

Input-Output Dynamic Properties of Complex Networks

3.1 INTRODUCTION

The behavior of the networked system has a strong relationship with topological structure of the networks. Therefore, it is imperative to investigate this connection. Recently, some researchers have started work in this direction. Initial work on the correlation between the control indices and the properties of a network was presented in [Ruths and Ruths, 2014; Torres and Roy, 2015]. In several applications, it is necessary to understand the influence of a node (or a set of nodes) on a specific node (or a set of specific nodes). When the system dynamics are LTI, transfer function between input and output nodes models input-output behavior of the system. A study to understand input-output behavior, for consensus problem, using transfer functions is presented in [Briegel et al., 2011]. In addition, Yoon et al. [Yoon and Tsumura, 2011], derived a formula of the transfer function for a specific type of consensus network e.g. cyclic network with various types of network structure such as undirected, directed networks and considering different reference nodes. Moreover, the authors in [Briegel et al., 2011] presented an analysis to relate the shortest distance and Gain Factor for Laplacian matrix.

The existing works discuss consensus problem and do not have a relationship between network structure and important indices. This chapter derives the input-output properties between nodes by considering, in a more general setup, non-uniform link weights for large number of systems. The system dynamics is captured by the weighted Adjacency matrix. In particular, the structural properties of the network are related with a Gain Factor, DC-Gain and system zeros. These relations provide an insight into a hidden connection between the system behavior and network structure. In this chapter, it has been proved that the sum of product of the weights of all available shortest paths between the control and output nodes is equal to the Gain Factor of the pole-zero-gain transfer function of the complex network. It also relates the DC-Gain for the peripheral nodes with the distance between the control and output nodes in the complex network and comment about the system zeros. In particular, non-minimum phase nature of a networked system is investigated. Finally, it builds a relationship between the position of the network zeros and input-output dynamic properties of the networked systems because the performance of the networked system is influenced by the network zeros.

The remaining sections of the chapter are organized as follows. Section 3.2 presents a model of the system and derive pole-zero-gain transfer function of a network. Section 3.3 unfolds relationship between network properties and control centric properties like Gain Factor, DC-Gain and system zeros. The simulation results are presented in Section 3.4. Finally, Section 3.5 summarizes the chapter.

Notation: In this chapter, the set of real numbers is denoted by $\mathbb{R}$, the set of natural numbers is denoted by $\mathbb{N}$, the set of positive real numbers is denoted by $\mathbb{R}_{>0}$, the set of positive natural numbers is denoted by $\mathbb{N}_{>0}$. The subscript c and o are used for representing control node and output node respectively. $G_{co}(s)$ or $G(s)$ is transfer function between the nodes c and o . m_z represent the number of network zeros and k_{gf} is the Gain Factor of the networked system. The r_{rd} represents the relative degree, d is the distance between nodes c and o , $diam$ is the diameter of

network, g is the DC-Gain of network transfer function, B_c and C_o are the standard basis vectors with c^{th} and o^{th} entry 1, respectively, and the all other entries are zero and $det(.)$ is the determinant of a matrix and $tr(.)$ is the trace of a matrix, $|\cdot|$ represents the absolute value, $\|\cdot\|$ represents the norm of a matrix and $adj(.)$ is the adjoint of a matrix.

3.2 PRELIMINARIES

This section reviews the basic concepts in algebraic graph theory [Godsil and Royle, 2013], and presents a model for complex networks.

3.2.1 System Model and Preliminaries

A complex network with n nodes can be represented by a graph $G := (V, E)$, where, $V := \{1, 2, \dots, n\}$ and $E \subseteq V \times V$ are the set of nodes and the set of edges respectively. The weighted Adjacency matrix of the network G is defined as A , where, the elements $a_{ij} \in \mathbb{R}_{>0} \cup \{0\}$ are non-zero if node i and j are adjacent, or, are 0 otherwise.

The dynamics of complex network can be defined as

$$\dot{x}(t) = Ax(t) + B_c u(t) \quad (3.1a)$$

$$y(t) = C_o x(t) \quad (3.1b)$$

where $A \in \mathbb{R}^{n \times n}$ is the Adjacency matrix of the networked system, $B_c \in \mathbb{R}^{n \times 1}$ and $C_o \in \mathbb{R}^{1 \times n}$ are input and output matrices respectively, and which are the standard basis vectors with c^{th} and o^{th} entry 1, respectively, and the all other entries are zero, $x \in \mathbb{R}^{n \times 1}$ and $y(t) \in \mathbb{R}$ are state vector and output respectively, and $u \in \mathbb{R}$ is control input in the networked system.

Here, the Adjacency matrix A of the networked system has eigenvalues $\lambda_i \in \mathbb{R}$. This work assumes that there are no self loops in the network therefore diagonal elements of A are zero. In this chapter, the complex networks shall refer as networked systems henceforth.

Now, the control node, output node, diameter and peripheral nodes in the networked system are defined as follows.

Definition 3.1. A node is said to be a control node if the input is injected to that node. It is denoted by subscript c .

Definition 3.2. A node is said to be an output node if the output is observed from that node. It is denoted by subscript o .

Definition 3.3. The diameter of the networked system is defined as $diam = \max_{o \in V} \{d\}$, where, d is the distance between the control node c and output node o with $o = 1, \dots, n-1$.

The diameter $diam$ of the networked system is the largest distance between the control node c and the output node o .

Definition 3.4. The control and output nodes are said to be the peripheral nodes if the distance between the control and output nodes is equal to $d = n-1$ in the network.

In this chapter, the control node c is shown in red color and the output node o in green color. This concept illustrates with the help of the complete and star networks as shown in Figure 3.1.

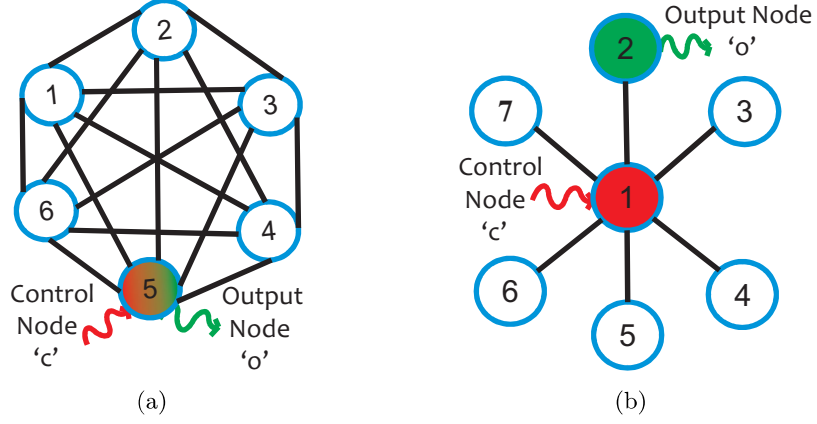


Figure 3.1: The control and output nodes have been marked in graph (a) Complete network with $c = o$, and (b) Star network with $c \neq o$

Transfer function between the control c and output o nodes becomes

$$G_{co}(s) = \frac{Y_o(s)}{U_c(s)} = k_{gf} \frac{s^{m_z} + b_{m_z-1}s^{m_z-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (3.2)$$

The pole-zero form of Eq.(3.2) becomes

$$G_{co}(s) = k_{gf} \frac{(s - z_1) \dots (s - z_{m_z})}{(s - p_1) \dots (s - p_n)} = k_{gf} \frac{\prod_{i=0}^{m_z} (s - z_i)}{\prod_{i=0}^n (s - p_i)} = C_o(sI - A)^{-1}B_c \quad (3.3)$$

where m_z and n represent the number of network zeros and the number of poles of the transfer function respectively, and k_{gf} is the Gain Factor of the transfer function for the system.

Now, the relative degree r_{rd} of the pole-zero-gain transfer function of the networked systems is defined as the difference between the order of the denominator and the numerator polynomials of the pole-zero-gain transfer function of the networked system. It relates the length of the shortest path between a pair of nodes to the relative degree of the networked system defined in Eq.(3.1).

Proposition 3.1. [Abad Torres and Roy, 2014] For the networked system defined in Eq.(3.1), the relative degree r_{rd} corresponding to a given pair of the c and o nodes is defined as

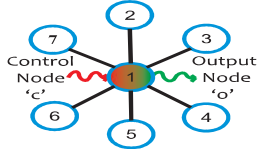
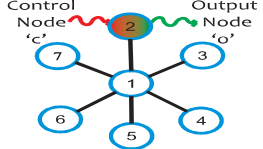
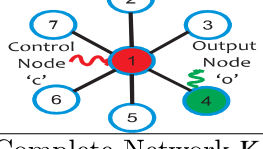
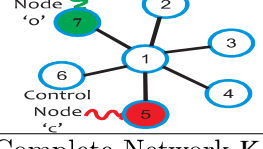
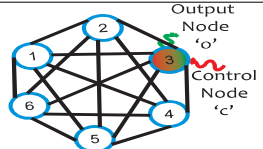
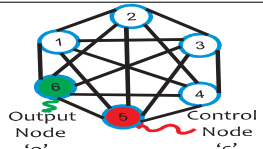
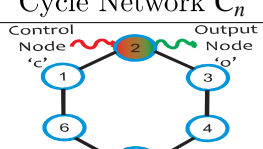
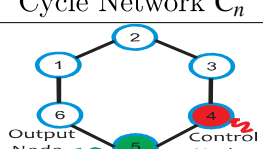
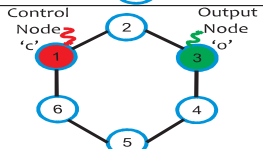
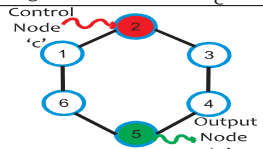
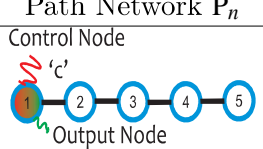
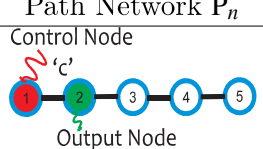
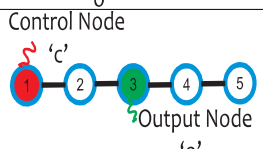
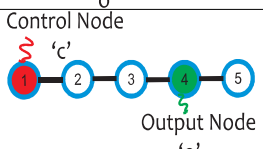
$$r_{rd} = d + 1 \quad (3.4)$$

where d i.e. $d = \text{dist}(c, o)$ be the shortest distance between the c and o nodes. The relative degree r_{rd} is 1 and 2 for Figure 3.1(a) and 3.1(b) respectively.

Proposition 3.2. [Abad Torres and Roy, 2014] The value $m_z = n - d - 1 = n - r_{rd}$ represents the number of finite zeros in the networked system defined in Eq.(3.1).

Choice of the control and output nodes influence transfer function between them. Table 3.1 lists some transfer functions for some networks for different control and output nodes. In Table 3.1, the transfer functions are written in terms of n , where n is even in the case of complete and cycle networks and odd in the case of star and path networks. Therefore, these transfer functions are general, i.e. the presented equations hold only for even values of n for complete and cycle networks and odd values of n for star and path networks. In particular, from the Table 3.1 it can be observed

Table 3.1 : Pole-zero-gain transfer functions for star networks S_n , complete networks K_n , cycle networks C_n and path networks P_n

Star Network S_n	Transfer Function	Star Network S_n	Transfer Function
	$\frac{s}{(s \pm \sqrt{n-1})}$		$\frac{(s \pm \sqrt{n-2})}{s(s \pm \sqrt{n-1})}$
	$\frac{1}{(s \pm \sqrt{n-1})}$		$\frac{1}{s(s \pm \sqrt{n-1})}$
Complete Network K_n	Transfer Function	Complete Network K_n	Transfer Function
	$\frac{s-(n-2)}{(s-(n-1))(s+1)}$		$\frac{1}{(s-(n-1))(s+1)}$
Cycle Network C_n	Transfer Function	Cycle Network C_n	Transfer Function
	$\frac{s(s \pm \sqrt{\frac{n}{2}})}{(s \pm (\frac{n}{2}-1))(s \pm 1)}$		$\frac{(s \pm \sqrt{\frac{n}{2}-1})}{(s \pm (\frac{n}{2}-1))(s \pm 1)}$
	$\frac{s}{(s \pm (\frac{n}{2}-1))(s \pm 1)}$		$\frac{2}{(s \pm (\frac{n}{2}-1))(s \pm 1)}$
Path Network P_n	Transfer Function	Path Network P_n	Transfer Function
	$\frac{(s \pm 1.618)(s \pm 0.618)}{s(s \pm \sqrt{\lceil \frac{n}{2} \rceil})(s \pm 1)}$		$\frac{(s \pm \sqrt{\lceil \frac{n}{2} \rceil})}{(s \pm \sqrt{\lceil \frac{n}{2} \rceil})(s \pm 1)}$
	$\frac{1}{s(s \pm \sqrt{\lceil \frac{n}{2} \rceil})}$		$\frac{1}{(s \pm \sqrt{\lceil \frac{n}{2} \rceil})(s \pm 1)}$

that the pole-zero-gain transfer functions of the Star Network S_n , the Complete Network K_n , the Cycle Network C_n and the Path Network P_n are non-minimum phase. A natural query raises is how the structure of network influences system dynamics.

To understand influence of network topology on system dynamics, the following illustrative example is used as shown in Figure 3.2.

Consider two networks of six nodes as shown in Figure 3.2. In these networks, consider the same distance between the control node $c = 4$ and output node $o = 1$. The topology of both are almost identical except a link (shown in red color in Figure 3.2(b)) between nodes 2 and 5 (in Figure 3.2(b) added compared to original Figure 3.2(a)). Now, calculate the Gain Factor k_{gf} and transfer

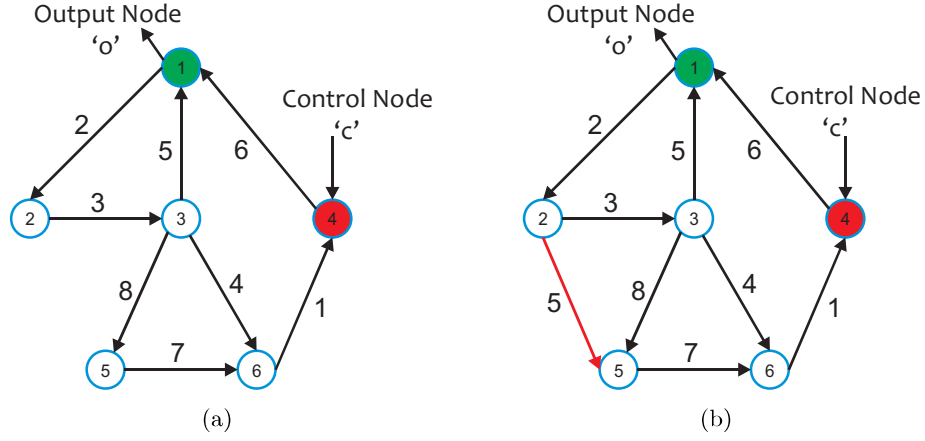


Figure 3.2 : Two networks of six nodes with same distance $d = 4$ between control node $c = 4$ (in red color) and output node $o = 1$ (in green color) but topology is different (a) The original network with 8 edges (b) One edge (shown in red color) between node 2 and node 5 added in original network in Figure 3.2(a)

function for both the networks. For network given in Figure 3.2(a), the Gain Factor k_{gf} is 24 and the transfer function is given in Table 3.2. For network given in Figure 3.2(b), the Gain Factor k_{gf} is 94 and the transfer function is given in Table 3.2. It can be observed with small change in structure, the Gain Factor and transfer function are different for both the networks. Therefore, it is necessary to understand role of the network structure to access the influence of a node (or a set of nodes) on a specific node (or a set of specific nodes). This motivates to investigate the connection between system behavior and the network structure. The comparison between the parameters calculated for the networks as shown in Figure 3.2 is summarized in Table 3.2.

Table 3.2 : Comparison between the parameters calculated for networks given in Figure 3.2

For original network shown in Figure 3.2(a)		For modified network shown in Figure 3.2(b)	
Uniform link weights (all weights unity)	Non-uniform link weights	Uniform link weights (all weights unity)	Non-uniform link weights
Transfer Function (TF) = $\frac{s+1}{s^6-s^3-s-1}$	Transfer Function (TF) = $\frac{24s+336}{s^6-30s^3-144s-2016}$	Transfer Function (TF) = $\frac{2s+1}{s^6-s^3-2s-1}$	Transfer Function (TF) = $\frac{94s+336}{s^6-30s^3-564s-2016}$
Gain Factor $k_{gf} = 1$	Gain Factor $k_{gf} = 24$	Gain Factor $k_{gf} = 2$	Gain Factor $k_{gf} = 94$

A small modification in topology has substantial influence on different indices. In a complex network, the structure has a strong influence on several important indices related with input-output dynamic properties. Furthermore, a relative position of the control node and output node and network structure has a strong influence on these indices. This chapter shows that the sum of the product of the weights of all shortest paths between the control and output nodes is equal to the Gain Factor of the pole-zero-gain transfer function of the complex network. It relates the DC-Gain for the peripheral nodes with the distance between the control and output nodes in the complex network and comment about the position of non-minimum phase zeros for collocated control and output nodes. Finally, it builds a relationship between the position of the network zeros and input-output dynamic properties of the networked systems.

3.3 INFLUENCE OF INPUT-OUTPUT DYNAMIC PROPERTIES ON CONTROL INDICES

This section relates the Gain Factor, DC-Gain and network zeros with the structure of a network. Understanding the pole-zero-gain transfer function characteristics in terms of the properties of network provides new insight about the dependency.

3.3.1 Gain Factor of Networked Systems

First a result which involves the power of matrix A is presented.

Theorem 3.1. For a given networked system, the control node c and output node o has shortest distance d , then the $(c,o)^{th}$ element of the powers of Adjacency matrix A holds

$$(A^q)_{co} = \begin{cases} w & \text{if } q \geq d \\ 0 & \text{if } q < d \end{cases} \quad (3.5)$$

where w represents the sum of product of the weights of walks between node c and node o which is defined as $w = \sum_{i=1}^{p_q} \prod_{j=1}^q w_{ij}$, where, p_q is the total number of walks of the length q and w_{ij} is weight of j^{th} edge of i^{th} walk and $q \geq 1$, $q \in \mathbb{N}_{>0}$.

Proof: This result can be proved by induction. The shortest distance from the control node c to the output node o is denoted as d . For $q = 1$ (since $A^1 = A$) result is true, where, the element w_{co} is equal to the weight of edges from node c to node o and it is also equal to the product of weights of walks of length 1 from node c to node o . Assume that the result is true for q . Let w_{co} be the $(c,o)^{th}$ element of A^q , and let a_{co} be the $(c,o)^{th}$ element of A . According to the inductive hypothesis w_{co} is the sum of product of weights of walks of the length q from the node c to the node o . Consider the $(c,o)^{th}$ element of $A^{q+1} = AA^q = a_{c1}w_{1o} + a_{c2}w_{2o} + \dots + a_{cn}w_{no} = \sum_{l=1}^n a_{cl}w_{lo}$, where, a_{ci} is the $(c,i)^{th}$ element of A and w_{io} is the $(i,o)^{th}$ element of A^q . Here $a_{c1}w_{1o}$ is equal to the product of weights of walk of the length q from node 1 to node o times the product of weight of walk of the length 1 from c to 1. This is same as to the product of weights of walks of length $q+1$ from c to o , where, node 1 is the second node. This logic applies for every l , i.e. $a_{cl}w_{lo}$ = product of weights of walks from c to o in which node l is the second node. Therefore, the sum gives the Sum of Product (SOP) of the weights of all possible walks from the c to o . Hence, for $q < d$, the $(c,o)^{th}$ element of all A^{q-1} is zero because there is no walk of length $(q-1)$ between the nodes c and o , and, therefore $(A^q)_{co}$ is also zero. For $q = d$, the $(c,o)^{th}$ element of the $(A^q)_{co}$ gives the sum of product of the weights of all possible walks of the length q between the control c and output o nodes. Hence, Theorem 3.1 is proved.

Corollary 3.1. For a given networked system with uniform link weights (all weights unity), the control node c and output node o has shortest distance d , then the $(c,o)^{th}$ element of the power (q) of Adjacency matrix A i.e. A^q is equal to the number of walks between the node c and node o of length q .

Now, the relationship between the Gain Factor k_{gf} of Eq.(3.3) and the sum of product of weights of all shortest paths between the control and output nodes is obtained in the next theorem.

Theorem 3.2. Consider the pole-zero-gain transfer function Eq.(3.3) for given control and output nodes then the Gain Factor k_{gf} of the networked system Eq.(3.1) becomes

$$k_{gf} = \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij} \quad (3.6)$$

where p_q is the total number of the paths of the shortest distance d and w_{ij} is weight of the edge j of path i .

Proof: The relative degree of the system is r_{rd} . The transfer function can be represented as

$$G_{co}(s) = \frac{k_{gf}s^{m_z} + b_{m_z-1}s^{m_z-1} + \dots + b_1s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0} \quad (3.7)$$

where $m_z = n - r_{rd}$ is the number of zeros in the system.

Also, the transfer function can be obtained in terms of matrices A , B_c and C_o

$$G_{co}(s) = C_o(sI_n - A)^{-1}B_c = \frac{C_o \text{adj}(sI_n - A)B_c}{\det(sI_n - A)} \quad (3.8)$$

where $\det(sI_n - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0$ is the characteristic polynomial equation of A , using Eq.(3.7) and Eq.(3.8)

$$k_{gf}s^{m_z} + b_{m_z-1}s^{m_z-1} + \dots + b_1s + b_0 = C_o \text{adj}(sI_n - A)B_c \quad (3.9)$$

The adjoint matrix $\text{adj}(sI_n - A)$ can be expressed by using the coefficient matrices Q_i of the numerator polynomial. Using the Leverrier-Faddeev algorithm [Hou, 1998], it becomes $\text{adj}(sI_n - A) = \sum_{i=0}^{n-1} Q_i s^i$. Here $Q_{n-j} = A^{j-1} + \sum_{\eta=1}^{j-1} \gamma_\eta A^{j-1-\eta}$, with $\gamma_\eta \in \mathbb{R}$ and $j = 1, 2, \dots, n$, where, $\gamma_\eta = -\frac{1}{\eta} \text{tr}(AQ_{n-\eta})$, $\eta = 1, 2, \dots, j-1$, where, $\text{tr}(AQ_{n-\eta})$ is the trace of the matrix $AQ_{n-\eta}$. This gives $k_{gf}s^{m_z} + b_{m_z-1}s^{m_z-1} + \dots + b_1s + b_0 = \sum_{i=0}^{n-1} C_o Q_i B_c s^i$. By comparing the coefficients of s , it becomes $C_o Q_i B_c = 0$ for $i > m_z$, and for $i = m_z$, the Gain Factor becomes $k_{gf} = C_o Q_{m_z} B_c$. The coefficient matrix Q_{m_z} can be expressed as

$$k_{gf} = C_o Q_{m_z} B_c = C_o A^{r_{rd}-1} B_c + \sum_{\eta=1}^{r_{rd}-1} \gamma_\eta C_o A^{r_{rd}-1-\eta} B_c \quad (3.10)$$

with $d = \text{dist}(o, c)$, and using Proposition 3.1, the Gain Factor becomes

$$k_{gf} = C_o A^d B_c + \sum_{\eta=1}^d \gamma_\eta C_o A^{d-\eta} B_c \quad (3.11)$$

In Eq.(3.11), the term $C_o A^{d-\eta} B_c$ corresponds to $(o, c)^{th}$ element of $A^{d-\eta}$. Using Theorem 3.1, the $(o, c)^{th}$ element of $A^{d-\eta}$ is zero as $d - \eta$ is less than the shortest distance between the control node c and output node o . Thus $k_{gf} = w = \sum_{i=1}^{p_q} \prod_{j=1}^d w_{ij}$, where, p_q is the total number of the paths of the shortest distance d and w_{ij} is weight of the path i of particular edge j of the path. So, the k_{gf} gives the sum of product of the weights of the shortest paths between the control node c and the output node o . This completes the proof of Theorem 3.2.

Corollary 3.2. For a given networked system with uniform link weights (all weights unity), consider the pole-zero-gain transfer function Eq.(3.3) for a given control node c and output node o then the Gain Factor k_{gf} of the networked system Eq.(3.1) is equal to the sum of number of walks between the node c and node o of length q .

In what follows next, the DC-Gain for the peripheral nodes of the network is obtained. It is a well known fact that large DC-Gain brings better tracking accuracy and disturbance rejection. When the open-loop DC-Gain is large, then the closed-loop DC-Gain is nearly equal to 1. This implies that the Steady State Error (SSE) becomes small.

3.3.2 DC-Gain of Peripheral Nodes

This subsection obtains the DC-Gain in terms of the distance between the control c and output o nodes in the networked system. Simplifying Eq.(3.3), the transfer function becomes

$$G_{co}(s) = C_o \frac{\text{adj}(sI - A)}{\det(sI - A)} B_c \quad (3.12)$$

At low frequency i.e. $s = 0$, from Eq.(3.12), the transfer function becomes

$$G_{co}(0) = C_o \frac{\text{adj}(-A)}{\det(-A)} B_c = C_o \frac{(-1)^{n-1} \text{adj}(A)}{(-1)^n \det(A)} B_c \quad (3.13)$$

Finally, the DC-Gain is obtained as

$$G_{co}(0) = g = (-1) C_o \frac{\text{adj}(A)}{\det(A)} B_c \quad (3.14)$$

In the following theorem, the relation between the distance of the c and o nodes and the DC-Gain of the transfer function for peripheral nodes is obtained.

Theorem 3.3. When $d = n - 1$, the DC-Gain g of the system

$$g = \frac{\text{SOP of weights of paths of length } (n - 1) \text{ between } o \text{ \& } c}{\det(A)} \quad (3.15)$$

where SOP is the sum of product.

Proof: Here B_c is the standard basis vector with c^{th} entry 1 and other entries are zero, i.e. $B_c = [0 \dots 1 \dots 0 \dots 0]^T$ and C_o be the standard basis vector with o^{th} entry 1 and other entries are zero, i.e. $C_o = [0 \dots 0 \dots 1 \dots 0]^T$. So, the DC-Gain of the networked system is defined as

$$g = (-1) C_o \frac{\text{adj}(A)}{\det(A)} B_c \quad (3.16)$$

Cayley-Hamilton theorem can be used to find the $\text{adj}(A)$ of the networked system [Decell, 1965]. It allows the adjugate (adjoint) of A to be represented in terms of traces and powers of A . So, the adjugate (adjoint) of A is denoted as

$$\text{adj}(A) = \sum_{\rho=0}^{n-1} A^\rho \sum_{\phi_1, \phi_2, \dots, \phi_{n-1}} \prod_{l=1}^{n-1} \frac{(-1)^{\phi_l+1}}{l^{\phi_l} \cdot \phi_l!} (\text{tr}(A^l))^{\phi_l} \quad (3.17)$$

where n is the dimension of A , $\text{tr}(A^l)$ is the trace of matrix A^l and the sum is taken over ρ and all sequences of $\phi_l \geq \mathbb{N}_{>0}$, satisfying the linear Diophantine equation [Mordell, 1969] defined as

$$\rho + \sum_{l=1}^{n-1} l \cdot \phi_l = n - 1 \quad (3.18)$$

where ρ is the index which comes from Eq.(3.17). The expand of Eq.(3.17) is in more generic form for a large number of nodes in the networked system. The Eq.(3.17) can be written as

$$\begin{aligned} \text{adj}(A) = I_n [& (-1)^n \frac{\text{tr}(A^{n-1})}{(n-1)} + \sum_{\phi=1; n-1-\phi \neq \phi}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-\phi-1} \frac{\text{tr}(A^{n-1-\phi}) \cdot \text{tr}(A^\phi)}{(n-1-\phi) \cdot (\phi)} + \\ & \sum_{j=2, \dots, \lfloor \frac{n-1}{2} \rfloor; n-1-\phi=\phi} (-1)^{n-j-1} \frac{(\text{tr}(A^{\phi-j+2}))^j}{(\phi-j+2)^j \cdot j!}] + (-A)[(n-1)^{th} \text{ order terms}] \end{aligned} \quad (3.19)$$

where $\phi = 1, 2, \dots, \lfloor (\frac{n-1}{2}) \rfloor$ and I_n is $n \times n$ identity matrix. Here $tr(A) = 0$ (because network does not have any self-loop) and trace of higher power of the matrix A are nonzero. The Eq.(3.19) is valid for $n \geq 2$ and for $n = 1$, the $adj(A) = 1$.

Let $\lambda_1, \lambda_2, \dots, \lambda_k, \dots, \lambda_{n-1}, \lambda_n$ be the eigenvalues of the matrix A . The trace of $\{A; A^2; \dots; A^k; \dots; A^{n-1}; A^n\}$ can be obtained in terms of $\{\lambda_1, \dots, \lambda_n; \lambda_1^2, \dots, \lambda_n^2; \dots; \lambda_1^k, \dots, \lambda_n^k; \dots; \lambda_1^{n-1}, \dots, \lambda_n^{n-1}; \lambda_1^n, \dots, \lambda_n^n\}$ respectively as

$$\begin{aligned} tr(A) &= \lambda_1 + \lambda_2 + \dots + \lambda_n = 0 \\ &\vdots, \dots, \vdots, \vdots, \dots, \vdots, \vdots, \dots, \vdots \\ tr(A^k) &= \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k \neq 0 \\ &\vdots, \dots, \vdots, \vdots, \dots, \vdots, \vdots, \dots, \vdots \\ tr(A^n) &= \lambda_1^n + \lambda_2^n + \dots + \lambda_n^n \neq 0 \end{aligned} \quad (3.20)$$

Define $\kappa_1 := tr(A), \kappa_2 := tr(A^2), \dots, \kappa_k := tr(A^k), \dots, \kappa_{n-1} := tr(A^{n-1})$, and $\kappa_n := tr(A^n)$, where, $\kappa_i \in \mathbb{R}, i = 1, 2, \dots, n$. Now, obtain the $adj(A)$ for n^{th} -order networked system in terms of eigenvalues of matrix A . Then Eq.(3.19) can be written as

$$\begin{aligned} adj(A) &= I_n [(-1)^n \frac{\kappa_{n-1}}{(n-1)} + \sum_{\phi=1; n-1-\phi \neq \phi}^{\lfloor (\frac{n-1}{2}) \rfloor} (-1)^{n-\phi-1} \frac{\kappa_{n-1-\phi} \cdot \kappa_\phi}{(n-1-\phi) \cdot (\phi)} + \\ &\sum_{j=2, \dots, \lfloor (\frac{n-1}{2}) \rfloor; n-1-\phi=\phi} (-1)^{n-j-1} \frac{(\kappa_{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}] + (-A)[(n-1)^{th} \text{ order terms}] \end{aligned} \quad (3.21)$$

where $\phi = 1, 2, \dots, \lfloor (\frac{n-1}{2}) \rfloor$.

Now, define $\kappa_{n-1} = \sum_{i=1}^n \lambda_i^{n-1}$, $\kappa_{n-1-\phi} = \sum_{i=1}^n \lambda_i^{n-1-\phi}$, $\kappa_\phi = \sum_{i=1}^n \lambda_i^\phi$, and $\kappa_{\phi-j+2} = \sum_{i=1}^n \lambda_i^{\phi-j+2}$, and substituting into Eq.(3.21), the $adj(A)$ becomes

$$\begin{aligned} adj(A) &= I_n [(-1)^n \frac{(\sum_{i=1}^n \lambda_i^{n-1})}{(n-1)} + \sum_{\phi=1; n-1-\phi \neq \phi}^{\lfloor (\frac{n-1}{2}) \rfloor} (-1)^{n-\phi-1} \frac{(\sum_{i=1}^n \lambda_i^{n-1-\phi}) \cdot (\sum_{i=1}^n \lambda_i^\phi)}{(n-1-\phi) \cdot (\phi)} + \\ &\sum_{j=2, \dots, \lfloor (\frac{n-1}{2}) \rfloor; n-1-\phi=\phi} (-1)^{n-j-1} \frac{(\sum_{i=1}^n \lambda_i^{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}] + (-A)[(n-1)^{th} \text{ order terms}] \end{aligned} \quad (3.22)$$

where $\phi = 1, 2, \dots, \lfloor (\frac{n-1}{2}) \rfloor$.

Expanding the Eq.(3.22) in terms of powers of the matrix A and use constants $\mu_0, \mu_1, \dots, \mu_k, \dots, \mu_{n-3}, \mu_{n-2}, \mu_{n-1}$ to simplify the expression gives

$$adj(A) = \mu_0 I_n + \mu_1 A + \dots + \mu_k A^k + \dots + \mu_{n-3} A^{n-3} + \mu_{n-2} A^{n-2} + \mu_{n-1} A^{n-1} \quad (3.23)$$

where $\mu_{n-1} = (-1)^{n-1}$, $\mu_{n-2} = (-1)^{n-2} (\sum_{i=1}^n \lambda_i^1)$, $\mu_{n-3} = -1(-1)^{n-3} \frac{(\sum_{i=1}^n \lambda_i^2)}{2}$, $\dots$, $\mu_k = ((-1)^k (-1)^{n-k} \frac{(\sum_{i=1}^n \lambda_i^{n-k-1})}{(n-k-1)}) + \sum_{\phi=1; n-k-1-\phi \neq \phi}^{\lfloor (\frac{n-k-1}{2}) \rfloor} (-1)^{n-k-\phi-1} \frac{(\sum_{i=1}^n \lambda_i^{n-k-1-\phi}) \cdot (\sum_{i=1}^n \lambda_i^\phi)}{(n-k-1-\phi) \cdot (\phi)} + \sum_{j=2, \dots, \lfloor (\frac{n-k-1}{2}) \rfloor, \phi=1, \dots, \lfloor (\frac{n-k-1}{2}) \rfloor; n-k-1-\phi=\phi} (-1)^{n-k-j-1} \frac{(\sum_{i=1}^n \lambda_i^{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}$, $\dots$, $\mu_1 = (-1)(-1)^{n-1} \frac{(\sum_{i=1}^n \lambda_i^{n-1-1})}{(n-1-1)} + \sum_{\phi=1; n-1-1-\phi \neq \phi}^{\lfloor (\frac{n-1-1}{2}) \rfloor} (-1)^{n-1-\phi-1} \frac{(\sum_{i=1}^n \lambda_i^{n-1-1-\phi}) \cdot (\sum_{i=1}^n \lambda_i^\phi)}{(n-1-1-\phi) \cdot (\phi)} + \sum_{j=2, \dots, \lfloor (\frac{n-1-1}{2}) \rfloor, \phi=1, \dots, \lfloor (\frac{n-1-1}{2}) \rfloor; n-1-1-\phi=\phi} (-1)^{n-1-j-1} \frac{(\sum_{i=1}^n \lambda_i^{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}$, and

$$\mu_0 = (-1)^n \frac{(\sum_{i=1}^n \lambda_i^{n-1})}{(n-1)} + \sum_{\phi=1; n-1-\phi \neq \phi}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-\phi-1} \frac{(\sum_{i=1}^n \lambda_i^{n-1-\phi}) \cdot (\sum_{i=1}^n \lambda_i^\phi)}{(n-1-\phi) \cdot (\phi)} \\ + \sum_{j=2, \dots, \lfloor \frac{n-1}{2} \rfloor, \phi=1, \dots, \lfloor \frac{n-1}{2} \rfloor; n-1-\phi=\phi} (-1)^{n-j-1} \frac{(\sum_{i=1}^n \lambda_i^{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}.$$

Here $\mu_0, \mu_1, \dots, \mu_k, \dots, \mu_{n-3}, \mu_{n-2}, \mu_{n-1}$ are constants. Using Eq.(3.16) and Eq.(3.23), the DC-Gain of the networked system is obtained as

$$g = (-1)C_o \frac{\mu_0 I_n + \mu_1 A + \dots + \mu_k A^k + \dots + \mu_{n-1} A^{n-1}}{\det(A)} B_c \quad (3.24)$$

Further, simplifying Eq.(3.24), the DC-Gain becomes

$$g = \frac{-1}{\det(A)} (C_o I_n B_c \mu_0 + C_o A B_c \mu_1 + \dots + C_o A^k B_c \mu_k + \dots + C_o A^{n-2} B_c \mu_{n-2} + C_o A^{n-1} B_c \mu_{n-1}) \quad (3.25)$$

The Eq.(3.25) can be expressed in matrix form as

$$g = \frac{-1}{\det(A)} [C_o I_n B_c \quad C_o A B_c \dots C_o A^k B_c \dots C_o A^{n-2} B_c \quad C_o A^{n-1} B_c] [\mu_0 \quad \mu_1 \dots \mu_k \dots \mu_{n-2} \quad \mu_{n-1}]^T \quad (3.26)$$

Further, rearranging Eq.(3.26), the g becomes

$$g = \frac{-1}{\det(A)} C_o [I_n B_c \quad A B_c \dots A^k B_c \dots A^{n-2} B_c \quad A^{n-1} B_c] [\mu_0 \quad \mu_1 \dots \mu_k \dots \mu_{n-2} \quad \mu_{n-1}]^T \quad (3.27)$$

Using definition of Controllability Matrix $CM = [I_n B_c \quad A B_c \dots A^k B_c \dots A^{n-2} B_c \quad A^{n-1} B_c]$. Now, the Eq.(3.27) can be rewritten as

$$g = \frac{-1}{\det(A)} C_o \sum_{q=0}^{n-1} A^q B_c \mu_q \quad (3.28)$$

The DC-Gain g of the networked system depends on the powers of the matrix A . Using the Theorem 3.1, if $q < d$ then $(o, c)^{th}$ element of A^q is zero and $q \geq d$ then $(o, c)^{th}$ element of A^q is nonzero.

In special case when $d = n - 1$, which is the largest possible distance between the nodes c and o then $(o, c)^{th}$ element of A^d is equal to sum of product of weights of paths of length $(n - 1)$.

For $d = n - 1$, using Eq.(3.28)

$$g = \frac{-1}{\det(A)} C_o A^{n-1} B_c \mu_{n-1} \quad (3.29)$$

Now, using μ_{n-1} from Eq.(3.23), the absolute DC-Gain g is obtained as

$$g = \frac{SOP \text{ of weights of paths of length } (n-1) \text{ between } o \text{ \& } c}{\det(A)} \quad (3.30)$$

It means that the DC-Gain g depends on the distance between the control c and output o nodes. This completes the proof of Theorem 3.3.

Remark 3.1. When $d = n - 1$ in the networked system then the diameter of network is also $diam = n - 1$. Therefore, the absolute DC-Gain g becomes $g = \frac{C_o A^{diam} B_c}{det(A)}$.

Now the constants μ_q in Eq.(3.28) can be expressed in terms of the closed walks in the network.

Definition 3.5. A closed walk is a walk that begins and ends at the same node [Biggs, 1993]. The sum of product of weights of closed walks of length q in a network equals to $tr(A^q)$. The proof is available in Lemma 2.5 of [Biggs, 1993].

The sum of product of weights of closed walks of length q is denoted as cw_q . The trace of a matrix A is the sum of its eigenvalues, so an alternative expression can be obtained as

$$cw_q := tr(A^q) = \sum_{i=1}^n A_{ii}^q = \sum_{i=1}^n \lambda_i^q \quad (3.31)$$

where $q = 1, 2, \dots, n - 1$.

Using the basic properties of trace of a matrix, a relationship between the constants μ_q and properties can be obtained as

$$\begin{aligned} cw_1 &:= tr(A^1) = \sum_{i=1}^n A_{ii}^1 = \sum_{i=1}^n \lambda_i^1 = \text{Sum of product of weights of closed walks of length 1.} \\ cw_2 &:= tr(A^2) = \sum_{i=1}^n A_{ii}^2 = \sum_{i=1}^n \lambda_i^2 = \text{Sum of product of weights of closed walks of length 2.} \\ cw_3 &:= tr(A^3) = \sum_{i=1}^n A_{ii}^3 = \sum_{i=1}^n \lambda_i^3 = \text{Sum of product of weights of closed walks of length 3. and so on...} \\ cw_k &:= tr(A^k) = \sum_{i=1}^n A_{ii}^k = \sum_{i=1}^n \lambda_i^k = \text{Sum of product of weights of closed walks of length } k. \text{ and so on...} \\ cw_{n-2} &:= tr(A^{n-2}) = \sum_{i=1}^n A_{ii}^{n-2} = \sum_{i=1}^n \lambda_i^{n-2} = \text{Sum of product of weights of closed walks of length } (n-2). \\ cw_{n-1} &:= tr(A^{n-1}) = \sum_{i=1}^n A_{ii}^{n-1} = \sum_{i=1}^n \lambda_i^{n-1} = \text{Sum of product of weights of closed walks of length } (n-1). \end{aligned}$$

$$\begin{aligned} &\text{Using Eq.(3.23), Eq.(3.31) and Definition 3.5, the constant terms i.e. } \mu_0, \mu_1, \dots, \mu_k, \dots, \mu_{n-3}, \mu_{n-2}, \mu_{n-1} \text{ are related to the network properties as } \mu_{n-1} = (-1)^{n-1}, \mu_{n-2} = (-1)^{n-2}(cw_1), \mu_{n-3} = \\ &-1(-1)^{n-3} \frac{(cw_2)}{2}, \dots, \mu_k = ((-1)^k (-1)^{n-k} \frac{(cw_{n-k-1})}{(n-k-1)}) + \sum_{\phi=1; n-k-1-\phi \neq \phi}^{\lfloor \frac{(n-k-1)}{2} \rfloor} (-1)^{n-k-\phi-1} \frac{(cw_{n-k-1-\phi}) \cdot (cw_\phi)}{(n-k-1-\phi) \cdot (\phi)} + \\ &\sum_{j=2, \dots, \lfloor \frac{(n-k-1)}{2} \rfloor, \phi=1, \dots, \lfloor \frac{(n-k-1)}{2} \rfloor; n-k-1-\phi=\phi} (-1)^{n-k-j-1} \frac{(cw_{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}, \dots, \\ &\mu_1 = (-1)(-1)^{n-1} \frac{(cw_{n-2})}{(n-2)} + \sum_{\phi=1; n-1-1-\phi \neq \phi}^{\lfloor \frac{(n-1-1)}{2} \rfloor} (-1)^{n-1-\phi-1} \frac{(cw_{n-2-\phi}) \cdot (cw_\phi)}{(n-2-\phi) \cdot (\phi)} \\ &+ \sum_{j=2, \dots, \lfloor \frac{(n-1-1)}{2} \rfloor, \phi=1, \dots, \lfloor \frac{(n-1-1)}{2} \rfloor; n-1-1-\phi=\phi} (-1)^{n-1-j-1} \frac{(cw_{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}, \text{ and } \mu_0 = (-1)^n \frac{(cw_{n-1})}{(n-1)} + \\ &\sum_{\phi=1; n-1-\phi \neq \phi}^{\lfloor \frac{(n-1)}{2} \rfloor} (-1)^{n-\phi-1} \frac{(cw_{n-1-\phi}) \cdot (cw_\phi)}{(n-1-\phi) \cdot (\phi)} + \sum_{j=2, \dots, \lfloor \frac{(n-1)}{2} \rfloor, \phi=1, \dots, \lfloor \frac{(n-1)}{2} \rfloor; n-1-\phi=\phi} (-1)^{n-j-1} \frac{(cw_{\phi-j+2})^j}{(\phi-j+2)^j \cdot j!}. \end{aligned}$$

So it can be observed that these constants are related to network properties i.e. sum of product of weights of the closed walk of length q , $q = 1, 2, \dots, n - 1$ in the networked system.

Corollary 3.3. For a system with uniform link weights (all weights unity), the distance d between the control node c and output node o is $d = n - 1$ then the DC-Gain g of the networked system Eq.(3.1) is equal to the sum of number of walks between the node c and node o of length $(n - 1)$ divided by $det(A)$ or $\frac{A^{n-1}}{det(A)}$.

The next section discusses about zeros of the transfer function.

3.3.3 Network Zeros Based on Graph-Theoretic Approach

The zeros provide the behavior and information regarding the frequency response of the system. A system is non-minimum phase if there are double zeros lying at the origin or if zeros are in the open right-half plane otherwise it is minimum phase [Kalman et al., 1969]. Therefore, the performance of the networked system is influenced by the network zeros. Fortunately, some interesting affirmations on the position of the network zeros can be built.

Definition 3.6. Two polynomials $N(s)$ and $D(s)$ are called coprime if $\gcd(N(s), D(s)) = 1$, where $\gcd$ is greatest common divisor.

Definition 3.7. The polynomial $D(s)$ is called monic if the coefficient of highest power of s in $D(s)$ is unity.

Theorem 3.4. Assume that polynomials $N(s)$ and $D(s)$ in Eq.(3.3) are coprime and $D(s)$ is monic for the networked system defined in Eq.(3.1). Then the networked system Eq.(3.1) has no zeros at the origin when

$$\text{Sum of product of weights of paths of length } (n-1) \neq \sum_{\eta=1}^{n-1} \frac{t_{\eta}}{\eta} C_o A^{n-(\eta+1)} B_c \quad (3.32)$$

where $t_{\eta} \in \mathbb{R}$ with $\eta = 1, 2, \dots, n-1$.

Proof: The numerator polynomial $N(s)$ of the pole-zero-gain transfer function Eq.(3.3) is denoted as $N(s) = C_o \text{adj}(sI_n - A) B_c$. Using the Leverrier-Faddeev algorithm, the $\text{adj}(sI_n - A) = \sum_{i=0}^{n-1} Q_i s^i$ is obtained. Here $Q_{n-r_{rd}} = A^{r_{rd}-1} + \sum_{\eta=1}^{r_{rd}-1} \gamma_{\eta} A^{r_{rd}-1-\eta}$ or $Q_{m_z} = A^d + \sum_{\eta=1}^d \gamma_{\eta} A^{d-\eta}$, where, m_z is the number of zeros, r_{rd} is the relative degree, d is the shortest distance between the control node c and the output node o and n is the number of nodes in the networked system. Here, the γ_{η} can be calculated as $\gamma_{\eta} = -\frac{1}{\eta} \text{tr}(A Q_{n-\eta})$, $\eta = 1, 2, \dots, r_{rd} - 1$, where, $\text{tr}(A Q_{n-\eta})$ gives the trace of the matrix $A Q_{n-\eta}$. Therefore, the numerator polynomial can be rewritten as $N(s) = k_{gf} s^{m_z} + b_{m_z-1} s^{m_z-1} + \dots + b_1 s + b_0 = C_o Q_0 B_c + C_o Q_1 B_c s + \dots + C_o Q_{m_z} B_c s^{m_z} = C_o \sum_{i=0}^{m_z} Q_i s^i B_c$. Using $m_z = n - r_{rd} = n - 1 - d$, the Q_0 is calculated as $Q_0 = A^{n-1} + \sum_{\eta=1}^{n-1} \gamma_{\eta} A^{n-1-\eta}$ or $Q_0 = A^{d+m_z} + \sum_{\eta=1}^{d+m_z} \gamma_{\eta} A^{d+m_z-\eta}$, where, $\gamma_{\eta} = -\frac{1}{\eta} \text{tr}(A Q_{d+m_z+1-\eta})$, $\eta = 1, 2, \dots, d + m_z$, where, $\text{tr}(A Q_{d+m_z+1-\eta})$ gives the trace of the matrix $A Q_{d+m_z+1-\eta}$. In this case, the value of $d + m_z$ remains constant in the networked system. However, with increase in d , number of zeros m_z reduces.

The constants are defined as follows

$$\begin{aligned} \sigma_1 &:= A^{n-1} \\ \sigma_2 &:= -A^{n-2} \text{tr}\left(\frac{\sigma_1}{A^{n-2}}\right) \\ \sigma_3 &:= -\frac{1}{2} A^{n-3} \text{tr}\left(\frac{\sigma_1 + \sigma_2}{A^{n-3}}\right) \\ &\vdots \\ \sigma_i &:= -\frac{1}{i-1} A^{n-i} \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_{i-1}}{A^{n-i}}\right) \\ &\vdots \\ \sigma_{n-1} &:= -\frac{1}{n-2} A^1 \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_{i-1} + \dots + \sigma_{n-2}}{A^1}\right) \\ \sigma_n &:= -\frac{1}{n-1} A^0 \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_{i-1} + \dots + \sigma_{n-1}}{A^0}\right) I_{n \times n} \end{aligned} \quad (3.33)$$

where $I_{n \times n}$ is the identity matrix of dimension $n \times n$, $\text{tr}(\cdot)$ is the trace function of matrix and here matrix A should be invertible matrix.

Using the above definitions

$$\begin{aligned} Q_0 = & A^{n-1} - A^{n-2} \text{tr}\left(\frac{\sigma_1}{A^{n-2}}\right) - \frac{1}{2} A^{n-3} \text{tr}\left(\frac{\sigma_1 + \sigma_2}{A^{n-3}}\right) - \dots - \frac{1}{i-1} A^{n-i} \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_{i-1}}{A^{n-i}}\right) - \dots \\ & - \frac{1}{n-2} A^1 \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_i + \dots + \sigma_{n-2}}{A^1}\right) - \frac{1}{n-1} A^0 \text{tr}\left(\frac{\sigma_1 + \sigma_2 + \dots + \sigma_i + \dots + \sigma_{n-1}}{A^0}\right) I_{n \times n} \end{aligned} \quad (3.34)$$

So, the matrix Q_0 can be written as

$$Q_0 = A^{n-1} - \left[\sum_{\eta=1}^{n-1} \frac{1}{\eta} A^{n-(\eta+1)} \text{tr}\left(\frac{\sum_{j=1}^{\eta} \sigma_j}{A^{n-(\eta+1)}}\right) \right] I_{n \times n} \quad (3.35)$$

Using Eq.(3.35), the value of b_0 is calculated as $b_0 = C_o Q_0 B_c$. Therefore, using Eq.(3.35), it becomes

$$b_0 = C_o \left(A^{n-1} - \left[\sum_{\eta=1}^{n-1} \frac{1}{\eta} A^{n-(\eta+1)} \text{tr}\left(\frac{\sum_{j=1}^{\eta} \sigma_j}{A^{n-(\eta+1)}}\right) \right] I_{n \times n} \right) B_c \quad (3.36)$$

where $\text{tr}\left(\frac{\sum_{j=1}^{\eta} \sigma_j}{A^{n-(\eta+1)}}\right)$ gives constant value for all η . Let $t_\eta = \text{tr}\left(\frac{\sum_{j=1}^{\eta} \sigma_j}{A^{n-(\eta+1)}}\right)$ with $t_\eta \in \mathbb{R}$, where, $\eta = 1, 2, \dots, n-1$.

Now Eq.(3.36) becomes

$$b_0 = C_o \left(A^{n-1} - \left[\sum_{\eta=1}^{n-1} \frac{t_\eta}{\eta} A^{n-(\eta+1)} \right] I_{n \times n} \right) B_c \quad (3.37)$$

Simplifying Eq.(3.37), it becomes

$$b_0 = C_o A^{n-1} B_c - \sum_{\eta=1}^{n-1} \frac{t_\eta}{\eta} C_o A^{n-(\eta+1)} B_c \quad (3.38)$$

The value of b_0 in Eq.(3.38), is not equal to zero if

$$\text{Sum of product of weights of paths of length } (n-1) \neq \sum_{\eta=1}^{n-1} \frac{t_\eta}{\eta} C_o A^{n-(\eta+1)} B_c \quad (3.39)$$

So, it is observed that if the value of b_0 is not equal to zero then there is no zero at the origin in the networked systems Eq.(3.1). This completes the proof of Theorem 3.4.

Lemma 3.1. The value of b_0 is not equal to zero if $\sum_{i=j=1}^n \alpha_j \gamma_i \neq 0$ otherwise $b_0 = 0$, where, α_j is the j^{th} element value of row vector α , where, α is the σ^{th} row of the controllability matrix CM and γ_i is the i^{th} element value of column vector $\gamma = [-\frac{t_{n-1}}{n-1} \quad -\frac{t_{n-2}}{n-2} \dots -\frac{t_i}{i-1} \dots -\frac{t_2}{2} \quad -\frac{t_1}{1} \quad 1]^T$ with $i = j = 1, 2, \dots, n$, $\gamma \in \mathbb{R}^{n \times 1}$.

Proof: From Eq.(3.38), b_0 becomes

$$b_0 = C_o \left[A^{n-1} B_c - \frac{t_1}{1} A^{n-2} B_c - \frac{t_2}{2} A^{n-3} B_c - \dots - \frac{t_i}{i-1} A^{n-i} B_c - \dots - \frac{t_{n-2}}{n-2} A^1 B_c - \frac{t_{n-1}}{n-1} A^0 B_c \right] \quad (3.40)$$

Now Eq.(3.40) can be written in matrix form as

$$b_0 = C_o [A^0 B_c \ A^1 B_c \dots A^{n-i} B_c \dots A^{n-3} B_c \ A^{n-2} B_c \ A^{n-1} B_c] \\ \left[-\frac{t_{n-1}}{n-1} \quad -\frac{t_{n-2}}{n-2} \dots -\frac{t_i}{i-1} \dots -\frac{t_2}{2} \quad -\frac{t_1}{1} \quad 1 \right]^T \quad (3.41)$$

Now, Controllability Matrix (CM) and a vector γ are defined as

$$CM = [B_c \ A^1 B_c \dots A^{n-i} B_c \dots A^{n-3} B_c \ A^{n-2} B_c \ A^{n-1} B_c] \text{ and} \quad (3.42a)$$

$$\gamma = \left[-\frac{t_{n-1}}{n-1} \quad -\frac{t_{n-2}}{n-2} \dots -\frac{t_i}{i-1} \dots -\frac{t_2}{2} \quad -\frac{t_1}{1} \quad 1 \right]^T \quad (3.42b)$$

From Eq.(3.42), it can be observed that the b_0 depends on the j^{th} element value of o^{th} row of the CM matrix and i^{th} element value of γ vector, where, $i = j = 1, 2, \dots, n$. Now the o^{th} row of the CM matrix is defined as $\alpha = [CM_{o1} \ CM_{o2} \dots CM_{oi} \dots CM_{o(n-1)} \ CM_{on}]$, where, CM_{oj} is the j^{th} element value of o^{th} row of CM matrix. Let α_j is the j^{th} element value of row vector α and γ_i is the i^{th} element value of column vector γ .

Therefore, the condition for $b_0 \neq 0$ in the numerator polynomial $N(s)$ of network transfer function can be obtained by relating it with the CM matrix. The value of b_0 is not equal to zero when $\sum_{i=j=1}^n \alpha_j \gamma_i \neq 0$ otherwise $b_0 = 0$. This completes the proof of Lemma 3.1.

Corollary 3.4. The networked systems Eq.(3.1) has no network zeros at the origin when $b_0 \neq 0$ or $(o, c)^{th}$ element of $Q_0 \neq 0$. Increasing or decreasing the shortest distance d between the control node c and the output node o may decrease or increase respectively, the number of network zeros m_z in the networked system.

The Corollary 3.4 can be explained with the help of some special networks such as complete, star, cycle and path networks as shown in the Table 3.1. It can be inferred from the Table 3.1, that if the shortest distance d between the control node c and output node o is increased (i.e. $d \uparrow$) then the number of network zeros m_z reduces (i.e. $m_z \downarrow$) and the relative degree r_{rd} increases (i.e. $r_{rd} = d + 1 \uparrow$).

In what follows next, a result related to collocated control c and output o nodes and effect on system zeros is presented.

Theorem 3.5. Assume that polynomials $N(s)$ and $D(s)$ in Eq.(3.3) are coprime and $D(s)$ is monic for causal (or unstable) networked system Eq.(3.1). Then for undirected systems, when the control node c and output node o are collocated i.e. $c = o$, the relationship between the poles and zeros can be given as

$$\lambda_1 \leq \beta_1 \leq \lambda_2 \leq \beta_2 \leq \dots \leq \beta_{n-2} \leq \lambda_{n-1} \leq \beta_{n-1} \leq \lambda_n \quad (3.43)$$

where λ_i are the roots of the denominator polynomial i.e. roots of $\det(sI_n - A) = 0$, here $i = 1, 2, \dots, n$ and β_j are roots of numerator polynomial $C_o \text{adj}(sI - A) B_c = 0$ i.e. system zeros.

Proof: As the node c and node o of the networked system to be identical i.e. $c = o$, the distance d between them is zero. So, the networked system has $m_z = n - 1$ zeros. The numerator polynomial of Eq.(3.3) is $N(s) = C_o \text{adj}(sI_n - A) B_c$, which equals to $(c, o)^{th}$ co-factor of the matrix $(sI_n - A)$. Therefore, the zeros can be obtained through $\det((sI_n - A)_{(c,o)}) = 0$. The zeros are the eigenvalues of the matrix $A_{(c,o)}$, where, $A_{(c,o)}$ is the $(c, o)^{th}$ co-factor matrix of the matrix A . As $N(s)$ and $D(s)$ are coprime by assumption and therefore there is no pole zero cancellation. Designate the eigenvalues of

$A_{(c,o)}$ in ascending order as $\beta_1 \leq \beta_2 \leq \dots \leq \beta_{n-2} \leq \beta_{n-1}$. Using the Interlacing Eigenvalues Theorem (see Theorem 8.1.7 of [Golub and Van Loan, 2012] and pp.103 – 104 of [Wilkinson and Wilkinson, 1965]), the Eq.(3.43) is obtained. From Eq.(3.43), it can be observed that the networked system is non-minimum phase. This completes the proof of Theorem 3.5.

In what follows next, the theory is verified with the help of the numerical examples.

3.4 NUMERICAL EXAMPLES

This section illustrates the Theorems 3.1 and 3.2 with the help of a network given in Example 3.1. It also illustrates the DC-Gain with the help of the numerical example of directed cycle network of five nodes given in Example 3.2.

Example 3.1. Consider a weighted directed networked system with five nodes as shown in Figure 3.3.

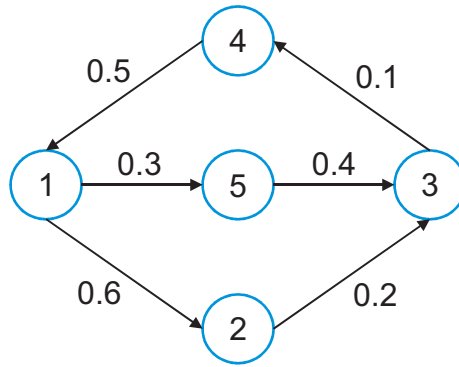


Figure 3.3 : A weighted directed networked system of five nodes with $c \neq o$

The Adjacency and control input-output matrices are given as:

$$A = \begin{bmatrix} 0 & 0.6 & 0 & 0 & 0.3 \\ 0 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & 0 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad B_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \quad C_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}^T$$

Consider the pair of the control node and output node as $\{2, 5\}$ and $\{5, 2\}$. In this example, the shortest distance between the control node and output node is $d = 4$. So, the $A^d = A^4$ is calculated as

$$A^4 = \begin{bmatrix} 0.012 & 0 & 0 & 0 & 0 \\ 0 & 0.006 & 0 & 0 & 0.012 \\ 0 & 0 & 0.012 & 0 & 0 \\ 0 & 0 & 0 & 0.012 & 0 \\ 0 & 0.003 & 0 & 0 & 0.006 \end{bmatrix} \quad (3.44)$$

In first case, consider the control and output nodes pair $\{c = 2, o = 5\}$ with $d = 4$. So, the $(5, 2)^{th}$ entry of A^4 is equal to the sum of product of the weights of all possible paths between node

$c = 2$ and node $o = 5$ of length 4. Here, it can be observed from Eq.(3.44) that the $(5,2)^{th}$ entry of A^4 is equal to 0.003 which is equal to the sum of product of the weights of the path between node $c = 2$ and node $o = 5$ of length 4 shown in Figure 3.3. In the second case, consider the control and output nodes pair $\{c = 5, o = 2\}$ with $d = 4$. So, the $(2,5)^{th}$ entry of A^4 is equal to the sum of product of the weights of all possible paths between node $c = 5$ and node $o = 2$ of length 4. Here, it can be observed from Eq.(3.44) that the $(2,5)^{th}$ entry of A^4 is equal to 0.012 which is equal to the sum of product of the weights of the path between node $c = 5$ and node $o = 2$ of length 4 shown in Figure 3.3. The Gain Factor k_{gf} is also calculated which is equal to the sum of product of the weights of all possible shortest paths between them. Therefore, for both the cases the Gain Factor k_{gf} is obtained as 0.003 and 0.012 respectively.

Example 3.2. Consider a directed cycle networked system of five nodes in this example. The Adjacency and control input-output matrices are given as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

In this network, assume first node as the control node and last node as the output node. So the distance between the control node 1 and output node 5 is equal to $d = 4$. This is also equal to the diameter of the network i.e. $d = \text{diam} = 4$. The determinant of the network matrix A is calculated i.e. $\det(A) = 1$, the DC-Gain g is obtained using Eq.(3.14) as

$$g = -C_o \text{adj}(A) B_c \quad (3.45)$$

where $c \neq o$. Next, the value of $\text{adj}(A)$ is calculated using Eq.(3.23), the $\text{adj}(A)$ becomes

$$\text{adj}(A) = \mu_0 I_n + \mu_1 A + \mu_2 A^2 + \mu_3 A^3 + \mu_4 A^4 \quad (3.46)$$

where $\mu_0 = -\frac{1}{4} \text{tr}(A^4) + \frac{1}{4 \cdot 2!} (\text{tr}(A^2))^2 = 0$, $\mu_1 = -\frac{1}{3} \text{tr}(A^3) = 0$, $\mu_2 = -\frac{1}{2} \text{tr}(A^2) = 0$, $\mu_3 = -\text{tr}(A) = 0$, and $\mu_4 = 1$. Put the values of $\mu_0, \mu_1, \mu_2, \mu_3$, and μ_4 into Eq.(3.46), the $\text{adj}(A)$ becomes

$$\text{adj}(A) = A^4 \quad (3.47)$$

Now, substitute the value of the power of matrix A into Eq.(3.47), the $\text{adj}(A)$ as follows

$$\text{adj}(A) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Now, the DC-Gain g is obtained using Eq.(3.45), the DC-Gain becomes

$$g = -C_o \text{adj}(A) B_c = -C_o \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} B_c \quad (3.48)$$

where $c \neq o$.

The distance from the control node i.e. $c = 1$ to the output node i.e. $o = 5$ is $d = 4$ in the networked system given in the Example 3.2. Now, the absolute g is obtained using Eq.(3.48) with the help of the input matrix $B_{c=1}$ and the output matrix $C_{o=5}$ i.e. $|g_{1 \rightarrow 5}| = 0$. The absolute DC-Gain $|g_{1 \rightarrow 5}|$ is equal to the $(5, 1)^{th}$ element of the matrix $\frac{A^d}{\det(A)} = \frac{A^4}{1} = 0$. This example verifies theoretical development of the Section 3.3.2.

3.5 SUMMARY

In a complex network, the network has a strong influence on input-output properties of a given node pair. The work proposed in this chapter has analyzed this relationship. It has been shown that the sum of the product of weights of all shortest paths between the control and output nodes is equal to the Gain Factor of the pole-zero-gain transfer function of the complex network. In particular, the DC-Gain for peripheral nodes interlaced with the distance between the control and output nodes in the systems has been investigated. Hence, it can be concluded that variation in distance between a pair of nodes has influence on Gain Factor and DC-Gain. Increase or decrease in distance (i.e. link weight) leads to increase or decrease in Gain Factor and DC-Gain respectively. In addition, the non-minimum phase properties of the networked system has been investigated. The proposed work helps in the understanding of input-output behavior of the networked systems. In next chapter, a pair of driver node (or control node) and output node will be analyzed based on sensitivity index.

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