two of its four engines function is an example of a 2-out-of-4 system.

Definition 1.2.25 *A system is said to be the* r*-out-of-*n *system if it functions as long as at least* r *of its* n *components function.* $\square$

Let $\tau_{r|n}(\boldsymbol{X})$ be the lifetime of an r-out-of-n system formed n independent components with the lifetime vector $\boldsymbol{X} = (X_1, \ldots, X_n)$. Then, its structure function, denoted by $\tau_{r|n}$, is given by

$$\tau_{r|n}(\boldsymbol{x}) = \begin{cases} 1, & \sum_{i=1}^{n} x_i \ge r \\ 0, & \sum_{i=1}^{n} x_i < r \end{cases}$$

and its reliability function is given by

$$\begin{aligned} P(\tau_{r|n}(\boldsymbol{X}) > t) &= h_{r|n}(\bar{F}_{X_1}(t), \ldots, \bar{F}_{X_n}(t)) \\ &= \sum_{j=0}^{n-r} \sum_{\{J : |J| = j\}} \left(\prod_{i \in J} (1 - \bar{F}_{X_i}(t)) \right) \left(\prod_{i' \notin J} \bar{F}_{X_{i'}}(t) \right), \end{aligned}$$

where $|J|$ is the cardinality of the set J. If $X_1, \ldots, X_n$ are iid random variables, then $h_{r|n}(\cdot)$ can be written as

$$h_{r|n}(p) = \sum_{i=r}^{n} \binom{n}{i} p^i (1-p)^{n-i}, \quad \text{for } p \in (0,1).$$

Two extreme cases of r-out-of-n systems are the series system (n-out-of-n) and the parallel system (1-out-of-n). In the literature, these systems have been extensively studied by numerous researchers [cf. Barlow and Proschan, 1975; Samaniego, 2007].

Let $\tau_{n|n}(\mathbf{X})$ and $\tau_{1|n}(\mathbf{X})$ be the lifetimes of a series and a parallel systems, respectively. Let these systems be formed by n dependent components with the lifetime vector $\mathbf{X} = (X_1, \ldots, X_n)$, where the dependence structure is described by an Archimedean copula with the generator ϕ. Then, their structure functions are given by

$$\phi_{\tau_{n|n}(\mathbf{x})} = \min\{x_1, \ldots, x_n\}$$

and

$$\phi_{\tau_{1|n}(\mathbf{x})} = \max\{x_1, \ldots, x_n\},$$

respectively. Moreover, their reliability functions are given by

$$P(\tau_{n|n}(\mathbf{X}) > t) = h_{n|n}(\bar{F}_{X_1}(t), \ldots, \bar{F}_{X_n}(t)) = \phi\left(\psi(\bar{F}_{X_1}(t)) + \cdots + \psi(\bar{F}_{X_n}(t))\right).$$

and

$$P(\tau_{1|n}(\mathbf{X}) > t) = h_{1|n}(\bar{F}_{X_1}(t), \ldots, \bar{F}_{X_n}(t)) = 1 - \phi\left(\psi(1 - \bar{F}_{X_1}(t)) + \cdots + \psi(1 - \bar{F}_{X_n}(t))\right),$$

respectively.

1.2.7 Usual order statistics

Consider a collection of n random variables $\{X_1, \ldots, X_n\}$. If we sort them in the ascending order of magnitude, we obtain a unique order relation, namely, $X_{1:n} \leq \cdots \leq X_{n:n}$, where $X_{1:n}$ is the smallest of $X_1, \ldots, X_n$; $X_{2:n}$ is the second smallest of $X_1, \ldots, X_n$ and so on. We call $X_{r:n}$ as the r-th smallest usual order statistic based on $X_1, \ldots, X_n$. Order statistics are widely applied in statistical inference, life testing, extreme value theory, image processing, and other areas [David and Nagaraja, 2003; Castilo, 1988; Pitas and Venetsanopoulos, 1992]. In reliability theory, order statistics are used to model the lifetime of a system. The lifetime of an ordinary r-out-of-n system is the same as the $(n-r+1)$-th order usual order statistic of the lifetimes of components of the system. In light of this, it is sufficient to study the $(n-r+1)$-th order usual order statistic for understanding the behaviour of an ordinary r-out-of-n system and vice versa. To know more on this topic, we refer the reader to Arnold *et al.* [1992].

Let $X_1, X_2, \ldots, X_n$ be n independent random variables. Then, the CDF of $X_{r:n}$ is given by

$$F_{X_{r:n}}(t) = 1 - \sum_{j=0}^{r-1} \sum_{\{J:|J|=j\}} \left(\prod_{i \in J} F_{X_i}(t)\right)\left(\prod_{i' \notin J} \bar{F}_{X_{i'}}(t)\right).$$

Further, if $X_1, X_2, \ldots, X_n$ are iid, then the above expression can be written as

$$\begin{aligned} F_{X_{r:n}}(t) &= \sum_{i=r}^{n} \binom{n}{r} F_{X_1}^{r}(t) \bar{F}_{X_1}^{n-r}(t) \\ &= \frac{n!}{(n-r)!(r-1)!} \int_{0}^{F_{X_1}(t)} u^{r-1}(1-u)^{n-r} du, \end{aligned}$$

and the corresponding PDF is given by

$$f_{X_{r:n}}(t) = \frac{n!}{(n-r)!(r-1)!} F_{X_1}^{r-1}(t) \bar{F}_{X_1}^{n-r}(t) f_{X_1}(t).$$

1.2.8 Sequential order statistics

In conventional modeling of the lifetimes of r-out-of-n systems, it is generally assumed that the failure of one component does not have any impact on the lifetimes of remaining surviving components. However, in most of the cases, this assumption oversimplifies to a given real-life scenario. For example, the load of an aircraft engine, when it fails, is transferred to the remaining surviving engines and consequently, the lifetimes of the remaining engines decrease. To model such phenomena, we need more generalized models that can capture the impact of failure of one component on others. To deal with this problem, Kamps [1995] introduced the notion of sequential order statistics (SOS) which is an extension of usual order statistics (OS). Consequently, Cramer and Kamps [1996] introduced sequential r-out-of-n systems as an extension of the ordinary r-out-of-n systems. As before, the lifetime of a sequential r-out-of-n system is the same as the $(n-r+1)$-th order sequential order statistic of the lifetimes of components of the system. In a sequential r-out-of-n system, when a component fails, the distributions of the residual lifetimes of the remaining components are assumed to be different from the distributions that they had previously. This distributional change can be viewed as a failure-related damage or an increment of pressure imposed on the surviving components. There are numerous papers written on this topic [see, e.g., Balakrishnan *et al.*, 2008; Burkschat, 2009; Burkschat *et al.*, 2010; Burkschat and Navarro, 2011; Cramer and Kamps, 1996, 2001, 2003; Kamps, 1995, and the references therein]. In what follows, we define the sequential order statistics.

Let $F_1, \ldots, F_n$ be n absolutely continuous cumulative distribution functions with $F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1)$. Consider a system of n components installed at time $t = 0$. Assume that all components of the system are functioning at the time of inception. Let $X_1^{(1)}, \ldots, X_n^{(1)}$ be n IID random variables, with the distribution function F_1, representing the lifetimes of n components. Then, the first component failure time is given by

$$X_{1:n}^{\star} = \min\left\{X_1^{(1)}, \ldots, X_n^{(1)}\right\}.$$

Given $X^{\star}_{1:n} = t_1$, the residual lifetimes of $(n-1)$ remaining components are equal in distribution with the residual lifetimes of $(n-1)$ IID components with age t_1 and with the distribution function F_2 (instead of F_1); here F_2 is assumed in place of F_1 as the failure of the first component has an impact on the performance of other components. Let the lifetimes of these IID components be represented by $X^{(2)}_1, \dots, X^{(2)}_{n-1}$. Then $X^{(2)}_j \sim F_2(\cdot|t_1)$, where $\bar{F}_2(x|t_1) = \bar{F}_2(x)/\bar{F}_2(t_1)$ for $x \geq t_1$. Moreover, $X^{(2)}_j \geq t_1$, for $j = 1, \dots, n-1$. Further, the second component failure time is given by

$$X^{\star}_{2:n} = \min\left\{X^{(2)}_1, \dots, X^{(2)}_{n-1}\right\}.$$

By proceeding in this manner, we assume that the i-th failure occurs at time t_i $(> t_{i-1})$, i.e., $X^{\star}_{i:n} = t_i$. Then, the residual lifetimes of the $(n-i)$ remaining components are equal in distribution with the residual lifetimes of $(n-i)$ IID components with age t_i and the distribution function F_{i+1}. Let the lifetimes of these IID components be represented by $X^{(i+1)}_1, \dots, X^{(i+1)}_{n-i}$. Then, $X^{(i+1)}_j \sim F_{i+1}(\cdot|t_i)$, where $\bar{F}_{i+1}(x|t_i) = \bar{F}_{i+1}(x)/\bar{F}_{i+1}(t_i)$ for $x \geq t_i$. Moreover, note that $X^{(i+1)}_j \geq t_i$, for $j = 1, \dots, n-i$. Then, the $(i+1)$-th component failure time is given by

$$X^{\star}_{i+1:n} = \min\left\{X^{(i+1)}_1, \dots, X^{(i+1)}_{n-i}\right\}.$$

Finally, if the $(n-1)$-th component failure occurs at time $t_{n-1} = X^{\star}_{n-1:n}$, then the last component failure time is given by $X^{\star}_{n:n}$ with the reliability function $\bar{F}_n(x|t_{n-1}) = \bar{F}_n(x)/\bar{F}_n(t_{n-1})$ for $x \geq t_{n-1}$. Then the random variables $X^{\star}_{1:n} \leq \cdots \leq X^{\star}_{n:n}$ are called the sequential order statistics (SOS) based on $F_1, \dots, F_n$. Moreover, the joint probability density function of all ordered sequential order statistics is given by

$$f_{X^{\star}_{1:n},\dots,X^{\star}_{n:n}}(x_1, \dots, x_n) = n! \prod_{i=1}^{n} \left(\frac{\bar{F}_i(x_i)}{\bar{F}_i(x_{i-1})}\right)^{n-i} \frac{f_i(x_i)}{\bar{F}_i(x_{i-1})},$$

where $0 = x_0 \leq x_1 \leq \cdots \leq x_n < \infty$. Further, the underlying structure of sequential order statistics can be viewed in terms of a triangular scheme as follows.

$$\begin{array}{ccccccccc}
X^{\star}_{1:n} & \leftarrow & X^{(1)}_1 & X^{(1)}_2 & \dots & X^{(1)}_{n-1} & X^{(1)}_n & \sim & F_1(\cdot) \\
X^{\star}_{2:n} & \leftarrow & X^{(2)}_1 & X^{(2)}_2 & \dots & X^{(2)}_{n-1} & & \sim & \frac{F_2(\cdot)-F_2(t_1)}{1-F_2(t_1)} \\
\vdots & & \vdots & \vdots & & & & \vdots & \\
X^{\star}_{n-1:n} & \leftarrow & X^{(n-1)}_1 & X^{(n-1)}_2 & & & & \sim & \frac{F_{n-1}(\cdot)-F_{n-1}(t_{n-2})}{1-F_{n-1}(t_{n-2})} \\
X^{\star}_{n:n} & \leftarrow & X^{(n)}_1 & & & & & \sim & \frac{F_n(\cdot)-F_n(t_{n-1})}{1-F_n(t_{n-1})}.
\end{array}$$

1.2.9 Random weighted r-out-of-n system

There are many real-life scenarios wherein components in a system may make distinct contributions to the survivability of the system. The performance of such a system can be determined not only by the survivability of its components, but also by the use of some appropriate measure (such as total capacity) of all contributions made by its components. However, these systems are structurally not the same with the so-called coherent systems since every component in a coherent system has equal contribution in the performance of the system. To overcome this problem, Wu and Chen [1994] introduced the notion of weighted r-out-of-n systems that capture such a disparity among its components. A system of n components, with associated weight vector $(w_1, \ldots, w_n)$, is said to be a weighted r-out-of-n system if it functions as long as the sum of the weights of all its failed components is at most r (> 0), and the system would fail when the sum is greater than r. This notion was further generalized in different directions (namely, two-stage weighted r-out-of-n systems, and so on) by [Chen and Yang, 2005; Samaniego and Shaked, 2008], and others. Different reliability studies of weighted r-out-of-n systems have been carried out by [Chen and Yang, 2005; Ding *et al.*, 2010; Eryilmaz, 2013, 2015, 2019; Franko and Tütüncü, 2015; Li and Zuo, 2008; Long *et al.*, 2008; Wang *et al.*, 2012; Zhang, 2018].

In a weighted r-out-of-n system, all the weights associated with its components are assumed to be fixed. However, this assumption may not be realistic in some situations. Indeed, the weights of the components may be random for different reasons; for example, shocks that they encounter, the environments in which they are operated in, etc. In this regard, Eryilmaz [2013] introduced the notion of random weighted r-out-of-n systems wherein the weights are assumed to be random. In what follows, we give the definition of a random weighted r-out-of-n system.

Definition 1.2.26 *Let T be a random variable representing the lifetime of a random weighted r-out-of-n system formed by n components with the lifetime vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ and the associated random weight vector $\boldsymbol{W} = (W_1, \ldots, W_n)$, where W_i represents the weight of the i-th component, $i = 1, \ldots, n$. Then, the total capacity (weight) of the system at time t (> 0), denoted by $\psi_{r,n}(t; \boldsymbol{X}, \boldsymbol{W})$, is given by*

$$\psi_{r,n}(t; \boldsymbol{X}, \boldsymbol{W}) \quad = \quad I\left(N_{t,n}(\boldsymbol{X}) \geq r\right) C_{t,n}(\boldsymbol{X}, \boldsymbol{W}),$$

where $I(\cdot)$ is the indicator random variable, $N_{t,n}(\boldsymbol{X}) = \sum_{i=1}^{n} I(X_i > t)$, and $C_{t,n}(\boldsymbol{X}, \boldsymbol{W}) = \sum_{i=1}^{n} W_i I(X_i > t)$. Thus, for a given threshold value $c > 0$, $\psi_{r,n}(t; \boldsymbol{X}, \boldsymbol{W}) \geq c$ holds if, and only if, both $\sum_{i=1}^{n} I(X_i > t) \geq r$ and $\sum_{i=1}^{n} W_i I(X_i > t) \geq c$ hold. Consequently, the failure time of the system with the given threshold c can be expressed as

$$T = \inf\left\{t : \psi_{r,n}(t; \boldsymbol{X}, \boldsymbol{W}) < c\right\}, t > 0.$$

Moreover, the reliability of the system is given by

$$P(T > t) = P\left(\psi_{r,n}(t; \boldsymbol{X}, \boldsymbol{W}) \geq c\right), \quad t > 0. \qquad \square$$

Some special cases of random weighted r-out-of-n systems are the ordinary r-out-of-n system ($c = r$ and $W_i \equiv 1$ with probability 1, for $i = 1, \ldots, n$), the fixed weighted c-out-of-n system ($r = 1$ and $W_i = w_i$ with probability 1, for $i = 1, \ldots, n$), etc.

1.2.10 Redundant component

The systems used in real-life are very complex in nature and are also quite expensive. So, one of the important problems in reliability theory is how one can improve the system reliability. One of the effective solutions to this problem is the allocation of one or more redundant component(s) into the system. There are mainly two types of redundancy notion. The commonly used one is the active (hot) redundancy in which the redundant component and the original component work together and consequently, the system lifetime is the maximum of the lifetimes of the original and redundant components. On the other hand, the cold redundancy is another strategy in which the redundant component is in inactive state with zero failure rate until the original component fails. It begins to function under the usual environment (in which the system is functioning) when the original component fails. Consequently, in this case, the system lifetime is the sum of the lifetimes of the original component and its corresponding redundant component. In the literature, numerous studies were conducted on different allocation policies of these two types of redundancies into a system [see Barlow and Proschan, 1965; Brito *et al.*, 2011; Boland *et al.*, 1988, 1992, 1994; Boland and El-Neweihi, 1995; Singh and Mishra, 1994; Mishra *et al.*, 2009, 2011a,b, and the references there in].

1.2.11 Reliability importance measure

It is common for a system to have several components that do not equally contribute to its functioning. Due to limited resources, we often go for a frugal approach in design, improvement, or maintenance of such a system. However, it may be extremely time-consuming, or possibly impossible, to construct a formal optimum strategy for utilizing the existing resources for every complex system. In such cases, it is preferable to distribute resources in proportion to the importance of the components of a system and to focus those resources on a relatively limited number of components that are crucial to the overall functioning of the system. The motivation for defining several importance measures naturally comes from this.

In reliability theory, importance measures are widely used to determine the relative importance of components in a system. Each component of a system has two facets: its place in the system and its failure probability. Similarly, a system has two distinguishing characteristics, namely, structure and reliability. The reliability of a system depends not only on the reliabilities of its components but also on its structure. Thus, the system reliability may vary depending on the positions of the components within the system and its components' reliabilities. Thus, it is essential to know the importance of each component of a system. Different importance measures serve this purpose. Importance measures are broadly classified into three groups [Birnbaum, 1969]. The first group is the set of structural importance measures, which evaluate the importance of various components of a system in accordance with their locations. A structural importance measure utilizes only the structural knowledge of the system. The second group contains the set of all reliability importance measures. These are used when the mission time of a system is implicit and fixed, and the components are therefore evaluated by their reliabilities at a fixed time point. In order to compute reliability importance measures, the mission duration and the reliabilities of the components must be known in advance. When a system and its components are designed to provide a service for an indefinite amount of time, the third category of importance measures (namely, lifetime importance measures) are used. The lifetime importance measures are sensitive to both the relative locations of components in the system and the lifetime distributions of those components.

The monograph by Kuo and Zhu [2012a] provides in-depth information about different characteristics and various applications of different importance measures. Among all existing reliability importance measures, the Birnbaum marginal reliability importance (MRI) and joint reliability importance (JRI) measures are the popular ones. The MRI is the rate of change in system reliability with respect to the changes in components' reliabilities. Increasing in system reliability is strongly correlated with the improvements in reliabilities of high-MRI components. The JRI measure quantifies how the interaction effect of two components impacts on the reliability of a system. It describes the relative importance of one component when the other is functioning. Depending on the sign of JRI, Hagstrom [1990] defined the notions of 'reliability complements' and 'reliability substitutes'. If the JRI is greater than zero, then the importance of one component is more while the other is functioning (synergy or complements); if the JRI is less than zero, then the importance of one component is less while the other is functioning (diminishing returns or substitutes); if the JRI is equal to zero, then the importance of one component is not changed while the other is functioning [see Armstrong, 1995]. Below we give the definitions of some importance measures that are used in this thesis [see Eryilmaz and Bozbulut, 2014; Rahmani *et al.*, 2016].

Definition 1.2.27 *Let $h(\boldsymbol{p})$ be the reliability of a coherent system with n independent components having reliabilities $\boldsymbol{p} = (p_1, \ldots, p_n)$. Then, the*

(a) *Birnbaum marginal reliability importance (MRI) measure for the i-th component of the system, denoted by MRI_i, is given by $MRI_i = h(\boldsymbol{p}|p_i = 1) - h(\boldsymbol{p}|p_i = 0)$;*

(b) *Joint reliability importance (JRI) measure for the i and j-th components of the system,* *denoted by* $JRI_{(i,j)}$*, is given by* $JRI_{(i,j)} = h(\boldsymbol{p}|p_i = 1, p_j = 1) - h(\boldsymbol{p}|p_i = 1, p_j = 0) - h(\boldsymbol{p}|p_i = 0, p_j = 1) + h(\boldsymbol{p}|p_i = 0, p_j = 0)$.

Remark 1.2.1 *One may note that* $JRI_{(k,l)} = MRI_{k,+l} - MRI_{k,-l}$*, where* $MRI_{k,+l}$ *(resp.* $MRI_{k,-l}$*) is the MRI of the* k*-th component when the* l*-th component is working (resp. failed).* □

Like stochastic orders, the orders that are defined based on different reliability importance measures are also equally important to establish various inequalities in reliability theory, information theory and associated fields. The definition of an ordering notion based on Birnbaum reliability importance measure is given below [Boland *et al.*, 1989; Meng, 1995].

Definition 1.2.28 *Let* $h(\boldsymbol{p})$ *be the reliability of a coherent system with* n *independent components having reliabilities* $\boldsymbol{p} = (p_1, \ldots, p_n)$*. Then, the* j*-th component is said to be more critical than* i*-th component in the sense of the Birnbaum MRI measure, denoted by* $i \overset{c}{<} j$*, if* $MRI_i < MRI_j$*.*

1.2.12 System signature

Coherent systems are uniquely identified/indexed by their structure functions. As the number of n-component coherent systems increases exponentially with n, indexing systems through their structure functions tends to be of limited benefit in problems that demand comparisons or optimizations among systems. To overcome this drawback, a different kind of index, the so-called *system signature*, was introduced in the literature [Samaniego, 2007]. The system signature is a probability vector with the benefits of being both readily managed and interpretable but less broad than a structure function. We below provide the definition of a system signature.

Definition 1.2.29 *Let* T *be a random variable representing the lifetime of a coherent system formed by* n *IID components. Then, the signature of this system, denoted by* $\boldsymbol{s}$*, is an* n*-dimensional probability vector whose* i*-th element* s_i *is given by*

$$s_i = P(T = X_{i:n}),\ i = 1, \ldots, n,$$

where $X_{i:n}$ *is the* i*-th order usual order statistic of* n *components' failure times, and* s_i *represents the probability that the* i*-th component failure causes the system failure. Moreover, the lifetime of the system is given by*

$$\bar{F}_T(t) = \sum_{i=1}^{n} \sum_{j=0}^{i-1} s_i \binom{n}{j} F^j(t) \bar{F}^{n-j}(t),$$

where $F(\cdot)$ *is the common distribution function of the lifetimes of the components.*

1.2.13 Some useful definitions

We below discuss some functions that will be used in subsequent chapters.

Definition 1.2.30 *A real-valued function* $f(\cdot)$ *is called star-shaped (resp. antistar-shaped) if* $f(x)/x$ *is increasing (resp. decreasing) in* x.

Definition 1.2.31 *A real-valued function* $f(\cdot)$ *is called super-additive (resp. sub-additive) if, for all* x, y, $f(x+y) \geq (resp. \leq\ ) f(x)+f(y)$. □

Next, we discuss the proportional hazard rate model (PHR) which is one of the commonly used semi-parametric models. This model has many applications in survival analysis, reliability theory and many other fields [see Marshall and Olkin, 2007]. A set of random variables $\{Z_1, \ldots, Z_n\}$ is said to follow the PHR model if, for $i = 1, \ldots, n$,

$$\bar{F}_{Z_i}(t) = (\bar{F}(t))^{\alpha_i}, \text{ for some } \alpha_i > 0 \text{ and for all } t > 0,$$

where $\bar{F}$ is the baseline survival function. Moreover, we denote this PHR model by $F_{Z_i} \sim \mathrm{PHR}(F; \alpha_i)$, for $i = 1, \ldots, n$.

1.3 A Brief Discussion on the Main Results of the Thesis

In reality, we deal with many complex systems such as televisions, computers, electronic circuits for mechanical systems, etc. In most cases, the underlying structures of these systems can be described either by ordinary coherent systems or by some generalized variants of coherent systems. Reliability theory broadly deals with various aspects of a system, namely, whether a system is good or bad, how long a system may survive, what the failure rate of a system is, etc. Despite the widespread use of complex systems in our daily lives, theoretical advances are not yet at a level where all aspects of a system can be well-described. Keeping this in mind, this thesis is devoted to study different reliability aspects of general coherent systems.

This thesis is divided into seven chapters. Chapter 1 is about the introduction and review of literature. Here, we discuss some basic definitions, notations and carry out a comprehensive literature review regarding the problems addressed in this thesis. A concise discussion of the main results devloped in Chapters 2-6 is presented below. Further, the concluding remarks are given in Chapter 7.

In modern industries, different kinds of systems are used that are not only expensive but also have quite complex structures. If such a system fails, it may cause catastrophic damage to the concerned industry. Thus, it is always a challenging problem to choose the most reliable system from a set of possible options. Stochastic orders are frequently employed, as effective tools, for dealing with this problem. Another important problem related to system reliability is how one can analyze the behaviour of the lifetime of a system that has been operational for a period of time. The notion of stochastic ageings can be utilized to deal with this problem.

As the majority of systems used in daily life have interdependent relationships between their components, the assumption of 'independent components' in the modeling of such systems often oversimplifies the actual scenarios. Copula is widely used to model the interdependence structure between the components of a system. Among all existing copulas, the family of Archimedean copulas is most often used because of its versatility in capturing dependence structures. In addition, Archimedean copulas have many interesting mathematical properties. Thus, in Chapters 2-4, we study various ordering and ageing properties of general coherent systems formed by dependent components with the dependence structures described by Archimedean copulas.

In Chapter 2, we consider different coherent systems formed by DID components with the dependence structures described by Archimedean copulas. We derive several stochastic comparison results in terms of different stochastic orders (namely, the usual stochastic order, the hazard rate order, the reversed hazard rate order, the likelihood ratio order, and the ageing faster orders in terms of the failure rate and the reversed failure rate) for these systems. Further, we investigate the closure properties of several ageing classes (namely, IFR, DFR, DRFR, ILR, DLR, IFRA, and DFRA) under the formation of r-out-of-n systems. In addition, we provide several numerical examples to demonstrate the developed results.

Traditional approaches of modeling the lifetimes of r-out-of-n systems assume that the failure of one component has no effect on the lifetimes of the remaining components of a system. However, in many circumstances, this assumption is not appropriate. In order to capture the failure effect of one component on others, Kamps [1995] introduced the notion of sequential order statistics (SOS) which is an extension of usual order statistics (OS). The GOS model, which includes several other models of ordered random variables (namely, order statistics with non-integral sample size, Pfeifer's records, k_n-records from non-identical distributions, progressively type-II censored order statistics, etc.), has a close relationship with the SOS model. In particular, the GOS model is a special case of the SOS model. Thus, the SOS model can be considered as a more general framework that encompasses almost all models of ordered random variables. However, one fundamental flaw of the SOS model is that the lifetimes of the components are assumed to be independent at each step (i.e., after each failure), which is a very stringent assumption in many applications. To overcome this drawback, Baratnia and Doostparast [2019] have recently introduced an extended SOS model, known as the developed sequential order statistics (DSOS), which captures both interdependence and failure dependence structures. In Chapter 3, we consider the DSOS model governed by the Archimedean copula and study some univariate stochastic comparison results in both one-sample and two-sample scenarios. Further, we study various ageing properties of DSOS.

In line with the univariate study, the multivariate ordering properties of the SOS, the GOS and the OS models were also extensively studied in the literature. It is important to highlight that all of the studies done on GOS and SOS models have been conducted with the IID assumption of the underlying random variables. In Chapter 4, we consider the DSOS model governed by the Archimedean copula and develop several multivariate stochastic comparison results for this model. Further, we introduce the notion of developed generalized order statistics (DGOS), which is a generalization of the GOS model involving dependent random variables. Then, we study various univariate and multivariate ordering properties of DGOS governed by the Archimedean copula. Further, we study the same problem for SOS with non-identical components. The main objective of this chapter is to study various ordering properties of different models of ordered random vectors with dependent components.

There are many real-life scenarios wherein the components of a system may have different contributions in functioning the system. The basic structures of these systems match with weighted coherent systems. Some of the commonly used weighted coherent systems are the weighted r-out-of-n systems, the weighted min-max system, the two-stage weighted r-out-of-n systems, etc. [see Wu and Chen, 1994; Chen and Yang, 2005; Samaniego and Shaked, 2008]. As a generalization of weighted r-out-of-n systems, Eryilmaz [2013] introduced the notion of random weighted r-out-of-n systems wherein the weights are assumed to be random. In the same spirit, the notion of random weighted coherent systems was defined later. There are many real-life systems that can be represented by random weighted coherent systems [see Eryilmaz, 2013]. The performance of a random weighted r-out-of-n system is measured by its total capacity (weight). However, this measure is not meaningful for an arbitrary random weighted coherent system [see Eryilmaz, 2013]. One of the major drawbacks of this measure is that it does not depends on the structure of a system. To overcome this drawback, in Chapter 5, we introduce a new performance measure (namely, the survival capacity) and subsequently, introduce three survivability notions for random weighted coherent systems. Further, we study the optimal active redundancy allocation strategy in a random weighted coherent system with three different survival mechanisms. In addition to this, we also study the optimal assembly method of random weights in random weighted coherent systems.

We continue our discussion on random weighted coherent systems in Chapter 6 as well. Here, we provide an efficient method for computing the reliability of a random weighted coherent system with three different survival mechanisms. Further, we provide a signature-based reliability representation for a random weighted coherent system and derive the Birnbaum MRI and JRI measures for the components of this system. In addition, we provide an algorithm for evaluating the same. We introduce a novel structure-based weighted importance measure for the components of a random weighted coherent system with Type-III survivability, and then give an algorithm for deriving this measure. Further, we do the reliability estimation for a random weighted coherent system based on two different simulated data sets.

Finally, we conclude our discussion by mentioning potential future research problems followed by a list of references.

To make each chapter as independent as possible, a few definitions, descriptions of various problems, and some notations may be repeated in the subsequent chapters.